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A Review on Uncertain Set

Xiangfeng Yang[1], Jinwu Gao[2,*]
[1]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
[2]School of Information, Renmin University of China, Beijing 100872, China

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Abstract

As a generalization of uncertain variable, uncertain set is a set-valued function on an uncertainty space. This paper reviews a concept of uncertain set to describe "unsharp concepts" that are essentially sets but their boundaries are not sharply described. Besides, we review uncertain logic that is a methodology for calculating the truth value for uncertain propositions via uncertain set theory. In addition, this paper also reviews uncertain inference rules that derive consequences from uncertain knowledge via the tool of conditional uncertain set.

Keywords: uncertainty theory, uncertain set, uncertain logic, uncertain inference

1 Introduction

Probability theory has become a branch of mathematics for modeling random phenomena. Before applying probability theory in practice, we should obtain a probability distribution that is close enough to the real frequency via statistics. Otherwise, the law of large numbers is no longer valid and probability theory is no longer applicable. As a matter of fact, we sometimes have no observed data because of the technological or economical difficulties. In this case, we have to invite domain experts to estimate their belief degree that each event will happen, but, *human beings usually estimate a much wider range of values than the object actually takes* [18]. Thus the belief degrees deviate far from the frequency. If we still take human belief degrees as probability distribution, we maybe cause a counterintuitive result that was given by Liu [15].

In order to model human belief degrees, uncertainty theory was established by Liu [7] in 2007 and refined by Liu [12] in 2010. Uncertainty theory has become a new branch of mathematics for modeling indeterminate phenomena based on normality, duality, subadditivity and product axioms. It has been applied to a wide range of areas, such as uncertain programming [10, 12], uncertain risk analysis [13, 22], uncertain differential equation [8, 2, 32], uncertain finance [9, 1, 19], uncertain optimal control [35], uncertain game [3, 27, 28], uncertain graph [34, 4], etc. For more detailed exposition of uncertainty theory with applications, the readers may consult Liu's recent book [18].

As a generalization of uncertain variable, Liu [11] first proposed uncertain set in order to model "unsharp concepts" that are essentially sets but their boundaries are not sharply described, such as "young students", "almost all", "tall" in 2011. Essentially, uncertain set is a set-valued function on an uncertainty space. In order to measure uncertain set, the membership function and inverse membership function of uncertain set were presented by Liu [16]. Besides, Liu [17] gave the definition of independence for uncertain sets, and Liu [16] provided a set operational law of uncertain sets via membership function, and an arithmetic operational law via inverse membership function. As an important feature of uncertain set, the expected value of uncertain set was proposed by Liu [11]. After then, Liu [12] gave a formula to calculate the expected value via membership function, and Liu [16] gave another formula via inverse membership function. In addition, the concept of variance and distance between uncertain sets were presented by Liu [14]. As an extension of expected value and variance, Yang and Gao [29] proposed moments and central moments of uncertain set and gave some formulas to calculate the moments and central moments by membership function. In order to measure the uncertainty of uncertain set, entropy of uncertain set was defined by Liu [14] as a measure of information

*Corresponding author.
Email: yxiangfeng@gmail.com (X. Yang), jgao@ruc.edu.cn (J. Gao).

deficiency in the form of logarithm function. Then, Yao and Ke [33] proved the linearity of entropy operator. However, logarithm entropy cannot measure the uncertainty associated with all uncertain set. As extensions of logarithm entropy, many researchers defined other entropies. For instance, Wang and Ha [25] proposed a quadratic entropy of uncertain set; Yao and Ke [33] suggested a cross entropy for comparing a membership function against a reference membership function; Peng and Li [23] investigated radical entropy of uncertain set; Lu and Wang [20] presented triangular entropy of uncertain set and Yao [30] introduced a sine entropy of uncertain set. Conditional uncertain set [18] plays a crucial role in uncertain inference theory that is derived from an uncertain set restricted to a conditional uncertainty space given an uncertain event. When an uncertain set is independent of the given uncertain event, Yao [31] proved that the conditional uncertain set in this special case has a membership function. Besides, Ning and Huang [21] proposed some inequalities in the framework of uncertain set theory, including the Markov inequality, the Chebyshev inequality, the Jensen inequality, the Hölder inequality, and the Minkowski inequality.

For the sake of determining membership functions, Liu [14] designed a questionnaire survey for collecting experts experimental data. Based on experts experimental data, Liu [14] also proposed linear interpolation method and principle of least squares to determine membership functions. When dealing with multiple experts' experimental data, Wang and Wang [26] introduced Delphi method for uncertain statistics.

Based on uncertain set theory, Liu [14] presented uncertain logic to deal with human language by using truth value formula for uncertain propositions. As an application of uncertain logic, Liu [14] gave a linguistic summarizer that provides a means for extracting linguistic summary from a collection of raw date. Upon conditional uncertain set, Liu [11] presented the basic uncertain inference rule in 2010. Then, Gao, Gao and Ralescu [5] extended the uncertain inference rule to deal with multiple antecedents and multiple if-then rules. After that, Liu [11] proposed the concept of uncertain system, and then gave the tool of uncertain controller. Peng and Chen [24] proved that uncertain systems are universal approximator and then demonstrated that the uncertain controller is a reasonable tool. As an application, Gao [5] balanced an inverted pendulum via the uncertain controller.

The remainder of this paper is organized as follows. In Section 2, some basic concepts and properties of uncertain set are reviewed. In Section 3, uncertain logic is discussed. Section 4 introduces uncertain inference and uncertain system, and the design of uncertain inference controller for an inverted pendulum is given. Section 5 concludes the paper with a brief summary.

