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Multi-Dimensional Uncertain Calculus with Liu Process

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Abstract

Uncertain calculus deals with the integral and differential of some uncertain processes. So far, uncertain integrals have been defined with respect to Liu process, renewal process, finite variation process, and multiple Liu processes. This paper presents an uncertain integral of a matrix of uncertain processes with respect to multi-dimensional Liu process, and verifies its linearity property. Then an uncertain differential of a multi-dimensional uncertain process with respect to a multi-dimensional Liu process is defined, and a fundamental theorem is derived. In addition, a concept of multi-dimensional uncertain differential equation is proposed, and solutions of some special types of multi-dimensional uncertain differential equations are given.

Keywords: Liu process, uncertain calculus, uncertain differential equation

1 Introduction

Uncertainty theory, established by Liu [10] in 2007 and refined by Liu [13] in 2010, is a branch of axiomatic mathematics to deal with human's belief degree based on normality, duality, subadditivity and product axioms. An uncertain measure is a set function satisfying these four axioms, which indicates the belief degree that an uncertain event will occur. Then a concept of uncertain variable was proposed by Liu [10] to model a quantity under human uncertainty. In order to describe uncertain variables, concepts of uncertainty distribution, expected value, variance, entropy as well as independence were presented, and an uncertainty theory was thus founded. So far, many researchers have contributed a lot in this area. Peng and Iwamura [20] gave a sufficient condition for a function being an uncertainty distribution. Liu and Ha [19] derived a formula to calculate the expected value of a function of uncertain variables. Yao [27] gave a formula to calculate the variance of an uncertain variable via its inverse uncertainty distribution. Dai and Chen [6] verified the linearity of the entropy operator for uncertain variables.

In our daily life, some uncertain quantities evolve with the time. In order to model these quantities, a concept of uncertain process was proposed by Liu [11] as a sequence of uncertain variables driven by the time. As an example, Liu [12] designed a Liu process, that is an uncertain process with independent and stationary normal increments. Meanwhile, Liu [12] proposed an uncertain integral of an uncertain process with respect to Liu process, which was generalized to an uncertain integral of multiple uncertain processes with respect to multiple Liu processes by Liu and Yao [15] in 2012. In addition, an uncertain integral of an uncertain process with respect to an uncertain renewal process was defined by Yao [24], and an uncertain integral with respect to finite variation process was defined by Chen [2].

Uncertain differential equation is a type of differential equation driven by an uncertain process. Uncertain differential equation driven by Liu process was proposed by Liu [11]. A sufficient condition for such a differential equation having a unique solution was first given by Chen and Liu [4], which was weakened by Gao [7] later. The concept of stability was first proposed by Liu [12], and a sufficient condition for an uncertain differential equation being stable was given by Yao et al. [30]. After that, concepts of stability in mean [31], stability in p-th moment [22], almost sure stability [16], and stability in distribution [32] were proposed. In addition, Gao and Yao [8] studied conditions of continuous dependence for the solution of an uncertain differential equation with respect to the initial value.

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The solution methods of uncertain differential equation have also drawn a lot of attentions from the researchers. In 2010, Chen and Liu [4] gave the analytic solution of a linear uncertain differential equation. After that, Liu [17] and Yao [25] gave some methods to solve some special types of nonlinear uncertain differential equations. However, it is impossible to obtain the analytic solutions of some types of uncertain differential equations. In this case, Yao and Chen [29] designed a numerical method to solve an uncertain differential equation via a concept of α-path. After that, Yao [26] extended this result, and obtained the uncertainty distributions of extreme value and time integral of the solution via the α-path.

Uncertain differential equation was first introduced to finance by Liu [12], where he proposed an uncertain stock model via the geometric Liu process, and derived its European option pricing formulas. After that, Chen [1] derived its American option pricing formulas, and Sun and Chen [23] derived its Asian option pricing formulas. Besides, Yao [28] gave a sufficient and necessary condition for an uncertain stock market being no-arbitrage. In addition, Peng and Yao [21] proposed a mean-reverting stock model, and Chen et al. [5] proposed a stock model with dividends. Uncertain interest rate model was first proposed by Chen and Gao [3], and was further investigated by Jiao and Yao [9]. Uncertain currency model was proposed by Liu et al. [18], of which the exchanged rate was described by an uncertain differential equation.

