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Approximate Solution of High Order Fuzzy Initial Value Problems

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Abstract

In this paper, the Homotopy Analysis Method (HAM) is used to solve high order (≥ 2) linear and non linear fuzzy initial value problems involving ordinary differential equations. This method allows for the solution of the differential equation to be calculated in the form of an infinite series in which the components can be easily calculated. The HAM utilizes a simple method to adjust and control the convergence region of the infinite series solution by using the convergence-control parameter. Two examples are solved to illustrate the capability of the HAM.

Keywords: fuzzy number, fuzzy differential equation, homotopy analysis method, convergence-control parameter

1 Introduction

Fuzzy ordinary differential equations are suitable mathematical models to model dynamical systems in which there exist uncertainties or vagueness. These models are used in various applications including population models [4], quantum optics gravity [16], and medicine [3, 9]. In recent years approximate analytical methods such as Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) have been used to solve fuzzy problems involving ordinary differential equations. In [18], the HPM was used to solve first order linear fuzzy initial value problems involving ordinary differential equations. The ADM was employed in [5, 8] to solve first order linear and non linear fuzzy initial value problems. Abbasbandy et al. [2] used the VIM to solve linear system of first order fuzzy initial value problems. The use of Homotopy Analysis Method (HAM) to solve first order fuzzy initial value problem is quite straightforward and will not be carried out in this paper. In this paper, our aim is to apply the HAM to solve linear and nonlinear higher order (≥ 2) fuzzy initial value problems directly without reducing to first order system. Furthermore, we also aim to study the convergence of this method for the approximate solution of fuzzy linear and nonlinear high order initial value problems. The HAM was proposed by Liao[26] in his PhD thesis and it is powerful technique to solve both linear and nonlinear problems.

In recent years, many researchers have used HAM to solve various types of linear and nonlinear problems in science and engineering [10, 11, 12, 13, 14, 19, 27, 28, 29, 30]. According to [19], approximate analytical method (of which HAM is an example) has an advantage over perturbation methods in that it is not dependent on small or large parameters. Perturbation methods are based on the existence of small or large parameters, and they cannot be applied to all nonlinear equations. According to [32], both perturbation techniques and non-perturbative methods cannot provide a simple procedure to adjust or control the convergence region and rate of given approximate series. One of advantages of the HAM is that this method allows for fine-tuning of the convergence region and rate of convergence by allowing the convergence control-parameter $\hbar$ to vary [6, 10]. It is to be noted that proper choice of the initial condition, the auxiliary linear operator $\mathcal{L}$, and convergence-control with parameter $\hbar$ will guarantee the convergence of the HAM solution series [7]. Homotopy analysis series solution will be convergent by considering two factors: the auxiliary linear operator and initial guess [25]. To the best of our knowledge, this is the first attempt at solving the high order FIVP directly without reducing to a first order system the HAM. This is the main novelty of our paper. Clearly also the solution of a high order FIVP without reduction to a first order system has an advantage in terms of computational savings and this is an additional motivation. The structure of this paper is as follows. In Section 2, some basic definitions and notations are given which will be used in other sections. In Section 3, a description of HAM is given as expanded by previous researchers in particular [10, 11, 12, 19]. In Section 4, structure of HAM is formulated for solving high order fuzzy initial value problems. In Section 5, we explain about the convergence of

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HAM for solving FIVP. In Section 6, we present two numerical examples and finally, in Section 7, we give the conclusion of this study.

2 Preliminaries

Definition 1 [15, 23] An arbitrary fuzzy number is represented by an ordered pair of functions $\mu(t)=(\underline{\mu}(t),\overline{\mu}(t))$ for all $r\in[0,1]$ which satisfies:
(i) $\mu(t)$ is normal, i.e., $\exists t_0\in\mathbb{R}$ with $\mu(t_0)=1$.
(ii) $\mu(t)$ is convex fuzzy set, i.e., $\mu(\lambda t+(1-\lambda)s)\geq\min\{\mu(t),\mu(s)\},\forall t,s\in\mathbb{R},\lambda\in[0,1]$.
(iii) $\mu(t)$ is a bounded left continuous non-decreasing function over [0,1].
(iv) $\mu(t)$ is a bounded left continuous non-increasing function over [0,1], $\underline{\mu}(t)\leq\overline{\mu}(t)$.

Let E be the set of all upper semi-continuous normal convex fuzzy numbers with r-level bounded intervals such that:

$$[\mu]_r=\{t\in\mathbb{R}:\mu\geq r\}.$$

Definition 2 [17] A mapping $f:T\rightarrow E$ for some interval $T\subseteq E$ is called a fuzzy process, and we denote r-level set by:

$$[\tilde{f}(t)]_r=\left[\underline{f}(t;r),\overline{f}(t;r)\right],t\in T,r\in[0,1].$$

The r-level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r-level sets based on the resolution identity theorem [36].

Definition 3 [33] The fuzzy integral of fuzzy process $\tilde{f}(t;r)$, $\int_a^b\tilde{f}(t;r)dt$ for $a,b\in T$ and $r\in[0,1]$, is defined by:

$$\int_a^b\tilde{f}(t;r)dt=\left[\int_a^b\underline{f}(t;r)dt,\int_a^b\overline{f}(t;r)dt\right].$$

Definition 4 [35] Each function $f:X\rightarrow Y$ induces another function $\tilde{f}:F(X)\rightarrow F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y)=\begin{cases}Sup_{x\in f^{-1}(y)}U(x), if\ y\in range\ (f)\\ 0 \qquad\qquad\qquad\quad , if\ y\in range\ (f).\end{cases}$$

This is called the Zadeh extension principle.

