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Why Trapezoidal and Triangular Membership Functions Work So Well: Towards a Theoretical Explanation

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Abstract

In fuzzy logic, an imprecise ("fuzzy") property is described by its membership function $\mu(x)$, i.e., by a function which describes, for each real number x, to what degree this real number satisfies the desired property. In principle, membership functions can be of different shape, but in practice, trapezoidal and triangular membership functions are most frequently used. In this paper, we provide an interval-based theoretical explanation for this empirical fact.

Keywords: fuzzy logic, trapezoidal membership function, triangular membership function

1 Formulation of the Problem

Need for fuzzy logic and membership functions. In many application areas, expert knowledge is described in terms of imprecise ("fuzzy") words from a natural language such as "small" or "large". To describe such words in computer-understandable terms, we can use fuzzy logic techniques (see, e.g., [1, 2, 3]). In fuzzy logic, each natural-language word is described by a *membership function* $\mu(x)$, a function that assigns, to every number x, the degree $\mu(x) \in [0,1]$ to what this number satisfies the corresponding property (e.g., the degree to which the number x is small).

α-cuts: a convenient alternative way to describe membership functions. Each membership function can be equivalently described by a family of its *α-cuts* $S_\alpha \stackrel{\text{def}}{=} \{x : \mu(x) \geq \alpha\}$ corresponding to different degrees $\alpha \in [0,1]$. In particular, the α-cut S_1 corresponding to $\alpha = 1$ is the set of all the real numbers x for which we are absolutely *sure* that x satisfies this property.

For $\alpha = 0$, the above definition cannot be applied, since $\mu(x) \geq 0$ for all real numbers x. Hence, the α-cut S_0 corresponding to $\alpha = 0$ is usually defined as the closure of the set $\{x : \mu(x) > 0\}$. The α-cut S_0 corresponding to $\alpha = 0$ is the set of all the real numbers x for which it is *possible* that x satisfies this property. Indeed, for values outside this α-cut, we have $\mu(x) = 0$, which means that these values cannot satisfy the desired property.

Once we know all the α-cuts, we can uniquely determine $\mu(x)$ as $\mu(x) = \sup\{\alpha : x \in S_\alpha\}$.

In principle, α-cuts can have different forms, but usually, all the α-cuts are intervals, i.e., $S_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$ for some real numbers $\underline{x}(\alpha)$ and $\overline{x}(\alpha)$.

How membership functions are elicited. Each membership function describes expert opinion; thus, to get appropriate numerical values, we need to elicit these values from the experts.

To fully describe a function $\mu(x)$, we need to know its values for all real numbers x. However, there are infinitely many real numbers, and we can only ask finitely many questions to each expert. Thus, we elicit some values and then interpolate. This elicitation-and-interpolation in often performed in terms of α-cuts. For example, we elicit the intervals $S_0 = \left[\underline{X}, \overline{X}\,\right]$ and $S_1 = [\underline{x}, \overline{x}]$, and then interpolate to arbitrary α.

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Empirical fact: trapezoidal membership functions usually work well. From the purely mathematical viewpoint, we can have many different shapes of membership functions. In most practical applications, however, simple "trapezoid" membership functions work well, for which we use linear interpolation to get both endpoints of the interval S_α: $S_\alpha = \left[\alpha \cdot \underline{x} + (1-\alpha)\cdot \underline{X}, \alpha \cdot \overline{x} + (1-\alpha)\cdot \overline{X}\,\right]$; see, e.g., [1, 2].

What we do in this paper. To the best of our knowledge, there has been no good theoretical explanation of why trapezoid functions work well. In this paper, we provide such an explanation.