In order to describe the irregular movement of a particle in a high dimensional space, a multi-dimensional Liu process was proposed by Zhang and Chen [33]. In this paper, we will propose an uncertain calculus with respect to multi-dimensional Liu process. The rest of this paper is organized as follows. Uncertain variable and traditional uncertain calculus will be introduced in Section 2 and Section 3, respectively. Then an uncertain integral of a matrix of uncertain processes with respect to a multi-dimensional Liu process will be introduced in Section 4, and an uncertain differential of a multi-dimensional uncertain process with respect to a multi-dimensional Liu process will be introduced in Section 5. Besides, a multi-dimensional uncertain differential equation will be introduced in Section 6, whose solution is a multi-dimensional uncertain process. At last, some remarks are made in Section 7.

2 Uncertain Variable

In this section, we introduce some basic concepts about uncertain variable, including uncertainty space, uncertainty distribution, expected value, and operational law.

Definition 1 ([10]) *Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}: \mathcal{L} \to [0,1]$ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1: *(Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.*
Axiom 2: *(Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.*
Axiom 3: *(Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$ we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Product uncertain measure was defined by Liu [12] in 2009, thus producing the the fourth axiom of uncertainty theory.
Axiom 4: *(Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots,$ respectively.

An uncertain variable is essentially a measurable function on an uncertainty space, and it is used to model an uncertain quantity.

Definition 2 ([10]) *An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set $\Re$ of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 3 ([10]) *The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

The uncertainty distribution Φ is said to be regular if its inverse function Φ^{-1} exists and is unique for each $\alpha \in (0,1)$. In this case, the inverse function Φ^{-1} is called the inverse uncertainty distribution, which plays an important role in the operation of independent uncertain variables.

Definition 4 ([12]) *The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{k=1}^{m}(\xi_i \in B_i)\right\} = \bigwedge_{k=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

Theorem 1 ([13]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions Φ_1, $\Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution*

$$\Phi^{-1}(r) = f\left(\Phi_1^{-1}(r), \cdots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \cdots, \Phi_n^{-1}(1-r)\right).$$

Definition 5 ([10]) *Let ξ be an uncertain variable. Then its expected value $E[\xi]$ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

For an uncertain variable ξ with an uncertainty distribution Φ, we have

$$E[\xi] = \int_0^{+\infty} (1-\Phi(r))\mathrm{d}r - \int_{-\infty}^{0} \Phi(r)\mathrm{d}r.$$

If the inverse uncertainty distribution Φ^{-1} exists, then

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

Definition 6 ([10]) *Let ξ be an uncertain variable with a finite expected value e. Then its variance is defined by*

$$V[\xi] = E\left[(\xi - e)^2\right].$$

We cannot derive the precise variance of an uncertain variable just from its uncertainty distribution, so Liu [10] suggested to accept

$$V[\xi] = \int_0^{+\infty} \left(1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x})\right)\mathrm{d}x$$

as a stipulation, where Φ is the uncertainty distribution of ξ. Based on this stipulation, Yao [27] gave the formula

$$V[\xi] = \int_0^1 \left(\Phi^{-1}(\alpha) - e\right)^2 \mathrm{d}\alpha$$

to calculate the variance via inverse uncertainty distribution.

3 Uncertain Calculus

In this section, we introduce some definitions about uncertain process and uncertain calculus. An uncertain process is a sequence of uncertain variables indexed by time or space.

Definition 7 ([11]) *Let T be an index set and $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set*

$$\{X_t \in B\} = \{\gamma \,|\, X_t(\gamma) \in B\}$$

is an event.

Definition 8 ([12]) *An uncertain process C_t is said to be a canonical Liu process if*
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \ x \in \Re.$$

Based on canonical Liu process, an uncertain integral named Liu integral was defined by Liu [12], thus offering a theory of uncertain calculus.

Definition 9 ([12]) *Let X_t be an uncertain process and C_t be a canonical Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then Liu integral of X_t is defined by

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be Liu integrable.