2 Uncertain Set

In this section, we first introduce four axioms of uncertainty theory and then review some concepts of uncertain set. Uncertain measure $\mathcal{M}$ is a real-valued set-function on a σ-algebra $\mathcal{L}$ over a nonempty set Γ which satisfies normality, duality, subadditivity and product axioms. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2.1 ([7]) *Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;
Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Besides, in order to provide the operational law, Liu [9] defined the product uncertain measure on the product σ-algebre $\mathcal{L}$ as follows.
Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

In 2010, Liu [11] first proposed uncertain set to model "unsharp concepts" that are essentially sets but their boundaries are not sharply described. Roughly speaking, an uncertain set is a set-valued function on an uncertainty space. In the next moment, we will introduce some concepts of uncertain set, membership function, independence, the operational law, expected value, variance, entropy, and conditional uncertain set.

Definition 2.2 ([11]) *An uncertain set is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets such that*

$$\{B \subset \xi\} = \{\gamma \in \Gamma \mid B \subset \xi(\gamma)\} \qquad \text{and} \qquad \{\xi \subset B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \subset B\}$$

are events for any Borel set B.

Example 2.1 Let the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with Borel algebra and Lebesgue measure. For each $\gamma \in [0,1]$, define $\xi(\gamma) = [\gamma - 1, 1 - \gamma]$. Then ξ is an uncertain set, and it is nonempty because $\xi(\gamma) \neq \emptyset$ for almost all $\gamma \in \Gamma$.

In order to describe uncertain set in practice, Liu [16] defined the concept of membership function that represents the membership degrees of any Borel set belonging to the uncertain set and the membership degrees of the uncertain set belonging to any Borel set in 2012.

Definition 2.3 ([16]) *An uncertain set ξ is said to have a membership function μ if for any Borel set B of real numbers, we have*

$$\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \mu(x),$$

$$\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \mu(x).$$

The above equations will be called measure inversion formulas.

Example 2.2 Suppose ξ is defined as Example 2.1. Then ξ has a membership function

$$\mu(x) = \begin{cases} 1 - |x|, & \text{if } x \in [-1,1] \\ 0, & \text{otherwise.} \end{cases}$$

We note that not all uncertain sets have membership functions. For instance, the uncertain set

$$\xi = \begin{cases} [1,5] \text{ with uncertain measure } 0.3 \\ [2,7] \text{ with uncertain measure } 0.7 \end{cases}$$

has no membership function. What uncertain sets have membership functions?
Case I: If an uncertain set ξ degenerates to a classical set A, then ξ has a membership function that is just the characteristic function of A.
Case II: Let ξ be an uncertain set taking values in a nested class of sets. That is, for any given γ_1 and $\gamma_2 \in \Gamma$, at least one of the following alternatives holds,

$$\xi(\gamma_1) \subset \xi(\gamma_2), \quad \xi(\gamma_2) \subset \xi(\gamma_2).$$

Then the uncertain set ξ has a membership function.

Example 2.3 An uncertain set ξ is called triangular if it has a membership function

$$\mu(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ \dfrac{x-c}{b-c}, & \text{if } b \le x \le c \end{cases}$$

denoted by (a, b, c) where a, b, c are real numbers with $a < b < c$.

A membership function μ is said to be regular if there exists a point x_0 such that $\mu(x_0) = 1$ and $\mu(x)$ is unimodal about the mode x_0. In 2011, Liu [14] proved that a real-valued function μ is a membership function of uncertain set if and only if $0 \le \mu(x) \le 1$.

Definition 2.4 ([16]) *Let ξ be an uncertain set with membership function μ. Then the set-valued function*

$$\mu^{-1}(\alpha) = \left\{x \in \Re \;\middle|\; \mu(x) \ge \alpha\right\}$$

is called the inverse membership function of ξ. Sometimes, the set $\mu^{-1}(\alpha)$ is called the α-cut of μ.

In 2011, Liu [16] proved that a function $\mu^{-1}(\alpha)$ is an inverse membership function if and only if it is a monotone decreasing set-valued function with respect to $\alpha \in [0,1]$. That is,

$$\mu^{-1}(\alpha) \subset \mu^{-1}(\beta), \quad \text{if } \alpha > \beta.$$

Example 2.4 The triangular uncertain set $\xi = (a, b, c)$ has an inverse membership function

$$\mu^{-1}(\alpha) = [(1-\alpha)a + \alpha b, \alpha b + (1-\alpha)c].$$

If ξ is an uncertain set with regular membership function μ, the functions

$$\mu_l^{-1}(\alpha) = \inf \mu^{-1}(\alpha)$$

and

$$\mu_r^{-1}(\alpha) = \sup \mu^{-1}(\alpha)$$

are called the left inverse membership function and right inverse membership function of uncertain set ξ, respectively.

Definition 2.5 ([18]) *The uncertain sets $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if for any Borel sets $B_1, B_2, \ldots, B_n$, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^* \subset B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\xi_i^* \subset B_i\right\}$$

and

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^* \subset B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\xi_i^* \subset B_i\right\}$$

where ξ_i^* are arbitrarily chosen from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \ldots, n$, respectively.

Definition 2.6 ([11]) *Let ξ and η be two uncertain sets on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$.*
(i) The union $\xi \cup \eta$ of uncertain sets ξ and η is

$$(\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma).$$

(ii) The intersection $\xi \cap \eta$ of uncertain sets ξ and η is

$$(\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma).$$

(iii) The complement ξ^c of the uncertain set ξ is

$$\xi^c(\gamma) = (\xi(\gamma))^c.$$

(iv) The function $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_n(\gamma)).$$

For the set operations of uncertain sets, Liu [16] proved the following theorem using membership function.

Theorem 2.1 ([16]) *Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then the union $\xi \cup \eta$ has a membership function*

$$\lambda_1(x) = \mu(x) \vee \nu(x),$$

the intersection $\xi \cap \eta$ has a membership function

$$\lambda_2(x) = \mu(x) \wedge \nu(x),$$

and the complement ξ^c has a membership function

$$\lambda_3(x) = 1 - \mu(x).$$

Moreover, for the arithmetic operations of uncertain sets, Liu [16] proved the following theorem using inverse membership function.

