Conservation of Energy Implies Conservation of Momentum: How We can Explain Conservation of Momentum to Before-Calculus Students

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Abstract

In solving physics problems, it is often important to use the laws of conservation of energy and momentum. While most people have intuitive understanding of energy and of its conservation, there is usually no intuition behind momentum, and known textbook derivations of conservation of momentum use calculus – which is usually taught after momentum. In this paper, we show how the law of conservation of momentum can be explained to before-calculus student: by using the fact that this law can be derived from the more intuitive conservation of energy if we consider energy in different coordinate systems. ©2014 World Academic Press, UK. All rights reserved.

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1 Conservation of Momentum: A "Stepchild" of Physics Education

It is important to learn conservation laws, especially conservation on energy and momentum. Conservation laws are very important in physics; see, e.g., [1]. They help in solving problems, starting with the simple problems about collisions.

There are three major conservation laws in mechanics: conservation of energy, conservation of momentum, and conservation of angular momentum. The first two conservation laws – of energy and momentum – are the most useful in the analyzing collisions of point bodies; conservation of angular momentum is important when we analyze more complex problems which involve rotating bodies.

Conservation of energy: brief reminder. Energy E is usually defined as the sum of kinetic energy $\sum_A m_A \cdot (\vec{v}_A)^2/2$ and all other types of energy, such as potential energy U and thermal energy Q:

$$E = \frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A)^2 + U + Q. \tag{1}$$

Conservation of energy means that if we consider the same (closed) system at two different moments of time $t < s$, then the total amount of energy is the same at both moments of time: $E(t) = E(s)$, i.e.,

$$\frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A(t))^2 + U(t) + Q(t) = \frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A(s))^2 + U(s) + Q(s). \tag{2}$$

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Comment. From the microscopic viewpoint, thermal energy is nothing else but kinetic energy of molecules, but from the macroscopic viewpoint, it is convenient to consider it separately.

Conservation of momentum: brief reminder. Momentum is defined as

$$\vec{P} \stackrel{\text{def}}{=} \sum_A m_a \cdot \vec{v}_A. \tag{3}$$

Conservation of momentum means that if we consider the same (closed) system at two different moments of time $t < s$, then the total amount of momentum is the same at both moments of time: $\vec{P}(t) = \vec{P}(s)$, i.e.,

$$\sum_A m_a \cdot \vec{v}_A(t) = \sum_A m_a \cdot \vec{v}_A(s). \tag{4}$$

Momentum is not as intuitive as energy. Most people have an intuitive understanding of energy and of energy conservation. In contrast, there is usually no good intuition behind the notion of momentum. To enhance the students' use of conservation of momentum, it is therefore desirable to provide a convincing explanation for the notion of momentum and for its conservation.

Usual derivations of the law of conservation of momentum use calculus. The usual textbook derivations of the law of conservation of momentum are based on the Third Newton's law: when one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction to that of the first body. By using the notation $\vec{F}_{AB}$ for the force exerted on the A-th body by the B-th body, we can describe this law as saying that

$$\vec{F}_{BA} = -\vec{F}_{AB}. \tag{5}$$

The total force $\vec{F}_A$ acting on the A-th body is equal to the sum of the forces caused by all the other bodies:

$$\vec{F}_A = \sum_B \vec{F}_{AB}. \tag{6}$$

According to the Second Newton's law, this force is equal to the mass m_A times acceleration $\vec{a}_A = \dfrac{d\vec{v}_A}{dt}$:

$$m_A \cdot \frac{d\vec{v}_A}{dt} = \vec{F}_A. \tag{7}$$

Thus, for the momentum $\vec{P}$, its time derivative is equal to

$$\frac{d\vec{P}}{dt} = \sum_A m_A \cdot \frac{\vec{v}_A}{dt} = \sum_A \vec{F}_A = \sum_A \sum_B \vec{F}_{AB}. \tag{8}$$

In this sum, for each pair (A, B), terms $\vec{F}_{AB}$ and $\vec{F}_{BA}$ cancel each other, and thus, we conclude that $\dfrac{d\vec{P}}{dt} = 0$, i.e., that the total momentum $\vec{P}$ does not change with time.

How can we explain conservation of momentum to students who did not have calculus yet?
Conservation of momentum is often taught to high school students *before* calculus. It is therefore desirable to come up with an alternative explanation for this conservation, an explanation accessible for before-calculus students. Such an explanation is provided in this paper. Specifically, we show that conservation of momentum can be derived from the conservation of energy.

2 Conservation of Momentum can be Derived from the Conservation of Energy

We can consider an alternative frame of reference associated with a moving body. The measured values of velocities depend on our choice of the frame of reference. From our viewpoint, of people standing on Earth, Earth is immobile, and velocities are measured relative to the Earth. From the viewpoint of astronauts on the Moon, Moon is immobile, while the Earth is moving with respect to their Lunar location.