For example, the canonical Liu process C_t is a Liu integrable process, and

$$\int_0^s C_t \mathrm{d}C_t = \frac{1}{2} C_s^2.$$

Definition 10 ([12]) *Let C_t be a canonical Liu process and Z_t be an uncertain process. If there exist uncertain processes μ_s and σ_s such that*

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s$$

for any $t \ge 0$, then Z_t is said to have a Liu differential

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t.$$

For example, the uncertain process C_t^2 has a Liu differential $\mathrm{d}C_t^2 = 2C_t \mathrm{d}C_t$. The uncertain process tC_t has a Liu differential $\mathrm{d}(tC_t) = C_t \mathrm{d}t + t\mathrm{d}C_t$.

Liu [12] verified the fundamental theorem of uncertain calculus, i.e., for a canonical Liu process C_t and a continuously differentiable function $h(t, c)$, the uncertain process $Z_t = h(t, C_t)$ has a Liu differential

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t.$$

Based on the fundamental theorem, Liu [12] proved the integration by parts theorem, i.e., for two Liu differentiable processes X_t and Y_t, the uncertain process $X_t Y_t$ has a Liu differential

$$\mathrm{d}(X_t Y_t) = Y_t \mathrm{d}X_t + X_t \mathrm{d}Y_t.$$

Definition 11 ([14]) *Uncertain processes $X_{1t}, X_{2t}, \ldots, X_{nt}$ are said to be independent if for any positive integer k and any times $t_1, t_2, \ldots, t_k$, the uncertain vectors*

$$\boldsymbol{\xi}_i = (X_{it_1}, X_{it_2}, \cdots, X_{it_k}), \quad i = 1, 2, \ldots, n$$

are independent, i.e., for any k-dimensional Borel sets $B_1, B_2, \ldots, B_n$, we have

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\}.$$

Definition 12 ([33]) *Let $C_{it}, i = 1, 2, \ldots, n$ be independent canonical Liu processes on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \cdots, C_{nt})^T$ is called an n-dimensional canonical Liu process on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$.*

The multi-dimensional canonical Liu process $\boldsymbol{C}_t$ is a multi-dimensional Lipschitz continuous uncertain process with stationary and independent increments.

4 Multi-Dimensional Uncertain Integral

In this section, we will study the uncertain integral of a matrix of uncertain processes with respect to a multi-dimensional canonical Liu process, and verify some properties of the uncertain integral.

Definition 13 *An $m \times n$ matrix $\boldsymbol{X}_t = [X_{ijt}]$ is called an uncertain matrix process if its elements X_{ijt} are uncertain processes for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.*

Definition 14 *Let $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \cdots, C_{nt})^T$ be an n-dimensional canonical Liu process, and $\boldsymbol{X}_t = [X_{ijt}]$ be an $m \times n$ uncertain matrix process whose elements X_{ijt} are integrable uncertain processes. Then the uncertain integral of $\boldsymbol{X}_t$ with respect to n-dimensional canonical Liu process $\boldsymbol{C}_t$ is defined by*

$$\int_a^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t = \begin{pmatrix} \sum_{j=1}^{n} \int_a^b X_{1jt} \mathrm{d}C_{jt} \\ \sum_{j=1}^{n} \int_a^b X_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \sum_{j=1}^{n} \int_a^b X_{mjt} \mathrm{d}C_{jt} \end{pmatrix}.$$

In this case, $\boldsymbol{X}_t$ is said to be uncertain integrable with respect to $\boldsymbol{C}_t$.

Example 1 Let $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \cdots, C_{nt})^T$ be an n-dimensional canonical Liu process, and $\boldsymbol{X}_t = (X_{1t}, X_{2t}, \cdots, X_{nt})$ be an n-dimensional integrable uncertain process. Then

$$\int_a^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t = \sum_{j=1}^{n} \int_a^b X_{jt} \mathrm{d}C_{jt}.$$

Example 2 Let C_t be a canonical Liu process, and $\boldsymbol{X}_t = (X_{1t}, X_{2t}, \cdots, X_{nt})^T$ be an n-dimensional integrable uncertain process. Then

$$\int_a^b \boldsymbol{X}_t \mathrm{d}C_t = \begin{pmatrix} \int_a^b X_{1t} \mathrm{d}C_t \\ \int_a^b X_{2t} \mathrm{d}C_t \\ \vdots \\ \int_a^b X_{mt} \mathrm{d}C_t \end{pmatrix}.$$