2 Analysis of the Problem

What we need. For each intermediate degree α, we need to describe a reasonable interpolation function $f(\mathbf{x}, \mathbf{X})$ which takes two intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{X} = \left[\underline{X}, \overline{X}\,\right]$ for which $\mathbf{x} \subseteq \mathbf{X}$ and returns the corresponding interval S_α with the property that

$$\mathbf{x} \subseteq f(\mathbf{x}, \mathbf{X}) \subseteq \mathbf{X}. \tag{1}$$

Let us describe other natural properties of this function.

Scale-invariance. The numerical values x of the corresponding physical quantity depend on the choice of a measuring unit. For example, if we use centimeters instead of meters, then all numerical values get multiplied by $\lambda = 100$: e.g., instead of 2 m, we get 200 cm. In general, each numerical value x is multiplied by the ratio $\lambda > 0$ of the old measuring unit to the new one, i.e., each value x replaced by the new value $x' = \lambda \cdot x$.

Similarly, if in the original units, a range was described by the interval $\mathbf{x} = [\underline{x}, \overline{x}]$, then in the new units, it is described by the correspondingly re-scaled interval $\lambda \cdot \mathbf{x} \stackrel{\text{def}}{=} \{\lambda \cdot x : x \in [\underline{x}, \overline{x}]\}$, i.e., by the interval

$$\lambda \cdot [\underline{x}, \overline{x}] = [\lambda \cdot \underline{x}, \lambda \cdot \overline{x}]\,. \tag{2}$$

It is reasonable to require that the transformation described by the interpolation function $f(\mathbf{x}, \mathbf{X})$ not change if we simply change the measuring unit. In other words, we require that if $S_\alpha = f(\mathbf{x}, \mathbf{X})$, then $\lambda \cdot S_\alpha = f(\lambda \cdot \mathbf{x}, \lambda \cdot \mathbf{X})$. Substituting $S_\alpha = f(\mathbf{x}, \mathbf{X})$ into this formula, we get the following description of scale-invariance:

$$f(\lambda \cdot \mathbf{x}, \lambda \cdot \mathbf{X}) = \lambda \cdot f(\mathbf{x}, \mathbf{X}). \tag{3}$$

Shift-invariance. For many quantities (e.g., for time), the numerical value x also depends on the starting point. For example, if we start measuring time not from year 0, but from the year 1789 (as the French revolution did), then each year x becomes a new year $x' = x - 1789$. In general, we have $x' = x + x_0$ for some constant shift x_0.

If in the original units, a range was described by the interval $\mathbf{x} = [\underline{x}, \overline{x}]$, then in the new units, it is described by the correspondingly shifted interval $\mathbf{x} + x_0 \stackrel{\text{def}}{=} \{x + x_0 : x \in [\underline{x}, \overline{x}]\}$, i.e., by the interval

$$\mathbf{x} + x_0 = [\underline{x} + x_0, \overline{x} + x_0]\,. \tag{4}$$

It is reasonable to require that the transformation described by the interpolation function $f(\mathbf{x}, \mathbf{X})$ not change if we simply change the starting point. In other words, we require that if $S_\alpha = f(\mathbf{x}, \mathbf{X})$, then $S_\alpha + x_0 = f(\mathbf{x} + x_0, \mathbf{X} + x_0)$. Substituting $S_\alpha = f(\mathbf{x}, \mathbf{X})$ into this formula, we get the following description of shift-invariance:

$$f(\mathbf{x} + x_0, \mathbf{X} + x_0) = f(\mathbf{x}, \mathbf{X}) + x_0. \tag{5}$$

Sign-invariance. For some quantities, e.g., for the electric current, the sign is just a convention, we could easily call positive charge negative, etc. If we change the sign, then the original numerical value x is replaced by the new value $x' = -x$.