Theorem 2.2 ([16]) *Let $\xi_1,\ \xi_2, \ldots, \xi_n$ be independent uncertain sets with inverse membership functions $\mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_n^{-1}$, respectively. If f is a measurable function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain set with inverse membership function,

$$\lambda^{-1}(\alpha) = f(\mu_1^{-1}(\alpha), \mu_2^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha)).$$

Theorem 2.3 *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain sets with regular membership functions $\mu_1, \mu_2, \ldots, \mu_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is monotone increasing with respect to $x_1, x_2, \ldots, x_m$ and monotone decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain set*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

has a regular membership function, and

$$\lambda_l^{-1}(\alpha) = f(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)),$$

$$\lambda_r^{-1}(\alpha) = f(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha) \cdots, \mu_{nl}^{-1}(\alpha))$$

where $\lambda_l^{-1}, \mu_{1l}^{-1}, \mu_{2l}^{-1}, \ldots, \mu_{nl}^{-1}$ are left inverse membership functions, and $\lambda_r^{-1},\ \mu_{1r}^{-1}, \mu_{2r}^{-1}, \ldots, \mu_{nr}^{-1}$ are right inverse membership functions of $\xi, \xi_1, \xi_2, \ldots, \xi_n$, respectively.

In 2010, Liu [11] proposed a concept of expected value that is an important characteristic of an uncertain set.

Definition 2.7 ([11]) *Let ξ be a nonempty uncertain set. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \succeq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \preceq x\}\mathrm{d}x$$

provided that at least one of the two integrals is finite.

After then, Liu [12] gave a formula to calculate the expected value via membership function, and Liu [16] gave another formula via inverse membership function.

Theorem 2.4 ([12]) *Let ξ be an uncertain set with regular membership function μ. Then*

$$E[\xi] = x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \mu(x)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \mu(x)\mathrm{d}x$$

where x_0 is a point such that $\mu(x_0) = 1$.

Theorem 2.5 ([16]) *Let ξ be a nonempty uncertain set with membership function μ. Then*

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)\right) \mathrm{d}\alpha$$

where $\inf \mu^{-1}(\alpha)$ and $\sup \mu^{-1}(\alpha)$ are the infimum and supremum.

Example 2.5 The expected value of triangular uncertain set $\xi = (a, b, c)$ is $E[\xi] = (a + 2b + c)/4$.

For a strictly monotone function, we have the following theorem to calculate the expected value via the left and right inverse membership function.

Theorem 2.6 ([16]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain sets with regular membership functions $\mu_1, \mu_2, \ldots, \mu_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

has an expected value

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\mu_l^{-1}(\alpha) + \mu_r^{-1}(\alpha)\right) \mathrm{d}\alpha$$

where $\mu_l^{-1}(\alpha)$ and $\mu_r^{-1}(\alpha)$ are determined by

$$\mu_l^{-1}(\alpha) = f(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)),$$

$$\mu_r^{-1}(\alpha) = f(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha) \cdots, \mu_{nl}^{-1}(\alpha)).$$

Besides, the expected value operator has linearity that was proved by Liu [16] .

Theorem 2.7 ([16]) *Let ξ and η be independent uncertain sets with finite expected value. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

The variance of uncertain set that was proposed by Liu [14] provides a degree of the spread of the membership function around its expected value.

Definition 2.8 ([14]) *Let ξ be a nonempty uncertain set with finite expected value e. Then the variance of ξ is defined by*

$$V[\xi] = E[(\xi - e)^2].$$

And a formal [18] was given to calculate variance via membership function.

Theorem 2.8 ([18]) *Let ξ be an uncertain set with membership function μ and expected value e. Then the variance of ξ is*

$$V[\xi] = \frac{1}{2}\int_0^{+\infty} \left(\sup_{(y-e)^2 \ge x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y) \right) \mathrm{d}x.$$

Example 2.6 The variance of symmetric triangular uncertain set (a, b, c) is $V[\xi] = (c - a)^2/24$.

Theorem 2.9 ([18]) *If ξ is an uncertain set with finite expected value, a and b are real numbers, then*

$$V[a\xi + b] = a^2 V[\xi].$$

Based on expected value and variance, Yang and Gao [29] introduced moments and central moments of uncertain set and gave some formulas to calculate the moments and central moments by membership function.

Definition 2.9 ([29]) *Let ξ be an uncertain set and let k be a positive integer. If $E[\xi^k]$ is finite, then $E[\xi^k]$ is called the k-th moment of ξ.*

Theorem 2.10 ([29]) *Let ξ be a nonempty uncertain set with membership function μ, and let k be a positive integer. Then the k-th moment of ξ is*

$$E[\xi^k] = \frac{1}{2}\int_0^{+\infty}\left(\sup_{y^k \geq x}\mu(y) + 1 - \sup_{y^k < x}\mu(y)\right)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{0}\left(\sup_{y^k \leq x}\mu(y) + 1 - \sup_{y^k > x}\mu(y)\right)\mathrm{d}x.$$

Definition 2.10 ([29]) *Let ξ be an uncertain set with finite expected value e, and let k be a positive integer. If $E[(\xi - e)^k]$ is finite, then $E[(\xi - e)^k]$ is called the k-th central moment of ξ.*

Theorem 2.11 ([29]) *Let ξ be an uncertain set with membership function μ and finite expected value e, and let k be a positive integer. Then the k-th central moment of ξ is*

$$E[(\xi - e)^k] = \frac{1}{2}\int_0^{+\infty}\left(\sup_{(y-e)^k \geq x}\mu(y) + 1 - \sup_{(y-e)^k < x}\mu(y)\right)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{0}\left(\sup_{(y-e)^k \leq x}\mu(y) + 1 - \sup_{(y-e)^k > x}\mu(y)\right)\mathrm{d}x.$$

Example 2.7 Let $\xi = (a, b, c)$ be a triangular uncertain set and $ac \geq 0$. Then the k-th central moment of ξ is

$$E[\xi^k] = \frac{1}{2(k+1)}\left[2b^k + \sum_{i=0}^{k-1} b^i (a^{k-i} + c^{k-i})\right].$$

In order to measure the uncertainty of uncertain set, entropy of uncertain set was defined by Liu [14] as a measure of information deficiency in the form of logarithm function in 2011.