Example 3 Let $\boldsymbol{C}_t = (C_{1t}, C_{2t})^T$ be a 2-dimensional canonical Liu process, and

$$\boldsymbol{X}_t = \begin{pmatrix} C_{1t} & 0 \\ 0 & C_{2t} \end{pmatrix}$$

be an uncertain matrix process. Then

$$\int_0^t \boldsymbol{X}_s \mathrm{d}\boldsymbol{C}_s = \frac{1}{2} \begin{pmatrix} C_{1t}^2 \\ C_{2t}^2 \end{pmatrix}.$$

Theorem 2 *Let $\boldsymbol{C}_t$ be an n-dimensional canonical Liu process, and $\boldsymbol{X}_t$ be an $m \times n$ integrable uncertain matrix process on $[a, b]$. Then $\boldsymbol{X}_t$ is uncertain integrable with respect to $\boldsymbol{C}_t$ on each subinterval of $[a, b]$. Moreover, if $c \in [a, b]$, then*

$$\int_a^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t = \int_a^c \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t + \int_c^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t.$$

Proof: Since $\boldsymbol{X}_t$ is uncertain integrable with respect to $\boldsymbol{C}_t$ on $[a, b]$, the uncertain process X_{ijt} is Liu integrable with respect to C_{jt} on $[a, b]$. Then X_{ijt} is Liu integrable with respect to C_{jt} on each subinterval of $[a, b]$. By Definition 14, the matrix process $\boldsymbol{X}_t$ is uncertain integrable with respect to $\boldsymbol{C}_t$ on each subinterval of $[a, b]$. Next, for each point $c \in [a, b]$, we have

$$\int_a^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t = \begin{pmatrix} \sum\limits_{j=1}^n \int_a^b X_{1jt} \mathrm{d}C_{jt} \\ \sum\limits_{j=1}^n \int_a^b X_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \sum\limits_{j=1}^n \int_a^b X_{mjt} \mathrm{d}C_{jt} \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^n \int_a^c X_{1jt} \mathrm{d}C_{jt} + \sum\limits_{j=1}^n \int_c^b X_{1jt} \mathrm{d}C_{jt} \\ \sum\limits_{j=1}^n \int_a^c X_{2jt} \mathrm{d}C_{jt} + \sum\limits_{j=1}^n \int_c^b X_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \sum\limits_{j=1}^n \int_a^c X_{mjt} \mathrm{d}C_{jt} + \sum\limits_{j=1}^n \int_c^b X_{mjt} \mathrm{d}C_{jt} \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{j=1}^n \int_a^c X_{1jt} \mathrm{d}C_{jt} \\ \sum\limits_{j=1}^n \int_a^c X_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \sum\limits_{j=1}^n \int_a^c X_{mjt} \mathrm{d}C_{jt} \end{pmatrix} + \begin{pmatrix} \sum\limits_{j=1}^n \int_c^b X_{1jt} \mathrm{d}C_{jt} \\ \sum\limits_{j=1}^n \int_c^b X_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \sum\limits_{j=1}^n \int_c^b X_{mjt} \mathrm{d}C_{jt} \end{pmatrix} = \int_a^c \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t + \int_c^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t.$$

The proof is thus completed.

Theorem 3 *Let $\boldsymbol{C}_t$ be an n-dimensional canonical Liu process, $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ be two $m \times n$ integrable uncertain matrix processes on $[a, b]$. Then for any real numbers α and β, we have*

$$\int_a^b (\alpha \boldsymbol{X}_t + \beta \boldsymbol{Y}_t) \mathrm{d}\boldsymbol{C}_t = \alpha \int_a^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t + \beta \int_a^b \boldsymbol{Y}_t \mathrm{d}\boldsymbol{C}_t.$$

