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Measure Generated by Joint Credibility Distribution Function*

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Abstract

This note addresses additive interval set functions constructed by joint credibility distributions. When joint credibility distributions can be represented as the product or minimum of their marginal credibility distributions, the constructed additive interval set functions are positive metrics so that Lebesgue-Stieltjes (L–S) measures can be generated by joint credibility distributions. In general case, the constructed additive interval set function has bounded variation, and can be represented as the difference of two positive additive interval set functions. Finally, we discuss the measurable sets with respect to positive metrics.

Keywords: interval set function, joint credibility distribution, bounded variation, measurable set

1 Main Results

We first recall some basic concepts about interval set functions [1]. Let $\Re^n$ be the Euclidean space, and S its nonempty open subset. In this section, we take S as a basic region (basic set), and just consider the closed subintervals $\boldsymbol{I} = [\boldsymbol{a}, \boldsymbol{b}]$ of basic region S, where $\boldsymbol{a} \leq \boldsymbol{b}$. If for any interval $\boldsymbol{I}$, we assign a real number $\alpha(\boldsymbol{I})$, then we define an interval set function $\alpha(\boldsymbol{I})$ on S. If $\boldsymbol{I}$ is the union of two separable intervals $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$, one has $\alpha(\boldsymbol{I}) = \alpha(\boldsymbol{I}_1) + \alpha(\boldsymbol{I}_2)$, then $\alpha(\boldsymbol{I})$ is called an additive interval set function. If $\boldsymbol{J}$ is a degenerated closed interval or empty set $\emptyset$, then one has $\alpha(\boldsymbol{J}) = 0$.

We now consider the joint credibility distribution of a fuzzy vector. The interested reader may refer to [2] and the references therein for the required knowledge about credibility measure theory. For an n-ary fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_1, \ldots, \xi_n)$, its joint credibility distribution is denoted as $\alpha_{\boldsymbol{\xi}}(\boldsymbol{x}) = \mathrm{Cr}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_n \leq x_n\}$, for any $\boldsymbol{x} \in S$. For each $1 \leq i \leq n$, the marginal credibility distribution of ξ_i is denoted as $\alpha_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \leq x_i\}$. According to [1], we can construct an additive interval set function by using the joint credibility distribution $\alpha_{\boldsymbol{\xi}}(\boldsymbol{x})$.

If $\boldsymbol{I} = [\boldsymbol{a}, \boldsymbol{b}]$ is a closed subinterval of S, where $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$, then an additive interval set function can be constructed as follows

$$\alpha_{\boldsymbol{\xi}}(\boldsymbol{I}) = \sum_{c} (-1)^{v(c)} \mathrm{Cr}\{\xi_1 \leq c_1, \xi_2 \leq c_2, \ldots, \xi_n \leq c_n\}, \tag{1}$$

where $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ runs over the entire endpoints of the interval $[\boldsymbol{a}, \boldsymbol{b}]$ (i.e., for all points $\boldsymbol{c}$ such that $c_k = a_k$ or $c_k = b_k$ for $1 \leq k \leq n$), and $v(c)$ is the number of components a_k in the vector $\boldsymbol{c}$. In what follows, we call $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ as the additive interval set function constructed by the joint credibility distribution $\mathrm{Cr}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_n \leq x_n\}$.

A joint credibility distribution $\alpha_{\boldsymbol{\xi}}(\boldsymbol{x})$ is called *monotone metric function* if the constructed additive interval set function $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ is positive in the sense that $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I}) \geq 0$ for any subinterval $\boldsymbol{I} \subseteq S$. In the following, we give two cases under which the joint credibility distributions are monotone metric functions. We state them in the following two theorems.

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Theorem 1. *If a joint credibility distribution $\alpha_{\boldsymbol{\xi}}(\boldsymbol{x})$ is the product of its marginal credibility distributions $\alpha_{\xi_i}(x_i)$ for any $\boldsymbol{x} \in S$, then the following additive interval set function*

$$\alpha_{\boldsymbol{\xi}}^{p}(\boldsymbol{I}) = \sum_{c}(-1)^{v(c)} \prod_{i=1}^{n} \mathrm{Cr}\{\xi_i \le c_i\} \tag{2}$$

is a positive metric. As a result, the joint credibility distribution $\alpha_{\boldsymbol{\xi}}(\boldsymbol{x})$ is a monotone metric function.