Definition 2.11 ([14]) *Let ξ be an uncertain set with membership function μ. Then its entropy is*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

From the above the definition, we can easy obtain that $H[\xi] \geq 0$ and equality holds if ξ is essentially a classical set. And the equation $H[\xi^c] = H[\xi]$ is true. If ξ is an uncertain set on the interval $[a, b]$, Liu [14] got

$$H[\xi] \leq (b - a)\ln 2$$

and equality holds if ξ has a membership function $\mu(x) = 0.5$ on $[a, b]$. After then, Yao and Ke [33] gave a formula to calculate entropy and proved the positive linearity of entropy.

Theorem 2.12 ([14]) *Let ξ be an uncertain set with regular membership function μ. Then*

$$H[\xi] = \int_0^1 (\mu_l^{-1}(\alpha) - \mu_r^{-1}(\alpha))\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha.$$

Theorem 2.13 ([14]) *Let ξ and η be independent uncertain sets with regular membership functions. Then for any real numbers a and b, we have*

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta].$$

However, logarithm entropy cannot measure the uncertainty associated with all uncertain set. As extensions of logarithm entropy, many researchers defined other entropies. For instance, Wang and Ha [25] proposed a quadratic entropy of uncertain set; Yao and Ke [33] suggested a cross entropy for comparing a membership function against a reference membership function; Peng and Li [23] investigated radical entropy of uncertain set; Lu and Wang [20] presented triangular entropy of uncertain set and Yao [30] introduced a sine entropy of uncertain set.

The distance of uncertain sets was proposed by Liu [14], and a formal was given to calculate the distance via membership function.

Definition 2.12 ([14]) *The distance between uncertain sets* ξ *and* η *is defined as*

$$d(\xi,\eta) = E[|\xi - \eta|].$$

Theorem 2.14 ([18]) *Let* ξ *and* η *be nonempty uncertain sets. Then the distance between* ξ *and* η *is*

$$d(\xi,\eta) = \frac{1}{2}\int_0^{+\infty}\left(\sup_{|y|\geq x}\lambda(y) + 1 - \sup_{|y|<x}\lambda(y)\right)\mathrm{d}x$$

where λ *is the membership function of* $\xi - \eta$.

Conditional uncertain set is derived from an uncertain set restricted to a conditional uncertainty space given an uncertain event that plays a crucial role in uncertain inference theory. Liu [18] firstly gave the definition of conditional uncertain set. And then, when an uncertain set is independent of the given uncertain event, Yao [31] proved that the conditional uncertain set in this special case has a membership function.

Definition 2.13 ([18]) *Let* ξ *be an uncertain set, and let* Λ *be an event with* $\mathcal{M}\{\Lambda\} > 0$*. Then the conditional uncertain set* ξ *given* Λ *is said to have a membership function* $\mu(x|\Lambda)$ *if for any Borel set* B*, we have*

$$\mathcal{M}\{B \subset \xi|\Lambda\} = \inf_{x\in B}\mu(x|\Lambda),$$

$$\mathcal{M}\{\xi \subset B|\Lambda\} = 1 - \sup_{x\in B^c}\mu(x|\Lambda).$$

Definition 2.14 ([31]) *An uncertain set* ξ *and an uncertain event* Λ *are said to be independent if for any Borel set* B*, the uncertain events* $\{B \subset \xi\}$ *and* Λ *are independent, and the uncertain events* $\{\xi \subset B\}$ *and* Λ *are also independent.*

Theorem 2.15 ([31]) *Let* ξ *be an uncertain set with a membership function* $\mu(x)$ *on an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$*, and* Λ *be an uncertain event independent of* ξ*. Then the conditional uncertain set* $\xi|_\Lambda$ *on the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|\Lambda\})$ *has a membership function*

$$\nu(x) = \begin{cases} \dfrac{\mu(x)}{\mathcal{M}\{\Lambda\}}, & \text{if } \mu(x) < \dfrac{1}{2}\mathcal{M}\{\Lambda\} \\ \dfrac{\mu(x) + \mathcal{M}\{\Lambda\} - 1}{\mathcal{M}\{\Lambda\}}, & \text{if } \mu(x) > 1 - \dfrac{1}{2}\mathcal{M}\{\Lambda\} \\ 0.5, & \text{otherwise.} \end{cases}$$

3 Uncertain Logic

Based on uncertain set theory, Liu [14] proposed uncertain logic to deal with human language in 2011. As an application of uncertain logic, Liu [14] gave a linguistic summarizer that provides a means for extracting linguistic summary from a collection of raw date. Firstly, we review some concepts including uncertain quantifier, uncertain subject, uncertain predicate and truth value. And then, a linguistic summary model was introduced.

Definition 3.1 ([14]) *Uncertain quantifier is an uncertain set representing the number of individuals.*

Example 3.1 The uncertain quantifier $\mathcal{Q}$ of "almost all" on the universe A may have a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq n-5 \\ \dfrac{x-n+5}{3}, & \text{if } n-5 \leq x \leq n-2 \\ 1, & \text{if } n-2 \leq x \leq n \end{cases} \tag{1}$$

where n is the cardinality of the universe A.

An uncertain quantifier is said to be unimodal if its membership function is unimodal. An uncertain quantifier is said to be monotone if its membership function is monotone. Especially, an uncertain quantifier is said to be increasing if its membership function is increasing; and an uncertain quantifier is said to be decreasing if its membership function is decreasing.