If in the original units, a range was described by the interval $\mathbf{x} = [\underline{x}, \overline{x}]$, then in the new units, it is described by the interval $-\mathbf{x} \stackrel{\text{def}}{=} \{-x : x \in [\underline{x}, \overline{x}]\}$, i.e., by the interval

$$-\mathbf{x} = [-\overline{x}, -\underline{x}]\,. \tag{6}$$

It is reasonable to require that the transformation described by the interpolation function $f(\mathbf{x},\mathbf{X})$ not change if we simply change the sign. In other words, we require that if $S_\alpha = f(\mathbf{x},\mathbf{X})$, then $-S_\alpha = f(-\mathbf{x},-\mathbf{X})$. Substituting $S_\alpha = f(\mathbf{x},\mathbf{X})$ into this formula, we get the following description of sign-invariance:

$$f(-\mathbf{x},-\mathbf{X}) = -f(\mathbf{x},\mathbf{X}). \tag{7}$$

Union-invariance. Our objective is to describe a word from natural language. Many such words combine several different meanings: for example, "small" may be subdivided into "very small", "somewhat small", etc. In this case, the interval $\mathbf{x}$ corresponding to the original term can be represented as a union of intervals $\mathbf{x}_\gamma$ corresponding to different meanings: $\mathbf{x} = \bigcup\limits_\gamma \mathbf{x}_\gamma$. Similarly, the interval $\mathbf{X}$ corresponding to the original term can be represented as a union of intervals $\mathbf{X}_\gamma$ corresponding to different meanings $\mathbf{X} = \bigcup\limits_\gamma \mathbf{X}_\gamma$. In such situation, we can interpolate in two different ways:

- take the unions $\mathbf{x} = \bigcup\limits_\gamma \mathbf{x}_\gamma$ and $\mathbf{X} = \bigcup\limits_\gamma \mathbf{X}_\gamma$ corresponding to the original terms, and then find the interpolation $f(\mathbf{x},\mathbf{X})$ based on these unions $\mathbf{x}$ and $\mathbf{X}$;
- alternatively, we can first perform the interpolation for each meaning, and then take the union $\bigcup\limits_\gamma f(\mathbf{x}_\gamma,\mathbf{X}_\gamma)$ of the resulting interpolations $f(\mathbf{x}_\gamma,\mathbf{X}_\gamma)$.

It is reasonable to require that these two methods lead to the same interpolation result:

$$f\left(\bigcup_\gamma \mathbf{x}_\gamma, \bigcup_\gamma \mathbf{X}_\gamma\right) = \bigcup_\gamma f(\mathbf{x}_\gamma,\mathbf{X}_\gamma). \tag{8}$$

Now, we can formulate our main result.

3 Definition and the Main Result

Definition 1. *For every interval $\mathbf{x} = [\underline{x},\overline{x}]$ and for all real numbers $\lambda > 0$ and x_0, we define:*

$$\lambda \cdot \mathbf{x} \stackrel{\text{def}}{=} \{\lambda \cdot x : x \in [\underline{x},\overline{x}]\}; \quad \mathbf{x} + x_0 \stackrel{\text{def}}{=} \{x + x_0 : x \in [\underline{x},\overline{x}]\}; \quad -\mathbf{x} \stackrel{\text{def}}{=} \{-x : x \in [\underline{x},\overline{x}]\}.$$

It is known that:

Proposition 1. $\lambda \cdot [\underline{x},\overline{x}] = [\lambda \cdot \underline{x}, \lambda \cdot \overline{x}]\,; \quad \mathbf{x} + x_0 = [\underline{x} + x_0, \overline{x} + x_0]\,; \quad -\mathbf{x} = [-\overline{x}, -\underline{x}]\,.$

Definition 2. *By an* interval interpolation function*, we mean a function $f(\mathbf{x},\mathbf{X})$ which is defined for every two intervals for which $\mathbf{x} \subseteq \mathbf{X}$ and for which $\mathbf{x} \subseteq f(\mathbf{x},\mathbf{X}) \subseteq \mathbf{X}$. We that an interval interpolation function is:*