Proof: It follows from Definition 14 that

$$\int_a^b (\alpha \boldsymbol{X}_t + \beta \boldsymbol{Y}_t) \mathrm{d}\boldsymbol{C}_t = \begin{pmatrix} \sum\limits_{j=1}^n \int_a^b (\alpha X_{1jt} + \beta Y_{1jt}) \mathrm{d}C_{jt} \\ \sum\limits_{j=1}^n \int_a^b (\alpha X_{2jt} + \beta Y_{2jt}) \mathrm{d}C_{jt} \\ \vdots \\ \sum\limits_{j=1}^n \int_a^b (\alpha X_{mjt} + \beta Y_{mjt}) \mathrm{d}C_{jt} \end{pmatrix} = \begin{pmatrix} \alpha \sum\limits_{j=1}^n \int_a^b X_{1jt} \mathrm{d}C_{jt} + \beta \sum\limits_{j=1}^n \int_a^b Y_{1jt} \mathrm{d}C_{jt} \\ \alpha \sum\limits_{j=1}^n \int_a^b X_{2jt} \mathrm{d}C_{jt} + \beta \sum\limits_{j=1}^n \int_a^b Y_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \alpha \sum\limits_{j=1}^n \int_a^b X_{mjt} \mathrm{d}C_{jt} + \beta \sum\limits_{j=1}^n \int_a^b Y_{mjt} \mathrm{d}C_{jt} \end{pmatrix}$$

$$= \alpha \begin{pmatrix} \sum_{j=1}^{n} \int_a^b X_{1jt} \mathrm{d}C_{jt} \\ \sum_{j=1}^{n} \int_a^b X_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \sum_{j=1}^{n} \int_a^b X_{mjt} \mathrm{d}C_{jt} \end{pmatrix} + \beta \begin{pmatrix} \sum_{j=1}^{n} \int_a^b Y_{1jt} \mathrm{d}C_{jt} \\ \sum_{j=1}^{n} \int_a^b Y_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \sum_{j=1}^{n} \int_a^b Y_{mjt} \mathrm{d}C_{jt} \end{pmatrix} = \alpha \int_a^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t + \beta \int_a^b \boldsymbol{Y}_t \mathrm{d}\boldsymbol{C}_t.$$

The theorem is verified.

5 Multi-Dimensional Uncertain Differential

In this section, we will study the uncertain integral of a multi-dimensional uncertain process with respect to a multi-dimensional canonical Liu process, and verify the fundamental theorem.

Definition 15 *Let $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \cdots, C_{nt})^T$ be an n-dimensional canonical Liu process, and $\boldsymbol{Z}_t = (Z_{1t}, Z_{2t}, \cdots, Z_{mt})^T$ be an m-dimensional uncertain process. If there exists an m-dimensional uncertain process $\boldsymbol{\mu}_t = (\mu_{1t}, \mu_{2t}, \cdots, \mu_{mt})$ and an $m \times n$ uncertain matrix process $\boldsymbol{\sigma}_t = [\sigma_{ijt}]$ such that*

$$\boldsymbol{Z}_t = \boldsymbol{Z}_0 + \int_0^t \boldsymbol{\mu}_s \mathrm{d}s + \int_0^t \boldsymbol{\sigma}_s \mathrm{d}\boldsymbol{C}_s$$

for any $t \geq 0$, then $\boldsymbol{Z}_t$ is said to be uncertain differentiable with respect to $\boldsymbol{C}_t$, and has an uncertain differential

$$\mathrm{d}\boldsymbol{Z}_t = \boldsymbol{\mu}_t \mathrm{d}t + \boldsymbol{\sigma}_t \mathrm{d}\boldsymbol{C}_t.$$

Example 4 Let $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \cdots, C_{nt})^T$ be an n-dimensional canonical Liu process. Since

$$\boldsymbol{C}_t = \int_0^t \mathrm{d}\boldsymbol{C}_s$$

for any $t \geq 0$, we have that the n-dimensional canonical Liu process $\boldsymbol{C}_t$ is uncertain differentiable, and has an uncertain differential $\mathrm{d}\boldsymbol{C}_t$.