Theorem 2. *If a joint credibility distribution $\alpha_{\boldsymbol{\xi}}(\boldsymbol{x})$ is the minimum of its marginal credibility distributions $\alpha_{\xi_i}(x_i)$ for any $\boldsymbol{x} \in S$, then the following additive interval set function*

$$\alpha_{\boldsymbol{\xi}}^{m}(\boldsymbol{I}) = \sum_{c}(-1)^{v(c)} \min_{i=1}^{n} \mathrm{Cr}\{\xi_i \le c_i\} \tag{3}$$

is a positive metric. As a result, the joint credibility distribution $\alpha_{\boldsymbol{\xi}}(\boldsymbol{x})$ is a monotone metric function.

In general situation, the additive interval set function $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ constructed by Eq.(1) is not necessary a positive metric. To deal with the general case, we write

$$\alpha_{\boldsymbol{\xi}}^{+}(\boldsymbol{I}) = \sup_{\boldsymbol{J} \subseteq \boldsymbol{I}} \alpha_{\boldsymbol{\xi}}(\boldsymbol{J}), \quad \alpha_{\boldsymbol{\xi}}^{-}(\boldsymbol{I}) = \sup_{\boldsymbol{J} \subseteq \boldsymbol{I}} \{-\alpha_{\boldsymbol{\xi}}(\boldsymbol{J})\}, \tag{4}$$

where $\boldsymbol{J} = \cup \boldsymbol{I}_v$ is the union of a finite number of separable closed intervals $\boldsymbol{I}_v$ such that $\alpha_{\boldsymbol{\xi}}(\boldsymbol{J}) = \sum \alpha_{\boldsymbol{\xi}}(\boldsymbol{I}_v)$. Since $\alpha_{\boldsymbol{\xi}}(\emptyset) = 0$, we know $\alpha_{\boldsymbol{\xi}}^{+}(\boldsymbol{I}) \ge 0$ and $\alpha_{\boldsymbol{\xi}}^{-}(\boldsymbol{I}) \ge 0$. The positive interval set functions $\alpha_{\boldsymbol{\xi}}^{+}(\boldsymbol{I})$ and $\alpha_{\boldsymbol{\xi}}^{-}(\boldsymbol{I})$ are called upper variation and lower variation of $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$, respectively. In addition, the following interval set function

$$|\alpha_{\boldsymbol{\xi}}|(\boldsymbol{I}) = \alpha_{\boldsymbol{\xi}}^{+}(\boldsymbol{I}) + \alpha_{\boldsymbol{\xi}}^{-}(\boldsymbol{I}) \tag{5}$$

is called the total variation or absolute variation of $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$. Note that $|\alpha_{\boldsymbol{\xi}}|(\boldsymbol{I}) < \infty$ for any closed subinterval of S. Therefore, the interval set function $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ has bounded variation over S. In this case, we have

$$\alpha_{\boldsymbol{\xi}}(\boldsymbol{I}) = \alpha_{\boldsymbol{\xi}}^{+}(\boldsymbol{I}) - \alpha_{\boldsymbol{\xi}}^{-}(\boldsymbol{I}). \tag{6}$$

Theorem 3. *For any fuzzy vector $\boldsymbol{\xi}$ with joint credibility distribution $\alpha_{\boldsymbol{\xi}}(\boldsymbol{x})$ on S, the additive interval set function $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ constructed by Eq.(1) has bounded variation over S. Moreover, the upper variation $\alpha_{\boldsymbol{\xi}}^{+}(\boldsymbol{I})$, lower variation $\alpha_{\boldsymbol{\xi}}^{-}(\boldsymbol{I})$ and total variation $|\alpha_{\boldsymbol{\xi}}|(\boldsymbol{I})$ of $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ are all positive metrics.*

Finally, we deal with the L-S measure generated by a positive metric $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$. Recall that a nonempty open set $G \subseteq S$ has a regular representation in the sense that G is the union of countably separable closed intervals, and each closed subset of G can be covered by a finite number of such closed intervals. If $\{\boldsymbol{I}_v\}$ is a regular representation of a nonempty open set G, then the L–S measure of G with respect to $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ is calculated by

$$\mu_{\alpha}(G) = \sum_{v} \alpha_{\boldsymbol{\xi}}(\boldsymbol{I}). \tag{7}$$

Furthermore, for any subset M of S, its outer L–S measure with respect to $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ is computed by

$$\mu_{\alpha}^{*}(M) = \inf_{G \supseteq M} \mu_{\alpha}(G), \tag{8}$$

where inf is taken for all open sets G such that $M \subseteq G \subseteq S$. As a consequence, a set $M \subseteq S$ is called measurable with respect to $\alpha_{\boldsymbol{\xi}}(\boldsymbol{I})$ iff

$$\inf_{G \supseteq M} \mu_{\alpha}^{*}(G - M) = 0. \tag{9}$$

If M is measurable, then its L–S measure is usually denoted by $\mu_{\alpha}(M)$ or $\alpha(M)$ for convenience.

References

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