

# Efficient Solutions for Uncertain Random Multiobjective Programming Problem

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Received 2 February 2014; Revised 12 September 2014

## Abstract

Based on the chance theory, which is founded for modeling complex systems with not only uncertainty but also randomness, this paper is devoted to studying a new kind of multiobjective programming problem where randomness and uncertainty exist simultaneously, called uncertain random multiobjective programming (URMOP) problem. Since an uncertain random variable does not admit an order relationship, different statistical characteristics can produce different relationship sense between two uncertain random variables. Starting from this idea, several different concepts of Pareto efficient solutions to URMOP problem are provided on the basis of statistical characteristics, such as expected-value efficiency, expected-value variance efficiency, maximum chance efficiency, etc.. Our study enables us to determine what type of efficient solutions should be considered in URMOP problem by each of these concepts.

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**Keywords:** uncertain random programming, multiobjective programming, Pareto efficiency, uncertainty theory

## 1 Introduction

Many real-life problems require considering and optimizing multiple and conflicting objectives from the multiobjective optimization point of view, leading us into the area of multiobjective programming (MOP) problem, which has been widely studied by researchers in a variety of fields, where the most intensive development of the theory and methods can be found in [3, 15, 4]. These studies are all centered on the deterministic environment. However, as we know, real-world problems are often indeterministic. For a particular problem considered, the parameters involved always take unknown or uncommensurable values at the moment of making the decision.

As pointed out by Liu [11], various kinds of indeterminacy can be categorized as randomness and uncertainty. When the parameters involved in the MOP problem are stochastic in nature, that is to say, those parameters are random variables and can be characterized using probability theory on the basis of historical data collection, the resulting problem is referred to as a stochastic MOP problem, which has been widely used in many real-world decision making problems with the gradual perfection of probability theory, including power systems planning [2], network design [1], traveling salesman problem [14], etc. But in our daily life, due to the economical or technological reasons, we sometimes meet with a situation where the historical data about the event are too small to estimate a probability distribution when making decisions. In this case, information about these problems is often available in the form of experts' belief degree, which can be categorized as uncertainty. Since human tends to overweight unlikely events [7], belief degree has a much larger variance than the real probability distribution. As a result, it is inappropriate to apply probability theory to the belief degree.

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In order to deal with the uncertainty arising from the belief degree, an uncertainty theory was founded by Liu [8] and refined by Liu [11] based on normality, duality, subadditivity and product axioms. When the parameters involved the MOP problem are uncertain variables, the resulting problem is called an uncertain MOP problem. Due to its close description and representation of real world situation, the uncertain MOP problem has been widely studied by researchers in a variety of fields, including both theory (e.g. [16] and [17]) and application (e.g. [5] and [6]). Though the stochastic MOP problem and uncertain MOP problem have been successfully applied to real-life problems, it is not enough to satisfy the practical needs because human uncertainty and objective randomness always simultaneously appear in a system. For instance, in a real redundancy allocation problem, some components' reliability distributions can be obtained via statistics, while some can only be obtained through experts' belief degree.

For modeling such systems with not only uncertainty but also randomness, Liu [12, 13] founded a chance theory on the basis of probability theory and uncertainty theory in 2013. A concept of chance space was proposed by Liu [12] as a product of probability space and uncertainty space, and a chance measure, as a generalization of probability measure and uncertain measure, was also defined to indicate the possibility that an uncertain random event happens. Then a concept of uncertain random variable was proposed to model the quantities under uncertain and random condition, and chance distribution, expected value and variance were employed to describe an uncertain random variable afterwards.

Obviously, the uncertain random MOP (URMOP) models can lead to very large scale problems. Though Liu [12] first introduced the chance theory into uncertain random programming problem in 2013, the literatures about uncertain random MOP problem are still very few. To the best of our knowledge, only Zhou [18] puts forward an efficient solution definition and two crisp equivalent models with respect to uncertain random MOP problem in the previous work, where the solution method proposed is to convert the uncertain random MOP problem into an expected-value MOP model, and the Pareto efficiency is defined based on the expected values of uncertain random objectives. However, given that, in general, an uncertain random variable does not admit an order relationship, a solution may be efficient in the Pareto sense of a URMOP problem for obtaining a specific value for the uncertain random variables which intervene in the problem and not for others. Therefore, in order to obtain the efficient solutions for URMOP problem, we need to define previously the specific concepts of efficient solution that will be used. In this paper, on the basis of uncertain random variable's statistical characteristics, e.g. expected value, variance, maximum chance, optimistic value, pessimistic value, and so on, several efficient solution concepts of URMOP are proposed and defined, such as expected value efficient solution, expected value variance efficient solution, minimum optimistic value efficient solution, minimum pessimistic value efficient solution, and etc..