Example 5 Let $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \cdots, C_{nt})^T$ be an n-dimensional canonical Liu process. Since

$$t\boldsymbol{C}_t = \int_0^t \boldsymbol{C}_s \mathrm{d}s + \int_0^t s\mathrm{d}\boldsymbol{C}_s$$

holds for any $t \geq 0$, the n-dimensional uncertain process $t\boldsymbol{C}_t$ is uncertain differentiable with respect to $\boldsymbol{C}_t$, and has an uncertain differential

$$\mathrm{d}(t\boldsymbol{C}_t) = \boldsymbol{C}_t \mathrm{d}t + t\mathrm{d}\boldsymbol{C}_t.$$

Example 6 Let $\boldsymbol{C}_t = (C_{1t}, C_{2t})^T$ be a 2-dimensional canonical Liu process, and $\boldsymbol{Z}_t = (C_{1t}^2, C_{2t}^2)^T$ be a 2-dimensional uncertain process. Since

$$\begin{pmatrix} C_{1t}^2 \\ C_{2t}^2 \end{pmatrix} = 2 \int_0^t \begin{pmatrix} C_{1s} & 0 \\ 0 & C_{2s} \end{pmatrix} \mathrm{d} \begin{pmatrix} C_{1s} \\ C_{2s} \end{pmatrix}$$

holds for any $t \geq 0$, the uncertain process $\boldsymbol{Z}_t$ is uncertain differentiable with respect to $\boldsymbol{C}_t$, and has an uncertain differential

$$\mathrm{d}\boldsymbol{Z}_t = 2 \begin{pmatrix} C_{1t} & 0 \\ 0 & C_{2t} \end{pmatrix} \mathrm{d}\boldsymbol{C}_t.$$

Theorem 4 *(Fundamental Theorem) Let $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \cdots, C_{nt})^T$ be an n-dimensional canonical Liu process, and let $\boldsymbol{h}(t, \boldsymbol{c}) = (h_1(t, \boldsymbol{c}), h_2(t, \boldsymbol{c}), \cdots, h_m(t, \boldsymbol{c}))^T$ be a continuously differentiable m-dimensional vector-valued function, where $\boldsymbol{c} = (c_1, c_2, \cdots, c_n)^T$ is an n-dimensional vector. Then the m-dimensional uncertain process $\boldsymbol{Z}_t = \boldsymbol{h}(t, \boldsymbol{C}_t)$ is uncertain differentiable with respect to $\boldsymbol{C}_t$, and has an uncertain differential*

$$\mathrm{d}\boldsymbol{Z}_t = \frac{\partial \boldsymbol{h}}{\partial t}(t, \boldsymbol{C}_t)\mathrm{d}t + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{c}}(t, \boldsymbol{C}_t)\mathrm{d}\boldsymbol{C}_t,$$

where

$$\frac{\partial \boldsymbol{h}}{\partial t}(t, \boldsymbol{C}_t) = \begin{pmatrix} \dfrac{\partial h_1}{\partial t}(t, C_{1t}, \cdots, C_{nt}) \\ \dfrac{\partial h_2}{\partial t}(t, C_{1t}, \cdots, C_{nt}) \\ \vdots \\ \dfrac{\partial h_m}{\partial t}(t, C_{1t}, \cdots, C_{nt}) \end{pmatrix}$$

and

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{c}}(t, \boldsymbol{C}_t) = \begin{pmatrix} \dfrac{\partial h_1}{\partial c_1}(t, C_{1t}, \cdots, C_{nt}) & \cdots & \dfrac{\partial h_1}{\partial c_n}(t, C_{1t}, \cdots, C_{nt}) \\ \dfrac{\partial h_2}{\partial c_1}(t, C_{1t}, \cdots, C_{nt}) & \cdots & \dfrac{\partial h_2}{\partial c_n}(t, C_{1t}, \cdots, C_{nt}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial h_m}{\partial c_1}(t, C_{1t}, \cdots, C_{nt}) & \cdots & \dfrac{\partial h_m}{\partial c_n}(t, C_{1t}, \cdots, C_{nt}) \end{pmatrix}.$$

Proof: Write $\Delta \boldsymbol{C}_t = \boldsymbol{C}_{t+\Delta t} - \boldsymbol{C}_t$. Then $\Delta \boldsymbol{C}_t$ and Δt are infinitesimals with the same order. Since $\boldsymbol{h}(t, \boldsymbol{c})$ is a continuously differentiable function, we have

$$\Delta \boldsymbol{Z}_t = \frac{\partial \boldsymbol{h}}{\partial t}(t, \boldsymbol{C}_t)\Delta t + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{c}}(t, \boldsymbol{C}_t)\Delta \boldsymbol{C}_t.$$