- scale-invariant *if it satisfies the property $f(\lambda \cdot \mathbf{x}, \lambda \cdot \mathbf{X}) = \lambda \cdot f(\mathbf{x},\mathbf{X})$ for all intervals $\mathbf{x}$ and $\mathbf{X}$ and for all real numbers $\lambda > 0$;*
- shift-invariant *if it satisfies the property $f(\mathbf{x} + x_0, \mathbf{X} + x_0) = f(\mathbf{x},\mathbf{X}) + x_0$ for all intervals $\mathbf{x}$ and $\mathbf{X}$ and for all real numbers x_0;*
- sign-invariant *if it satisfies the property $f(-\mathbf{x},-\mathbf{X}) = -f(\mathbf{x},\mathbf{X})$ for all intervals $\mathbf{x}$ and $\mathbf{X}$; and*
- union-invariant *if whenever $\mathbf{x} = \bigcup\limits_\gamma \mathbf{x}_\gamma$ and $\mathbf{X} = \bigcup\limits_\gamma \mathbf{X}_\gamma$ for $\mathbf{x}_\gamma \subseteq \mathbf{X}_\gamma$, the property $f\left(\bigcup\limits_\gamma \mathbf{x}_\gamma, \bigcup\limits_\gamma \mathbf{X}_\gamma\right) = \bigcup\limits_\gamma f(\mathbf{x}_\gamma,\mathbf{X}_\gamma)$ is satisfied.*

Proposition 2. *Every scale-, shift-, sign-, and union-invariant interval interpolation function has the form*

$$f\left([\underline{x}, \overline{x}], [\underline{X}, \overline{X}]\right) = \left[\alpha \cdot \underline{x} + (1-\alpha) \cdot \underline{X}, \alpha \cdot \overline{x} + (1-\alpha) \cdot \overline{X}\right] \quad (9)$$

for some real number $\alpha \in [0, 1]$.

Discussion. In other words, the interval interpolation function has the same form as interpolation corresponding to the trapezoidal membership functions. Thus, this result provides a theoretical explanation of the practical success of trapezoidal membership functions.

Comments.

- Triangular membership functions are a particular case of the trapezoidal ones, corresponding to degenerate intervals $\mathbf{x} = [x, x]$.
- As one can easily check, the interval interpolation function (9) is indeed scale-, shift-, sign-, and union-invariant.
- Strictly speaking, the value α whose existence is proven in Proposition 2 does not have to coincide with the fuzzy level α; however, we can always re-scale the scale of the degree $[0, 1]$ by using the value α from the formula (9) as the new degree of certainty. After this re-scaling, we get exactly the trapezoidal membership function.

Proof of Proposition 2.

1°. Our objective is to describe the value $f\left([\underline{x}, \overline{x}], [\underline{X}, \overline{X}]\right)$ for arbitrary intervals $\mathbf{x} = [\underline{x}, \overline{x}] \subseteq [\underline{X}, \overline{X}] = \mathbf{X}$, i.e., for arbitrary numbers $\underline{x}$, $\overline{x}$, $\underline{X}$, and $\overline{X}$ for which

$$\underline{X} \le \underline{x} \le \overline{x} \le \overline{X}.$$

To find the desired value $f\left([\underline{x}, \overline{x}], [\underline{X}, \overline{X}]\right)$, let us first use union-invariance.

The interval $\mathbf{x} = [\underline{x}, \overline{x}]$ can be represented as the union of three intervals $\mathbf{x} = \mathbf{x}_1 \bigcup \mathbf{x}_2 \bigcup \mathbf{x}_3$ where $\mathbf{x}_1 = [\underline{x}, \underline{x}]$, $\mathbf{x}_2 = [\underline{x}, \overline{x}]$, and $\mathbf{x}_3 = [\overline{x}, \overline{x}]$. Correspondingly, the interval $\mathbf{X} = [\underline{X}, \overline{X}]$ can be represented as the union of three intervals $\mathbf{X} = \mathbf{X}_1 \bigcup \mathbf{X}_2 \bigcup \mathbf{X}_3$ where $\mathbf{X}_1 = [\underline{X}, \underline{x}]$, $\mathbf{X}_2 = [\underline{x}, \overline{x}]$, and $\mathbf{X}_3 = [\overline{x}, \overline{X}]$. Here, $\mathbf{x}_i \subseteq \mathbf{X}_i$ for all i and thus,