The rest of this paper is organized as follows. In Section 2, some useful definitions and properties about uncertain variable and uncertain random variable with application to URMOP problem are introduced. Then several new efficient solution concepts of the URMOP problem are proposed in Section 3, where the MOP problem with random uncertain variables is also discussed on the basis of these concepts. Finally, a brief summary is given and some open points are stated for future research work in Section 4.

## 2 Preliminary

In this section, we review some basic concepts about uncertain variable and uncertain random variable, which are used throughout the paper.

### 2.1 Uncertain Variable

Let  $\Gamma$  be a nonempty set, and  $\mathcal{L}$  a  $\sigma$ -algebra over  $\Gamma$ . Each element  $\Lambda$  in  $\mathcal{L}$  is called an event. A set function  $\mathcal{M}$  from  $\mathcal{L}$  to  $[0, 1]$  is called an uncertain measure if it satisfies the following axioms [8]:

**Axiom 1.** (*Normality Axiom*)  $\mathcal{M}\{\Gamma\} = 1$  for the universal set  $\Gamma$ .

**Axiom 2.** (*Duality Axiom*)  $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$  for any event  $\Lambda$ .

**Axiom 3.** (*Subadditivity Axiom*) For every countable sequence of events  $\Lambda_1, \Lambda_2, \dots$ , we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

In 2009, Liu [9] proposed the fourth axiom of uncertainty theory called product axiom.

**Axiom 4.** (*Product Axiom*) Let  $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$  be uncertainty spaces for  $k = 1, 2, \dots$ . The product uncertain measure  $\mathcal{M}$  is an uncertain measure satisfying

$$\mathcal{M} \left\{ \prod_{k=1}^{\infty} \Lambda_k \right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k \{ \Lambda_k \}$$

where  $\Lambda_k$  are arbitrarily chosen events from  $\mathcal{L}_k$  for  $k = 1, 2, \dots$ , respectively. The triplet  $(\Gamma, \mathcal{L}, \mathcal{M})$  is referred to as a uncertainty space, in which an uncertain variable is defined as follows [8]:

**Definition 1** ([8]) *An uncertain variable is a measurable function  $\xi$  from an uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  to the set of real numbers, i.e., for any Borel set  $B$  of real numbers, the set*

$$\{ \xi \in B \} = \{ \gamma \in \Gamma \mid \xi(\gamma) \in B \}$$

*is an event.*

**Definition 2** ([9]) *The uncertain variables  $\xi_1, \xi_2, \dots, \xi_n$  are said to be independent if*

$$\mathcal{M} \left\{ \bigcap_{i=1}^n (\xi_i \in B_i) \right\} = \bigwedge_{i=1}^n \mathcal{M} \{ \xi_i \in B_i \}$$

*for any Borel sets  $B_1, B_2, \dots, B_n$  of real numbers.*

**Definition 3** ([8]) *The uncertainty distribution  $\Phi$  of an uncertain variable  $\xi$  is defined by*

$$\Phi(x) = \mathcal{M} \{ \xi \leq x \}$$

*for any real number  $x$ .*

**Definition 4** ([10]) *Let  $\xi$  be an uncertain variable with regular uncertainty distribution  $\Phi$ . Then the inverse function  $\Phi^{-1}$  is called the inverse uncertainty distribution of  $\xi$ .*

**Definition 5** ([8]) *Let  $\xi$  be an uncertain variable. Then the expected value of  $\xi$  is defined by*

$$E[\xi] = \int_0^{\infty} \mathcal{M} \{ \xi \geq x \} dx - \int_{-\infty}^0 \mathcal{M} \{ \xi \leq x \} dx$$