Letting $\Delta t \to 0$, we have that $\boldsymbol{Z}_t$ is uncertain differentiable with respect to $\boldsymbol{C}_t$, and it has an uncertain differential

$$\mathrm{d}\boldsymbol{Z}_t = \frac{\partial \boldsymbol{h}}{\partial t}(t, \boldsymbol{C}_t)\mathrm{d}t + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{c}}(t, \boldsymbol{C}_t)\mathrm{d}\boldsymbol{C}_t.$$

Example 7 Consider the uncertain differential of $\boldsymbol{Z}_t = (C_{1t}^2, C_{2t}^2)^T$ with respect to $\boldsymbol{C}_t = (C_{1t}, C_{2t})^T$. In this case, we have $\boldsymbol{h}(t, \boldsymbol{c}) = (c_1^2, c_2^2)^T$. It is clear that

$$\frac{\partial \boldsymbol{h}}{\partial t}(t, \boldsymbol{c}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{c}}(t, \boldsymbol{c}) = 2\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

It follows from the fundamental theorem that

$$\mathrm{d}\boldsymbol{Z}_t = 2\begin{pmatrix} C_{1t} & 0 \\ 0 & C_{2t} \end{pmatrix}\mathrm{d}\boldsymbol{C}_t.$$

Corollary 1 ([15]) *Let $(C_{1t}, C_{2t}, \cdots, C_{nt})^T$ be an n-dimensional canonical Liu process, and let $h(t, c_1, c_2, \cdots, c_n)$ be a continuously differentiable function. Then the uncertain process $Z_t = h(t, C_{1t}, C_{2t}, \cdots, C_{nt})$ is uncertain differentiable and has an uncertain differential*

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_{1t}, C_{2t}, \cdots, C_{nt})\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial h}{\partial c_i}(t, C_{1t}, C_{2t}, \cdots, C_{nt})\mathrm{d}C_{it}.$$

Proof: Take $m = 1$ in the fundamental theorem, and the corollary is verified.

Theorem 5 *Let $\boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \ldots, \boldsymbol{X}_{nt}$ be differentiable vector-valued uncertain processes, and $\boldsymbol{h}(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$ be a continuously differentiable vector-valued function. Then the uncertain process $\boldsymbol{X}_t = \boldsymbol{h}(t, \boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \cdots, \boldsymbol{X}_{nt})$ is uncertain differentiable and has an uncertain differential*

$$\mathrm{d}\boldsymbol{X}_t = \frac{\partial \boldsymbol{h}}{\partial t}(t, \boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \cdots, \boldsymbol{X}_{nt})\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_i}(t, \boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \cdots, \boldsymbol{X}_{nt})\mathrm{d}\boldsymbol{X}_{it}.$$

Proof: Since the function $\boldsymbol{h}$ is continuously differentiable, the infinitesimal increment of $\boldsymbol{X}_t$ has a first-order approximation

$$\Delta\boldsymbol{X}_t = \frac{\partial \boldsymbol{h}}{\partial t}(t, \boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \cdots, \boldsymbol{X}_{nt})\Delta t + \sum_{i=1}^{n} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_i}(t, \boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \cdots, \boldsymbol{X}_{nt})\Delta\boldsymbol{X}_{it}.$$

Letting $\Delta t \to 0$, we have

$$\mathrm{d}\boldsymbol{X}_t = \frac{\partial \boldsymbol{h}}{\partial t}(t, \boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \cdots, \boldsymbol{X}_{nt})\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_i}(t, \boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \cdots, \boldsymbol{X}_{nt})\mathrm{d}\boldsymbol{X}_{it}.$$

The theorem is thus proved.