It is well known that motion is *relative*, in the sense that physical laws do not change if we switch to a frame of reference based on a body which is moving in constant direction with a constant speed $\vec{u}$. This *principle of relativity* is not mostly associated with Einstein's Relativity Theory, but in reality, this principle has been explicitly formulated already by Galileo Galilei in 1632 [2], several decades prior to 1687 Newton's laws [3]. Galileo observed that if we are inside a smoothly moving ship, without a possibility to look outside, then no matter what experiments we perform, we will not be able to tell whether the ship is moving or standing still.

Let us consider energy conservation in a moving frame. Since all the laws of physics are valid in a moving frame of reference, in particular, the law of conservation of energy should also hold in the moving frame, i.e., we should have $E'(t) = E'(s)$, where E' denotes the value of the energy in a moving frame. When we change to a moving frame, potential and thermal energy does not change, and masses do not change, the only thing that changes are velocities. So, in the new coordinate frame, energy conservation takes the form:

$$\frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}'_A(t))^2 + U(t) + Q(t) = \frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}'_A(s))^2 + U(s) + Q(s). \tag{9}$$

The fact that the new frame of reference is associated with a moving body means that this moving body, which has velocity $\vec{u}$ in the original frame, has velocity 0 in the new frame. In general, a body with velocity $\vec{v}_A$ in the original frame has velocity $\vec{v}'_A = \vec{v}_A - \vec{u}$ in the new frame. Substituting this expression into the formula (9), we get

$$\frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A(t) - \vec{u})^2 + U(t) + Q(t) = \frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A(s) - \vec{u})^2 + U(s) + Q(s). \tag{10}$$

Deriving conservation of momentum. In general, for any two vectors $\vec{a}$ and $\vec{b}$, we have

$$(\vec{a} - \vec{b})^2 = (\vec{a})^2 - 2\vec{a} \cdot \vec{b} + (\vec{b})^2, \tag{11}$$

where $\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i \cdot b_i$ denotes a scalar (dot) product of the two vectors. Thus, the equation (10) takes the form

$$\frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A(t))^2 - \sum_A m_A \cdot \vec{v}_A(t) \cdot \vec{u} + \frac{1}{2} \cdot \sum_A m_A \cdot (\vec{u})^2 + U(t) + Q(t)$$

$$= \frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A(s))^2 - \sum_A m_A \cdot \vec{v}_A(s) \cdot \vec{u} + \frac{1}{2} \cdot \sum_A m_A \cdot (\vec{u})^2 + U(s) + Q(s). \tag{12}$$

Cancelling the equal terms $\sum_A m_A \cdot (\vec{u})^2 / 2$ in both sides of (12), we get

$$\frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A(t))^2 - \sum_A m_A \cdot \vec{v}_A(t) \cdot \vec{u}(t) + U(t) + Q(t)$$

$$= \frac{1}{2} \cdot \sum_A m_A \cdot (\vec{v}_A(s))^2 - \sum_A m_A \cdot \vec{v}_A(s) \cdot \vec{u}_A(s) + U(s) + Q(s). \tag{13}$$

Subtracting this equality from the original energy conservation equality (2), we conclude that

$$\sum_A m_A \cdot \vec{v}_A(t) \cdot \vec{u} = \sum_A m_A \cdot \vec{v}_A(s) \cdot \vec{u}, \tag{14}$$

i.e., that

$$\vec{P}(t) \cdot \vec{u} = \vec{P}(s) \cdot \vec{u}, \tag{15}$$

where we denoted the sum $\sum_A m_A \cdot \vec{v}_A$ by $\vec{P}$. One can see that this is exactly what is usually known as momentum (3).

The equality (15) must hold for every possible vector $\vec{u}$. In particular, as $\vec{u}$, we can take a unit vector $\vec{e}_i$ along the i-th coordinate: $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$, and $\vec{e}_3 = (0, 0, 1)$. For these vectors $\vec{u}$, the formula (15) takes the form

$$P_i(t) = P_i(s). \tag{16}$$

Thus, the vectors $\vec{P}(t)$ and $\vec{P}(s)$ have the same coordinates and are, therefore, equal:

$$\vec{P}(t) = \vec{P}(s). \tag{17}$$

Conclusion. We started with the law of conservation of energy, and when we considered it in two different frames of references, we were able to derive the law of conservation of momentum (17). In contrast to the usual textbook derivations, this derivation does not use calculus, it only uses vectors – but vectors are needed anyway, since momentum itself is a vector. Thus, we get the desired derivation of the law of conservation of momentum for before-calculus students.

Acknowledgments

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