*provided that at least one of the two integrals is finite.*

**Theorem 1** ([10]) *Let  $\xi$  be an uncertain variable with regular uncertainty distribution  $\Phi$ . If the expected value exists, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha.$$

**Theorem 2** ([10]) *Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain variables with regular uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If the function  $f(x_1, x_2, \dots, x_n)$  is strictly increasing with respect to  $x_1, x_2, \dots, x_m$  and strictly decreasing with respect to  $x_{m+1}, x_{m+2}, \dots, x_n$ , then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain variable with inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \Phi_{m+2}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)).$$

**Example 1** An uncertain variable  $\xi$  is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \frac{x-a}{2(b-a)}, & \text{if } a \leq x \leq b \\ \frac{x+c-2b}{2(c-b)}, & \text{if } b \leq x \leq c \\ 1, & \text{if } x \geq c \end{cases}$$

denoted by  $\mathcal{Z}(a, b, c)$  where  $a, b, c$  are real numbers with  $a < b < c$ . Then the zigzag uncertain variable  $\xi$  has an expected value as follows:

$$E[\xi] = \frac{a + 2b + c}{4}.$$

## 2.2 Uncertain Random Variable

Let  $(\Gamma, \mathcal{L}, \mathcal{M})$  be an uncertainty space and let  $(\Omega, \mathcal{A}, \text{Pr})$  be a probability space. Then the product

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr}) = (\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \text{Pr})$$

is called a *chance space*.

**Definition 6** ([12]) *Let  $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr})$  be a chance space, and let  $\Theta \in \mathcal{L} \times \mathcal{A}$  be an event. Then the chance measure of  $\Theta$  is defined as*

$$\text{Ch}\{\Theta\} = \int_0^1 \text{Pr}\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq x\} dx.$$

**Definition 7** ([12]) *An uncertain random variable is a measurable function  $\xi$  from a chance space  $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr})$  to the set of real numbers such that  $\{\xi \in B\}$  is an event in  $\mathcal{L} \times \mathcal{A}$  for any Borel set  $B$ .*

**Definition 8** ([12]) *Let  $\xi$  be an uncertain random variable. Then its chance distribution is defined by*

$$\Phi(x) = \text{Ch}\{\xi \leq x\}$$

for any real number  $x \in \mathfrak{R}$ .

**Definition 9** ([12]) *Let  $\xi$  be an uncertain random variable. Then its expected value is defined by*

$$E[\xi] = \int_0^\infty \text{Ch}\{\xi \geq x\} dx - \int_{-\infty}^0 \text{Ch}\{\xi \leq x\} dx$$

provided that at least one of the two integrals is finite.

**Definition 10** ([12]) *Let  $\xi$  be an uncertain random variable with finite expected value  $e$ . Then the variance of  $\xi$  is defined by*

$$V[\xi] = E[(\xi - e)^2].$$

**Theorem 3** ([12]) *Let  $\xi_1, \xi_2, \dots, \xi_n$  be uncertain random variables on the chance space  $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr})$ , and let  $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$  be a measurable function. Then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain random variable determined by*

$$\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \dots, \xi_n(\gamma, \omega))$$

for all  $(\gamma, \omega) \in \Gamma \times \Omega$ .

**Theorem 4** ([13]) *Let  $\eta_1, \eta_2, \dots, \eta_m$  be independent random variables with probability distributions  $\Psi_1, \Psi_2, \dots, \Psi_m$ , respectively, and let  $\tau_1, \tau_2, \dots, \tau_n$  be uncertain variables (not necessarily independent). Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$$

has an expected value

$$E[\xi] = \int_{\mathfrak{R}^m} E[f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_n)] d\Psi_1(y_1) \cdots \Psi_m(y_m),$$

where  $E[f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_n)]$  is the expected value of the uncertain variable  $f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_n)$  for any real numbers  $y_1, y_2, \dots, y_m$ .