Definition 3.2 ([14]) *Let $\mathcal{Q}$ be an uncertain quantifier. Then the negated quantifier $\neg\mathcal{Q}$ is the complement of $\mathcal{Q}$ in the sense of uncertain set, i.e.,*

$$\neg\mathcal{Q} = \mathcal{Q}^c.$$

Theorem 3.1 ([18]) *Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is λ. Then the negated quantifier $\neg\mathcal{Q}$ has a membership function*

$$\neg\lambda(x) = 1 - \lambda(x).$$

Example 3.2 Let $\mathcal{Q}$ be the uncertain quantifier "almost all" defined by (1). Then its negated quantifier $\neg\mathcal{Q}$ has a membership function

$$\neg\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le n-5 \\ \dfrac{n-x-2}{3}, & \text{if } n-5 \le x \le n-2 \\ 0, & \text{if } n-2 \le x \le n. \end{cases}$$

Definition 3.3 ([14]) *Let $\mathcal{Q}$ be an uncertain quantifier. Then the dual quantifier of $\mathcal{Q}$ is*

$$\mathcal{Q}^* = \forall - \mathcal{Q}.$$

Theorem 3.2 ([18]) *Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is λ. Then the dual quantifier $\mathcal{Q}^*$ has a membership function*

$$\lambda^*(x) = \lambda(n-x)$$

where n is the cardinality of the universe A.

Example 3.3 Let $\mathcal{Q}$ be the uncertain quantifier "almost all" defined by (1). Then its dual quantifier $\mathcal{Q}^*$ has a membership function

$$\lambda^*(x) = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ \dfrac{5-x}{3}, & \text{if } 2 \le x \le 5 \\ 0, & \text{if } 5 \le x \le n. \end{cases}$$

Definition 3.4 ([14]) *Uncertain subject is an uncertain set containing some specified individuals.*

Example 3.4 "Young students" is an uncertain subject that is an uncertain set on the universe of "all students", whose membership function may be defined by

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ \dfrac{x-15}{5}, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ \dfrac{45-x}{10}, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases}$$

Definition 3.5 ([14]) *Uncertain predicate is an uncertain set representing a property that the subjects have in common.*

Example 3.5 The word "tall" is an uncertain predicate that has a membership function

$$\mu(x)=\begin{cases}0, & \text{if } x\le 180\\ \dfrac{x-180}{5}, & \text{if } 180\le x\le 185\\ 1, & \text{if } 185\le x\le 195\\ \dfrac{200-x}{5}, & \text{if } 195\le x\le 200\\ 0, & \text{if } x\ge 200.\end{cases} \tag{2}$$

Definition 3.6 ([14]) *Let P be an uncertain predicate. Then its negated predicate $\neg P$ is the complement of P in the sense of uncertain set, i.e.,*

$$\neg P = P^c.$$

Theorem 3.3 ([18]) *Let P be an uncertain predicate with membership function μ. Then its negated predicate $\neg P$ has a membership function*

$$\neg\mu(x) = 1-\mu(x).$$

Example 3.6 Let P be the uncertain predicate "tall" defined by (2). Then its negated predicate $\neg P$ has a membership function

$$\neg\mu(x)=\begin{cases}1, & \text{if } x\le 180\\ \dfrac{185-x}{5}, & \text{if } 180\le x\le 185\\ 0, & \text{if } 185\le x\le 195\\ \dfrac{x-195x}{5}, & \text{if } 195\le x\le 200\\ 1, & \text{if } x\ge 200.\end{cases}$$

Definition 3.7 ([14]) *Assume that $\mathcal{Q}$ is an uncertain quantifier, S is an uncertain subject, and P is an uncertain predicate. Then the triplet*

$$(\mathcal{Q}, S, P) = \text{“}\mathcal{Q} \text{ of } S \text{ are } P\text{”}$$

is called an uncertain proposition.

Example 3.7 "Almost all young students are tall" is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is "almost all", the uncertain subject S is "young students" and the uncertain predicate P is "tall".

Besides, a logic equivalence theorem was proved by Liu [14].

Theorem 3.4 ([14]) *Let $(\mathcal{Q}, S, P)$ be an uncertain proposition. Then*

$$(\mathcal{Q}^*, S, P) = (\mathcal{Q}, S, \neg P).$$

The truth value of $(\mathcal{Q}, S, P)$ should be the uncertain measure the "$\mathcal{Q}$ of S are P", i.e.,

$$T(\mathcal{Q}, S, P) = \mathcal{M}\{\mathcal{Q} \text{ of } S \text{ are } P\}.$$

However, it is impossible to deduce the value of $\mathcal{M}\{\mathcal{Q} \text{ of } S \text{ are } P\}$ from the information of $\mathcal{Q}$, S and P within uncertain set theory. Thus, Liu [14] gave the definition of truth value as follows.

Definition 3.8 ([14]) *Let $(\mathcal{Q}, S, P)$ be an uncertain proposition in which $\mathcal{Q}$ is a unimodal uncertain quantifier with membership function λ, S is an uncertain subject with membership function ν, and P is an uncertain predicate with membership function μ. Then the truth value of $(\mathcal{Q}, S, P)$ with respect to the universe A is*

$$T(\mathcal{Q}, S, P) = \sup_{0\le\omega\le 1}\left(\omega \wedge \sup_{K\in\mathbb{K}_\omega}\inf_{a\in K}\mu(a) \wedge \sup_{K\in\mathbb{K}_\omega^*}\inf_{a\in K}\neg\mu(a)\right) \tag{3}$$

where

$$\mathbb{K}_\omega = \{K \subset S_\omega \,|\, \lambda(|K|) \ge \omega\},$$
$$\mathbb{K}^*_\omega = \{K \subset S_\omega \,|\, \lambda(|S_\omega| - |K|) \ge \omega\},$$
$$S_\omega = \{a \in A \,|\, \nu(a) \ge \omega\}.$$

Noted that the truth value formula (3) is vacuous if the individual feature data of the universe A are not available.