Example 8 Suppose that $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ are m-dimensional vector-valued differentiable uncertain process. Then

$$\mathrm{d}(\boldsymbol{X}_t^T\boldsymbol{Y}_t) = \boldsymbol{Y}_t^T\mathrm{d}\boldsymbol{X}_t + \boldsymbol{X}_t^T\mathrm{d}\boldsymbol{Y}_t.$$

Proof: Taking $h(t, \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T\boldsymbol{y}$, we have

$$\frac{\partial h}{\partial t} = 0, \quad \frac{\partial h}{\partial \boldsymbol{x}} = \boldsymbol{y}^T, \quad \frac{\partial h}{\partial \boldsymbol{y}} = \boldsymbol{x}^T.$$

It follows from Theorem 5 that

$$\mathrm{d}(\boldsymbol{X}_t^T\boldsymbol{Y}_t) = \boldsymbol{Y}_t^T\mathrm{d}\boldsymbol{X}_t + \boldsymbol{X}_t^T\mathrm{d}\boldsymbol{Y}_t.$$

6 Multi-Dimensional Uncertain Differential Equation

Definition 16 *Let $\boldsymbol{C}_t$ be an n-dimensional canonical Liu process. Suppose $\boldsymbol{f}(t, \boldsymbol{x})$ is a vector-valued function from $T \times \Re^n$ to $\Re^m$, and $\boldsymbol{g}(t, \boldsymbol{x})$ is a matrix-valued function from $T \times \Re^n$ to the set of $m \times n$ matrices. Then*

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{f}(t, \boldsymbol{X}_t)\mathrm{d}t + \boldsymbol{g}(t, \boldsymbol{X}_t)\mathrm{d}\boldsymbol{C}_t \tag{1}$$

is called a multi-dimensional uncertain differential equation driven by a multi-dimensional canonical Liu process. A solution is an m-dimensional uncertain process that satisfies (1) identically at each t in $\Re$.

Remark 1 When $m = 1$, the uncertain differential equation (1) degenerates to the following equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sum_{i=1}^{n} g_i(t, X_t)\mathrm{d}C_{it}.$$

Example 9 Consider a 2-dimensional uncertain differential equation

$$\mathrm{d}\boldsymbol{X}_t = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}\mathrm{d}t + \begin{pmatrix} v_{11t} & v_{12t} \\ v_{21t} & v_{22t} \end{pmatrix}\mathrm{d}\boldsymbol{C}_t$$

with an initial value $\boldsymbol{X}_0 = (0, 0)^T$, where $\boldsymbol{C}_t = (C_{1t}, C_{2t})^T$ is a 2-dimensional canonical Liu process. It is easy verified that the equation has a solution

$$\boldsymbol{X}_t = \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} \int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{11s}\mathrm{d}C_{1s} + \int_0^t v_{12s}\mathrm{d}C_{2s} \\ \int_0^t u_{2s}\mathrm{d}s + \int_0^t v_{21s}\mathrm{d}C_{1s} + \int_0^t v_{22s}\mathrm{d}C_{2s} \end{pmatrix}.$$

Example 10 Consider a 2-dimensional uncertain differential equation

$$\mathrm{d}\boldsymbol{X}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{X}_t \mathrm{d}C_t$$

with an initial value $\boldsymbol{X}_0 = (0,1)^T$, where C_t is a canonical Liu process. It is easy verified that the equation has a solution

$$\boldsymbol{X}_t = \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} \sin(C_t) \\ \cos(C_t) \end{pmatrix}.$$

Example 11 Let $\boldsymbol{C}_t = (C_{1t}, C_{2t})^T$ be a 2-dimensional canonical process. Consider an uncertain differential equation

$$\mathrm{d}X_t = X_t \mathrm{d}t + (X_t, X_t)\mathrm{d}\boldsymbol{C}_t = X_t \mathrm{d}t + X_t \mathrm{d}C_{1t} + X_t \mathrm{d}C_{2t}$$

with an initial value $X_0 = 1$. It is easy verified that the equation has a solution

$$X_t = \exp(t + C_{1t} + C_{2t}).$$

7 Conclusions

This paper founded an uncertain calculus with respect to a multi-dimensional Liu process, and proposed a multi-dimensional uncertain differential equation driven by multi-dimensional Liu process. In detail, an uncertain integral of a matrix of uncertain processes with respect to a multi-dimensional Liu process was defined, and its linearity property was verified. Then an uncertain integral of a multi-dimensional uncertain process with respect to a multi-dimensional Liu process was defined, and the fundamental theorem was provided. In addition, a multi-dimensional uncertain differential equation was proposed, and solutions of some special types of multi-dimensional uncertain differential equations were given. Further research will cover the existence and uniqueness of the solution of a multi-dimensional uncertain differential equation as well as its stability.

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