$$f(\mathbf{x}, \mathbf{X}) = f(\mathbf{x}_1, \mathbf{X}_1) \bigcup f(\mathbf{x}_2, \mathbf{X}_2) \bigcup f(\mathbf{x}_3, \mathbf{X}_3). \quad (10)$$

In the following text, we will compute these three interpolations results $f(\mathbf{x}_i, \mathbf{X}_i)$ one by one.

2°. Let us start with $f(\mathbf{x}_2, \mathbf{X}_2)$. Here, $\mathbf{x}_2 = \mathbf{X}_2 = [\underline{x}, \overline{x}]$. By definition of an interval interpolation function, we have $\mathbf{x}_2 \subseteq f(\mathbf{x}_2, \mathbf{X}_2) \subseteq \mathbf{X}_2 = \mathbf{x}_2$. Thus,

$$f(\mathbf{x}_2, \mathbf{X}_2) = \mathbf{x}_2 = [\underline{x}, \overline{x}]. \quad (11)$$

3°. Let us now compute $f(\mathbf{x}_3, \mathbf{X}_3) = f\left([\overline{x}, \overline{x}], [\overline{x}, \overline{X}]\right)$. Here, $\overline{x} \le \overline{X}$, i.e., either $\overline{x} = \overline{X}$, or $\overline{x} < \overline{X}$. Let us consider these two cases one by one.

3.1°. If $\overline{x} = \overline{X}$, then, similar to Part 2 of this proof, we get

$$f(\mathbf{x}_3, \mathbf{X}_3) = f\left([\overline{x}, \overline{x}], [\overline{x}, \overline{x}]\right) = [\overline{x}, \overline{x}]. \quad (12)$$

3.2°. If $\overline{x} < \overline{X}$, then, by applying shift-invariance with $x_0 = \overline{x}$, we conclude that

$$f(\mathbf{x}_3, \mathbf{X}_3) = f\left([\overline{x}, \overline{x}], [\overline{x}, \overline{X}]\right) = f\left([0, 0], [0, \overline{X} - \overline{x}]\right) + \overline{x}. \quad (13)$$

Now, by applying scale-invariance with $\lambda = \overline{X} - \overline{x} > 0$, we conclude that

$$f\left([0, 0], [0, \overline{X} - \overline{x}]\right) = \left(\overline{X} - \overline{x}\right) \cdot f([0, 0], [0, 1]). \quad (14)$$

By definition of the interval interpolation function, the interval $[\underline{y},\overline{y}] \stackrel{\text{def}}{=} f([0,0],[0,1])$ satisfies the property $[0,0] \subseteq f([0,0],[0,1]) = [\underline{y},\overline{y}] \subseteq [0,1]$. The first inclusion implies that $\underline{y} \le 0$, the second that $\underline{y} \ge 0$ and $\overline{y} \le 1$. Thus, we have $\underline{y} = 0$, and, by defining $\alpha \stackrel{\text{def}}{=} 1 - \overline{y}$, we conclude that

$$f([0,0],[0,1]) = [0, 1-\alpha] \tag{15}$$

for some $\alpha \in [0,1]$.