Theorem 3.5 ([14]) *Let $(\mathcal{Q}, S, P)$ be an uncertain proposition in which $\mathcal{Q}$ is a unimodal uncertain quantifier with membership function λ, S is an uncertain subject with membership function ν, and P is an uncertain predicate with membership function μ. Then the truth value of $(\mathcal{Q}, S, P)$ is*

$$T(\mathcal{Q}, S, P) = \sup_{0\le\omega\le 1} (\omega \wedge \Delta(k_\omega) \wedge \Delta^*(k^*_\omega))$$

where

$$k_\omega = \min\{x \,|\, \lambda(x) \ge \omega\},$$
$$\Delta(k_\omega) = \textit{the } k_\omega\textit{-th largest value of } \{\mu(a_i) \,|\, a_i \in S_\omega\},$$
$$k^*_\omega = |S_\omega| - \max\{x \,|\, \lambda(x) \ge \omega\},$$
$$\Delta^*(k^*_\omega) = \textit{the } k^*_\omega\textit{-th largest value of } \{1 - \mu(a_i) \,|\, a_i \in S_\omega\}.$$

As an application of uncertain logic, linguistic summarizer is a human language statement that is concise and easy-to-understand by humans. For instance, "almost all young students are tall" is a linguistic summary of students' ages and heights.

One problem of data mining is to choose an uncertain quantifier $\mathcal{Q} \in \mathbb{Q}$, an uncertain subject $S \in \mathbb{S}$ and an uncertain predicate $P \in \mathbb{P}$ such that

$$T(\mathcal{Q}, S, P) \ge \beta$$

for the universe A $= \{a_1, a_2, \cdots, a_n\}$, where $\mathbb{Q}$ is collection of uncertain quantifiers, $\mathbb{S}$ is collection of uncertain subjects, $\mathbb{P}$ is collection of uncertain predicates and β is a confidence level. Thus we have a linguistic summarizer,

$$\begin{cases} \text{Find } \mathcal{Q}, S \text{ and } P \\ \text{subject to:} \\ \qquad \mathcal{Q} \in \mathbb{Q}, \quad S \in \mathbb{S}, \quad P \in \mathbb{P} \\ \qquad T(\mathcal{Q}, S, P) \ge \beta. \end{cases}$$

Each solution $(\overline{\mathcal{Q}}, \overline{S}, \overline{P})$ is a linguistic summary "$\overline{\mathcal{Q}}$ of $\overline{S}$ are $\overline{P}$".

4 Uncertain Inference

Uncertain inference [11] is a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertain set. This section will review a series of uncertain inference rules, uncertain system, and an application to an inverted pendulum system. We first introduce the following inference rules.

Inference Rule 4.1 ([11]) *Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. Assume a rule "if $\mathbb{X}$ is an uncertain set ξ then $\mathbb{Y}$ is an uncertain set η. From $\mathbb{X}$ is a constant a we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \eta|_{a\in\xi}$$

which is the conditional uncertain set of η given $a \in \xi$. The inference rule is represented by

$$\begin{array}{l} \textit{Rule: If } \mathbb{X} \textit{ is } \xi \textit{ then } \mathbb{Y} \textit{ is } \eta \\ \textit{From: } \mathbb{X} \textit{ is a constant } a \\ \hline \textit{Infer: } \mathbb{Y} \textit{ is } \eta^* = \eta|_{a\in\xi} \end{array}$$

Theorem 4.1 ([11]) *Let ξ and η are independent uncertain sets with membership functions μ and ν, respectively. If ξ^* is a constant a, then η^* has a membership function*

$$\nu^*(y) = \begin{cases} \dfrac{\nu(y)}{\mu(a)}, & \text{if } \nu(y) < \dfrac{\mu(a)}{2} \\ \dfrac{\nu(y)+\mu(a)-1}{\mu(a)}, & \text{if } \nu(y) > 1 - \dfrac{\mu(a)}{2} \\ 0.5, & \text{otherwise.} \end{cases}$$

Based on the above inference rule, Gao et al. [5] also discussed inference rule with multiple antecedents and with multiple if-then rules in 2010.

Inference Rule 4.2 ([5]) *Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ be three concepts. Assume a rule "if $\mathbb{X}$ is an uncertain set ξ and $\mathbb{Y}$ is an uncertain set η then $\mathbb{Z}$ is an uncertain set τ". From $\mathbb{X}$ is a constant a and $\mathbb{Y}$ is a constant b we infer that $\mathbb{Z}$ is an uncertain set*

$$\tau^* = \tau|_{(a\in\xi)\cap(b\in\eta)}$$

which is the conditional uncertain set of τ given $a \in \xi$ and $b \in \eta$. The inference rule is represented by

$$\begin{array}{l} \text{Rule: If } \mathbb{X} \text{ is } \xi \text{ and } \mathbb{Y} \text{ is } \eta \text{ then } \mathbb{Z} \text{ is } \tau \\ \text{From: } \mathbb{X} \text{ is } a \text{ and } \mathbb{Y} \text{ is } b \\ \hline \text{Infer: } \mathbb{Z} \text{ is } \tau^* = \tau|_{(a\in\xi)\cap(b\in\eta)} \end{array}$$

Theorem 4.2 ([5]) *Let ξ, η, τ are independent uncertain sets with membership functions μ, ν, λ, respectively. If ξ^* is a constant a and η^* is a constant b, then τ^* has a membership function*

$$\lambda^*(z) = \begin{cases} \dfrac{\lambda(z)}{\mu(a)\wedge\nu(b)}, & \text{if } \lambda(z) < \dfrac{\mu(a)\wedge\nu(b)}{2} \\ \dfrac{\lambda(z)+\mu(a)\wedge\nu(b)-1}{\mu(a)\wedge\nu(b)}, & \text{if } \lambda(z) > 1 - \dfrac{\mu(a)\wedge\nu(b)}{2} \\ 0.5, & \text{otherwise.} \end{cases}$$