### 3 Efficient Solutions for URMOP Problem

Consider the URMOP problem as follows:

$$\begin{cases} \min_{x \in R^n} & f(x, \xi(\gamma, \omega)) = (f_1(x, \xi(\gamma, \omega)), f_2(x, \xi(\gamma, \omega)), \dots, f_p(x, \xi(\gamma, \omega))) \\ \text{subject to:} & g_i(x, \xi(\gamma, \omega)) \leq 0, i = 1, 2, \dots, m \end{cases} \quad (1)$$

where the objective functions and constraints depend on a vector of uncertain random variables  $\xi(\gamma, \omega)$  defined on the chance space  $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \text{Pr})$ ; the chance distributions of uncertain random variables are known and independent from the decision variables of the problem,  $x_1, \dots, x_n$ ; and the functions

$$f_1(x, \xi(\gamma, \omega)), f_2(x, \xi(\gamma, \omega)), \dots, f_p(x, \xi(\gamma, \omega))$$

are defined on  $R^n \times \Gamma \times \Omega$ .

Note that since the uncertain random constraints  $g_i(x, \xi(\gamma, \omega)) \leq 0$  do not define a crisp feasible set in problem (1), they should be converted into chance constraints hold with confidence levels  $\alpha_1, \alpha_2, \dots, \alpha_m$ , that is,

$$\text{Ch}\{g_i(x, \xi(\gamma, \omega)) \leq 0\} \geq \alpha_i, i = 1, 2, \dots, m$$

which is a crisp feasible set, denoted as  $D \subset R^n$ .

The review of uncertain random programming problem [13, 18] shows that the solution of such problems involves always transforming the problem into a deterministic one, which is called the equivalent deterministic problem. This transformation is carried out by using some statistical characteristics of the uncertain random variables involved in the problem to remove the uncertain and random ambiguity, such as expected value. In fact, different statistical characteristics of uncertain random variable may draw forth different values of the uncertain random objectives, and further produce different Pareto senses of URMOP. Starting from this idea, we can define the following concepts of efficient solution for original URMOP problem (1):

**Definition 11 (Expected value efficient solution)** ([18]) *A Pareto efficient solution  $x^* \in D$  is said to be expected value Pareto efficient to the uncertain random MOP problem (1) if there is no feasible solution  $x$  such that*

$$E[f_j(x, \xi(\gamma, \omega))] \leq E[f_j(x^*, \xi(\gamma, \omega))], j = 1, 2, \dots, p$$

and  $E[f_j(x, \xi(\gamma, \omega))] < E[f_j(x^*, \xi(\gamma, \omega))]$  for at least one index  $j$ .

Though the expected value efficiency has been applied to the URMOP widely, there are still many other statistical characteristics of uncertain random variable applicable and worthy of study. For instance, when the designer is risk averse, the obtained solution in URMOP has to be done with high confidence levels, that is to say, solutions with large deviation are not desirable. In this case, we want to obtain the optimal solutions with high expected value and confidence level simultaneously. As an extension to the concepts of expected value efficient solution, we can define the concept of expected value variance efficient solution. In this case, the concept of efficiency arises from the construction of a problem with  $2p$  objectives involving the expected value and the variance of each uncertain random objective.

**Definition 12 (Expected value variance efficient solution)** *A Pareto efficient solution  $x^* \in D$  is said to be expected value variance Pareto efficient to the uncertain random MOP problem (1) if it is a Pareto efficient solution to the problem*

$$\min_{x \in D} f(x, \xi(\gamma, \omega)) = (E[f_1(x, \xi(\gamma, \omega))], \dots, E[f_p(x, \xi(\gamma, \omega))], V[f_1(x, \xi(\gamma, \omega))], \dots, V[f_p(x, \xi(\gamma, \omega))])$$

which includes the expected value and the variance of the uncertain random objective functions in problem (1).

Since the square root function is strictly increasing, the set of efficient solutions does not vary in the problem if we substitute variance for standard deviation.

From Definitions 11 and 12, we can see that both definitions record certain statistical characteristics of the uncertain random objectives. The expected value criterion generates central trend efficient solutions. The combination of both expected value and variance leads us to the expected value-variance criterion, which provides us with efficient solutions in which the uncertain random objectives are more concentrated around

their expected value and, for this reason, can be considered a criterion of “not very risky” efficiency. The application of these two criteria requires knowing only the expected value and variance of the uncertain random objectives and, for this reason, are easily applicable criteria. In fact, the expected value and expected value-variance deviation efficiency are often used in applied works, such as [6] and [5], where the expected value efficiency and expected value-variance deviation efficiency are considered in UAV mission planning problem and redundancy allocation problem, respectively.