Substituting the formula (15) into the expression (14), we conclude that

$$f\left([0,0],\left[0,\overline{X}-\overline{x}\right]\right) = \left(\overline{X}-\overline{x}\right)\cdot[0,1-\alpha] = \left[0,(1-\alpha)\cdot\left(\overline{X}-\overline{x}\right)\right]. \tag{16}$$

Substituting expression (16) into the formula (13), we conclude that

$$f(\mathbf{x}_3,\mathbf{X}_3) = \left[0,(1-\alpha)\cdot\left(\overline{X}-\overline{x}\right)\right]+\overline{x} = \left[\overline{x},(1-\alpha)\cdot\left(\overline{X}-\overline{x}\right)+\overline{x}\right] = \left[\overline{x},\alpha\cdot\overline{x}+(1-\alpha)\cdot\overline{X}\right]. \tag{17}$$

3.3°. One can easily check that, when $\overline{x} = \overline{X}$, the right-hand side of the formula (17) is equal to the right-hand side $[\overline{x},\overline{x}]$ of the formula (12). Thus, the formula (17) holds for both possible cases: when $\overline{x} = \overline{X}$ and when $\overline{x} < \overline{X}$.

4°. Finally, let us now compute $f(\mathbf{x}_1,\mathbf{X}_1) = f([\underline{x},\underline{x}],[\underline{X},\underline{x}])$. Here, $\underline{X} \le \underline{x}$, i.e., either $\underline{X} = \underline{x}$, or $\underline{X} < \underline{x}$. Let us consider these two cases one by one.

4.1°. If $\underline{X} = \underline{x}$, then, similar to Part 2 of this proof, we get

$$f(\mathbf{x}_1,\mathbf{X}_1) = f\left([\underline{x},\underline{x}],[\underline{x},\underline{x}]\right) = [\underline{x},\underline{x}]. \tag{18}$$

4.2°. If $\underline{X} < \underline{x}$, then, by applying shift-invariance with $x_0 = \underline{x}$, we conclude that

$$f(\mathbf{x}_1,\mathbf{X}_1) = f\left([\underline{x},\underline{x}],[\underline{X},\underline{x}]\right) = f\left([0,0],[-(\underline{x}-\underline{X}),0]\right)+\underline{x}. \tag{19}$$

Now, by applying scale-invariance with $\lambda = \underline{x} - \underline{X} > 0$, we conclude that

$$f\left([0,0],[-(\underline{x}-\underline{X}),0]\right) = (\underline{x}-\underline{X})\cdot f([0,0],[-1,0]). \tag{20}$$

Due to sign-invariance, we get

$$f([0,0],[-1,0]) = -f([0,0],[0,1]) = -[0,1-\alpha] = [-(1-\alpha),0]. \tag{21}$$

Substituting the formula (21) into the expression (20), we conclude that

$$f\left([0,0],[-(\underline{x}-\underline{X}),0]\right) = (\underline{x}-\underline{X})\cdot[-(1-\alpha),0] = [-(1-\alpha)\cdot(\underline{x}-\underline{X}),0]. \tag{22}$$

Substituting expression (22) into the formula (19), we conclude that

$$f(\mathbf{x}_1,\mathbf{X}_1) = [-(1-\alpha)\cdot(\underline{x}-\underline{X}),0]+\underline{x} = [\underline{x}-(1-\alpha)\cdot(\underline{x}-\underline{X}),\underline{x}] = [\alpha\cdot\underline{x}+(1-\alpha)\cdot\underline{X},\underline{x}]. \tag{23}$$

4.3°. One can easily check that, when $\underline{X} = \underline{x}$, the right-hand side of the formula (23) is equal to the right-hand side $[\underline{x},\underline{x}]$ of the formula (18). Thus, the formula (23) holds for both possible cases: when $\underline{X} = \underline{x}$ and when $\underline{X} < \underline{x}$.

5°. Now, we have the expressions (23), (11), and (17) for $f(\mathbf{x}_1,\mathbf{X}_1)$, $f(\mathbf{x}_2,\mathbf{X}_2)$, and $f(\mathbf{x}_3,\mathbf{X}_3)$. Substituting these expressions into the formula (10), we get the desired formula (9). The proposition is proven.

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