Inference Rule 4.3 ([5]) *Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. Assume two rules "if $\mathbb{X}$ is an uncertain set ξ_1 then $\mathbb{Y}$ is an uncertain set η_1" and "if $\mathbb{X}$ is an uncertain set ξ_2 then $\mathbb{Y}$ is an uncertain set η_2". From $\mathbb{X}$ is a constant a we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \frac{\mathcal{M}\{a\in\xi_1\}\cdot\eta_1|_{a\in\xi_1}}{\mathcal{M}\{a\in\xi_1\}+\mathcal{M}\{a\in\xi_2\}} + \frac{\mathcal{M}\{a\in\xi_2\}\cdot\eta_2|_{a\in\xi_2}}{\mathcal{M}\{a\in\xi_1\}+\mathcal{M}\{a\in\xi_2\}}.$$

The inference rule is represented by

$$\begin{array}{l} \text{Rule 1: If } \mathbb{X} \text{ is } \xi_1 \text{ then } \mathbb{Y} \text{ is } \eta_1 \\ \text{Rule 2: If } \mathbb{X} \text{ is } \xi_2 \text{ then } \mathbb{Y} \text{ is } \eta_2 \\ \text{From: } \mathbb{X} \text{ is a constant } a \\ \hline \text{Infer: } \mathbb{Y} \text{ is } \eta^* \end{array}$$

Theorem 4.3 ([5]) *Let $\xi_1, \xi_2, \eta_1, \eta_2$ are independent uncertain sets with continuous membership functions $\mu_1, \mu_2, \nu_1, \nu_2$, respectively, then the inference rule yields*

$$\eta^* = \frac{\mu_1(a)}{\mu_1(a)+\mu_2(a)}\eta_1^* + \frac{\mu_2(a)}{\mu_1(a)+\mu_2(a)}\eta_2^*$$

where η_i^* are uncertain sets whose membership functions are

$$\nu_i^*(y) = \begin{cases} \dfrac{\nu_i(y)}{\mu_i(a)}, & \text{if } \nu_i(y) < \dfrac{\mu_i(a)}{2} \\ \dfrac{\nu_i(y)+\mu_i(a)-1}{\mu_i(a)}, & \text{if } \nu_i(y) > 1 - \dfrac{\mu_i(a)}{2} \\ 0.5, & \text{otherwise.} \end{cases}$$

Inference Rule 4.4 ([5]) *Let $\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_m$ and $\mathbb{Y}$ be concepts. Assume rules "if $\mathbb{X}_1$ is ξ_1 and $\cdots$ and $\mathbb{X}_m$ is ξ_m then $\mathbb{Y}$ is η_i" for $i = 1, 2, \ldots, k$. From $\mathbb{X}_1$ is a_1 and $\cdots$ and $\mathbb{X}_m$ is a_m, we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i|_{(a_1 \in \xi_{i1}) \cap (a_2 \in \xi_{i2}) \cap \cdots \cap (a_m \in \xi_{im})}}{c_1 + c_2 + \cdots + c_k}$$

where the coefficients are determined by

$$c_i = \mathcal{M}\{(a_1 \in \xi_{i1}) \cap (a_2 \in \xi_{i2}) \cap \cdots \cap (a_m \in \xi_{im})\}$$

for $i = 1, 2, \ldots, k$. The inference rule is represented by

Rule 1: If $\mathbb{X}_1$ is ξ_{11} and $\cdots$ and $\mathbb{X}_m$ is ξ_{1m} then $\mathbb{Y}$ is η_1
Rule 2: If $\mathbb{X}_1$ is ξ_{21} and $\cdots$ and $\mathbb{X}_m$ is ξ_{2m} then $\mathbb{Y}$ is η_2
$\cdots$
Rule k: If $\mathbb{X}_1$ is ξ_{k1} and $\cdots$ and $\mathbb{X}_m$ is ξ_{km} then $\mathbb{Y}$ is η_k
From: $\mathbb{X}_1$ is a_1 and $\cdots$ and $\mathbb{X}_m$ is a_m

---

Infer: $\mathbb{Y}$ is η^*

Theorem 4.4 ([5]) *Assume $\xi_{i1}, \xi_{i2}, \ldots, \xi_{im}, \eta_i$ are independent uncertain sets with membership functions $\mu_{i1}, \mu_{i2}, \ldots, \mu_{im}, \nu_i$, $i = 1, 2, \ldots, k$, respectively. If $\xi_1^*, \xi_2^*, \ldots, \xi_m^*$ are constants $a_1, a_2, \ldots, a_m$, respectively, then the inference rule yields*

$$\eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i^*}{c_1 + c_2 + \cdots + c_k}$$

where η_i^* are uncertain sets whose membership functions are given by

$$\nu_i^*(y) = \begin{cases} \dfrac{\nu_i(y)}{c_i}, & \text{if } \nu_i(y) < \dfrac{c_i}{2} \\ \dfrac{\nu_i(y) + c_i - 1}{c_i}, & \text{if } \nu_i(y) > 1 - \dfrac{c_i}{2} \\ 0.5, & \text{otherwise,} \end{cases}$$

and c_i are constants determined by

$$c_i = \min_{1 \le l \le m} \mu_{il}(a_l)$$

for $i = 1, 2, \ldots, k$, respectively.

In 2010, Liu [11] proposed uncertain system that is a function from its inputs to outputs based on the uncertain inference rule. Generally speaking, an uncertain system consists of 5 parts:

1. inputs that are crisp data to be fed into the uncertain system;
2. a rule-base that contains a set of if-then rules provided by the experts;
3. an uncertain inference rule that infers uncertain consequents from the uncertain antecedents;
4. an expected value operator that converts the uncertain consequents to crisp values;
5. outputs that are crisp data yielded from the expected value operator.