However, in many real situations, such as the engineering design problem, portfolio selection problem, etc., the application of uncertain random multiobjective programming often requires the collaboration of the decision-makers (DM), who have to fix an aspiration level or a chance for each one of the uncertain random objectives. For this reason, we will now deal with the application of maximum chance (or the minimum risk) to uncertain random multiobjective programming problems. Given the URMOP problem, we can apply the maximum chance criterion to each uncertain random objective separately. In this case, the DM must fix a priori an aspiration level,  $u_k$ , for each uncertain random objective function and find the vector  $x$ , in which the chance measure of the  $k$ th objective function not being greater than the aspiration level fixed is maximum:  $\text{Ch}\{f_k(x, \xi(\gamma, \omega)) \leq u_k\}$ . The separate application of this criterion to each uncertain random objective function in the URMOP problem leads us to the following definition of efficiency:

**Definition 13 (Maximum chance efficient solution for levels  $u_1, \dots, u_p$ )** A Pareto efficient solution  $x^* \in D$  is said to be a vectorial maximum chance solution for levels  $u_1, \dots, u_p$  to the uncertain random MOP problem (1) if it is a Pareto efficient solution to the problem

$$\max_{x \in D} f(x, \xi(\gamma, \omega)) = (\text{Ch}\{f_1(x, \xi(\gamma, \omega)) \leq u_1\}, \dots, \text{Ch}\{f_p(x, \xi(\gamma, \omega)) \leq u_p\})$$

which includes the chance measures of the uncertain random objective functions in problem (1) under aspiration levels.

Note that, according to the definition of chance measure, the chance measures of uncertain random objective functions under aspiration levels can be calculated as follows

$$\text{Ch}\{f_i(x, \xi(\gamma, \omega)) \leq u_i\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | f_i(x, \xi(\gamma, \omega)) \leq u_i\} \geq x\} dx.$$

As an extension to Definition 13, we fix a chance measure for the uncertain random objective function,  $\alpha_k$ , and look for the smaller value  $u_k$  for which we can assert that, given the fixed chance measure, the uncertain random objective function does not exceed the level. From the application of this criterion to each uncertain random objective function in the problem (1), we can define the following concept of efficiency.

**Definition 14 (Efficient solution with chance measures  $\alpha_1, \dots, \alpha_p$ :  $\alpha$ -efficient solution)** A Pareto efficient solution  $x^* \in D$  is said to be a vectorial maximum chance solution for levels  $u_1, \dots, u_p$  to the uncertain random MOP problem (1) if it is a Pareto efficient solution to the problem

$$\begin{cases} \min_{x, u} & (u_1, u_2, \dots, u_p) \\ \text{subject to:} & \text{Ch}\{f_k(x, \xi(\gamma, \omega)) \leq u_k\} \geq \alpha_k, \\ & k = 1, 2, \dots, p, x \in D. \end{cases}$$

Besides the maximum chance efficiency and  $\alpha$ -efficiency, there are two more criteria involved the DM's preference, which can be referred to the optimistic criterion and pessimistic criterion. Before employing these two criteria to the URMOP problem, we need to define the optimistic value and pessimistic value of uncertain random variable first.

**Definition 15 (Optimistic value of uncertain random variable)** Let  $\xi(\gamma, \omega)$  be an uncertain random variable defined on chance space  $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ , and  $\alpha \in (0, 1]$ , then the optimistic value of  $\xi(\gamma, \omega)$  is defined by

$$\xi_{\text{sup}}(\alpha) = \sup\{r | \text{Ch}\{\xi(\gamma, \omega) \geq r\} \geq \alpha\}$$

where  $\text{Ch}\{\xi(\gamma, \omega) \geq r\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | \xi(\gamma, \omega) \geq r\} \geq x\} dx.$

**Definition 16 (Pessimistic value of uncertain random variable)** Let  $\xi(\gamma, \omega)$  be an uncertain random variable defined on chance space  $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ , and  $\alpha \in (0, 1]$ , then the pessimistic value of  $\xi(\gamma, \omega)$  is defined by

$$\xi_{\inf}(\alpha) = \inf\{r | \text{Ch}\{\xi(\gamma, \omega) \leq r\} \geq \alpha\}$$

where  $\text{Ch}\{\xi(\gamma, \omega) \leq r\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | \xi(\gamma, \omega) \leq r\} \geq x\} dx$ .