According to inference rule, if we infer n uncertain sets $\eta_1^*, \eta_2^*, \ldots, \eta_n^*$, from the m deterministic inputs $(\alpha_1, \alpha_2, \cdots, \alpha_m)$, and obtain n outputs $(\beta_1, \beta_2, \cdots, \beta_n)$ where $\beta_j = E[\eta_j^*]$ for $j = 1, 2, \ldots, n$, then we get a function from inputs $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ to outputs $(\beta_1, \beta_2, \cdots, \beta_n)$. Write this function by f, that is,

$$(\beta_1, \beta_2, \cdots, \beta_n) = f(\alpha_1, \alpha_2, \cdots, \alpha_m).$$

Then we get an uncertain system f.

In 2014, Peng and Chen [24] proved that uncertain systems are universal approximators. That is, uncertain system is capable of approximating any continuous function on a compact set to arbitrary accuracy.

Theorem 4.5 ([24]) *For any given continuous function g on a compact set $D \subset \Re^m$, and any given $\varepsilon > 0$, there exists an uncertain system f such that*

$$\sup_{(\alpha_1,\alpha_2,\cdots,\alpha_m)\in D} \| f(\alpha_1,\alpha_2,\cdots,\alpha_m) - g(\alpha_1,\alpha_2,\cdots,\alpha_m) \| < \varepsilon.$$

For a general function, Peng and Chen [24] also proved the following theorem.

Theorem 4.6 ([24])*For any given function g on a compact set $D \subset \Re^m$, and any given $\varepsilon > 0$, there exists an uncertain system f such that*

$$\left(\int_D \| f(\alpha_1,\alpha_2,\cdots,\alpha_m) - g(\alpha_1,\alpha_2,\cdots,\alpha_m) \|^2 \,\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_m\right)^{1/2} < \varepsilon$$

where $\int_D \| g(\alpha_1,\alpha_2,\cdots,\alpha_m) \|^2 \,\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_m < \infty$ in the sense of Lebesgue integral.

As a successful application, Gao [6] balanced an inverted pendulum by using the uncertain controller in 2013. The uncertain controller has two inputs ("angle" and "angular velocity") and one output ("force"). Three of them will be represented by uncertain sets labeled by

"*negative large*"	NL
"*negative small*"	NS
"*zero*"	Z
"*positive small*"	PS
"*positive large*"	PL

The membership function of those uncertain sets are shown in Figures 1, 2 and 3.

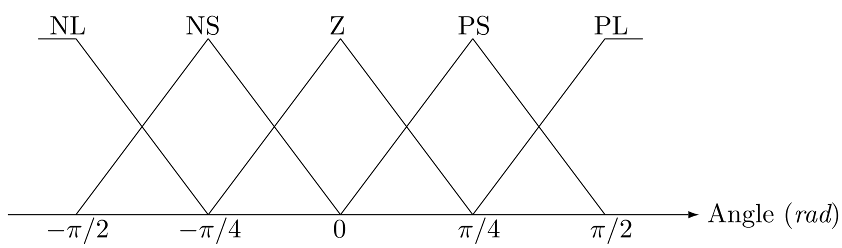

Figure 1: Membership functions of "Angle"

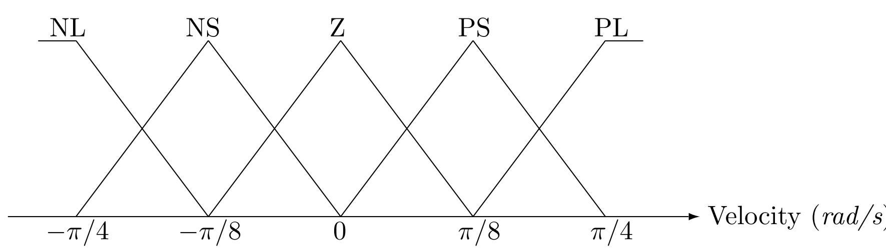

Figure 2: Membership functions of "Velocity"

In order to balance the inverted pendulum, the 25 if-then rules in Table 1 are accepted. In the first row and the first column, we use the following if-then rule.

"If the angle is negative large and the angle velocity is negative large, then the force is positive large".

A lot of simulation results show that the uncertain controller may balance the inverted pendulum successfully.

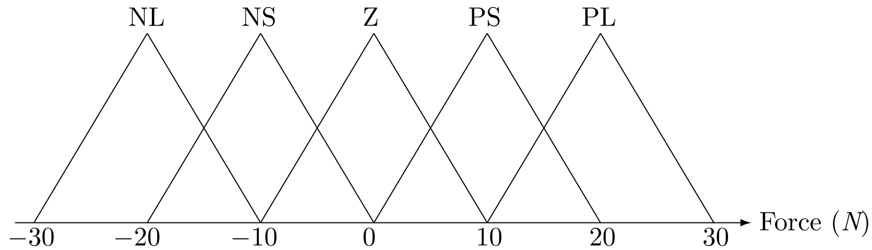

Figure 3: Membership functions of "Force"

Table 1: Rule Base with 5 × 5 If-Then Rules

Angle \ Velocity	NL	NS	Z	PS	PL
NL	PL	PL	PL	PS	Z
NS	PL	PL	PS	Z	NS
Z	PL	PL	PL	PS	Z
PS	PS	Z	NS	NL	NL
PL	Z	NS	PL	NL	NL

5 Conclusion

Uncertain set is used to describe "unsharp concepts" that are essentially sets but their boundaries are not sharply described. This paper reviewed some concepts of uncertain set. At the same time, we reviewed uncertain logic that is a methodology for calculating the truth value for uncertain propositions via uncertain set theory. In addition, this paper also reviewed uncertain inference rules that derive consequences from uncertain knowledge via the tool of conditional uncertain set.

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