Finally, we give the concepts of efficiency for two criteria mentioned above. As we will see next, in order to define these two concepts, the definitions of optimistic value and pessimistic value are applied to each uncertain random objective, respectively.

**Definition 17 (Minimum optimistic value efficient solution)** A Pareto efficient solution  $x^* \in D$  is said to be minimum optimistic value Pareto efficient to the uncertain random MOP problem (1) if there is no feasible solution  $x$  such that

$$f_j(x, \xi(\gamma, \omega))_{\sup}(\alpha) \leq f_j(x^*, \xi(\gamma, \omega))_{\sup}(\alpha), j = 1, 2, \dots, p$$

and  $f_j(x, \xi(\gamma, \omega))_{\sup}(\alpha) < f_j(x^*, \xi(\gamma, \omega))_{\sup}(\alpha)$  for at least one index  $j$ ,  $\alpha \in (0, 1]$ .

**Definition 18 (Minimum pessimistic value efficient solution)** A Pareto efficient solution  $x^* \in D$  is said to be minimum pessimistic value Pareto efficient to the uncertain random MOP problem (1) if there is no feasible solution  $x$  such that

$$f_j(x, \xi(\gamma, \omega))_{\inf}(\alpha) \leq f_j(x^*, \xi(\gamma, \omega))_{\inf}(\alpha), j = 1, 2, \dots, p$$

and  $f_j(x, \xi(\gamma, \omega))_{\inf}(\alpha) < f_j(x^*, \xi(\gamma, \omega))_{\inf}(\alpha)$  for at least one index  $j$ ,  $\alpha \in (0, 1]$ .

**Remark 1** On the basis of efficiency notions proposed above, we can solve the uncertain random multiobjective programming problem according to the real context of problem to satisfy its need in practical application. For instance, in the multiobjective machine scheduling problem with uncertain and random parameters, when there are no DMs' preferences available, and we want to minimize the makespan in the long run, we can employ the expected value efficiency to convert the original uncertain random multiobjective programming problem into a deterministic problem, where the objectives are the expected values of original uncertain random objectives, and then obtain expected value Pareto solutions through the solution methods in deterministic multiobjective programming problem, such as linear weighted methods, ideal point method, and etc.. When there are some DMs providing their aspiration levels about the makespan, we can employ maximum chance efficiency to convert the original problem into a deterministic problem, where the objectives are the maximum chances of original uncertain random objectives under DM's aspiration levels, and then obtain maximum chance Pareto solutions through the solution methods in deterministic multiobjective programming problem.

## 4 Conclusions

The general purpose of this study was to propose several new efficient solution concepts to describe the Pareto efficiency in uncertain random multiobjective programming problem. Based on our results, it was possible to deal with attaining efficient solutions from a different perspective since, given a particular problem, we were able to see from the established definitions which concept of Pareto efficiency was the most appropriate or the one that best fitted the preferences of the decision maker. Though the expected value efficiency has been widely used in URMOP, in our view, the maximum chance efficient solution for levels  $u_1, \dots, u_p$ , efficient solution with chance measures  $\alpha_1, \dots, \alpha_p$  or  $\alpha$ -efficient solution, minimum optimistic value efficient solution, and minimum pessimistic value efficient solution may generate "better" solutions, since these concepts require the collaboration of the DM, who has to fix an aspiration level or a chance for each one of the uncertain random criteria, that is to say, it determines the risk that he or she is willing to take in each of the uncertain random objectives. For this reason, we thought that these criteria were very suitable for solving URMOP problems in spite of being less used than the expected value efficiency. To be studied in future, the relationships between these efficient solutions and the exact solution methods of these Pareto efficient solutions should be studied.

## Acknowledgements

This work was supported by National Natural Science Foundation under Grant No. 71171199, and Natural Science Foundation of Shanxi Province under Grant No. 2013JM1003.

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