3 General Structure of HAM for Solving IVP

To describe of the HAM, we consider the following differential equation:

$$\mathcal{N}[y(t)]=0 \tag{3.1}$$

where $\mathcal{N}$ is a nonlinear operator, t is an independent variable and, $y(t)$ is an unknown function. Liao [26] constructs the zero-order deformation equation:

$$\mathcal{H}(t;p)=(1-p)\mathcal{L}[\emptyset(t;p)-y_0(t)]-p\hbar H(t)\mathcal{N}[\emptyset(t;p)] \tag{3.2}$$

where $p\in[0,1]$ is an embedding parameter, $\hbar\neq0$ is a convergence-control parameter. Function $H(t)\neq0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $y_0(t)$ is an initial guess of $y(t)$ and $\emptyset(t;p)$ the auxiliary function that should be satisfied in the initial conditions function. It should be pointed out that the convergence-control parameter $\hbar$ and auxiliary function $H(t)$ play important role to adjust and control the convergence of the series solution.

Clearly when $p=0$,

$$\mathcal{H}(t;p)=\mathcal{L}[\emptyset(t;0)-y_0(t)]=0, \tag{3.3}$$

and $p=1$, it can be concluded that

$$\mathcal{H}(t;1)=p\hbar H(t)\mathcal{N}[\emptyset(t;1)]=0. \tag{3.4}$$

Thus by requiring [19],

$$\mathcal{H}(t;p)=0, \tag{3.5}$$

we can obtain

$$(1-p)\mathcal{L}[\emptyset(t;p)-y_0(t)]=p\hbar H(t)\mathcal{N}[\emptyset(t;p)]. \tag{3.6}$$

If $p=0$ or $p=1$, the homotopy equations vary to [19]:

$$\emptyset(t;0)=y_0(t),\ \emptyset(t;1)=y(t). \tag{3.7}$$

As p increases from 0 to 1 the solution $\emptyset(t;p)$ varies from the initial guess $y_0(t)$ to the solution $y(t)$. By expanding $\emptyset(t;p)$ as a Taylor series with respect to p, we can obtain

$$\emptyset(t;p) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)\, p^m \tag{3.8}$$

where

$$y_m = \frac{1}{m!}\frac{\partial \emptyset(t;p)}{\partial p^m}\bigg|_{p=0}. \tag{3.9}$$

As we discussed, the auxiliary linear operator $\mathcal{L}$, the initial guesses $y_0(t)$, the convergence control parameter $\hbar$ and the auxiliary function $H(t)$ are very important for homotopy series solution. Note that if $p = 1$, then we have

$$\emptyset(t;1) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \tag{3.10}$$

which is must be one of the solutions of the original equation. It is to be noted that if $\hbar = -1$ and $H(t) = 1$ then Eq.(3.2) becomes:

$$(1-p)\mathcal{L}[\emptyset(t;p) - y_0(t)] + pH(t)\mathcal{N}[\emptyset(t;p)] = 0 \tag{3.11}$$

which is used in the HPM [18]. According to (3.9), the governing equations can be deduced from the zero-order deformation equations (3.3). We define the vectors

$$\vec{y}_i = \{y_0(t)y_1(t), \ldots, y_m(t)\}. \tag{3.12}$$

Differentiating (3.6) m times with respect to p and then setting $p = 0$ and after that dividing them by $m!$, we obtain the m^{th}-order deformation equation

$$\mathcal{L}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar \mathcal{R}_m(\vec{y}_{m-1}), \tag{3.13}$$

where

$$\mathcal{R}_m(\vec{y}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\emptyset(t;p)]}{\partial p^{m-1}}\bigg|_{p=0},$$

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 0. \end{cases} \tag{3.14}$$

Details of the convergence analysis of the HAM are discussed in [26, 31].

4 Fuzzification and Defuzzification of HAM for High Order FIVP

In Section 3 we gave details and structure of HAM for the approximate solution of high order FIVP. The HAM is applied to approximately solve high order linear and nonlinear FIVP. Toward this end, consider the following n^{th}-order FIVP subject to the initial condition:

$$\tilde{y}^{(n)}(t) = f\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \ldots, \tilde{y}^{(n-1)}(t)\right) + \widetilde{w}(t), \quad t \in [t_0, T] \tag{4.1}$$

subject to the initial conditions

$$\tilde{y}(t_0) = \tilde{y}_0, \tilde{y}'(t_0) = \tilde{y}_0{}', \ldots, \tilde{y}^{(n-1)}(t_0) = \tilde{y}_0{}^{(n-1)}. \tag{4.2}$$

According to Section 2, here $\tilde{y}$ is a fuzzy function of the crisp variable t, $f(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \ldots, \tilde{y}^{(n-1)}(t))$, is fuzzy function of the crisp variable t and the fuzzy variable y, $y^{(n)}$ is the fuzzy derivative of the $\tilde{y}(t), \tilde{y}'(t)$, $\tilde{y}''(t), \ldots, \tilde{y}^{(n-1)}(t)$ and $\tilde{y}(t_0), \tilde{y}'(t_0), \ldots, \tilde{y}^{(n-1)}(t_0)$ are convex fuzzy numbers. We denote the fuzzy function y by $\tilde{y} = [\underline{y}, \overline{y}]$, for $t \in [t_0, T]$ and $r \in [0,1]$, it means that the r-level set of $y(t)$ can be defined as:

$$[\tilde{y}(t)]_r = \left[\underline{y}(t;r), \overline{y}(t;r)\right],\ [\tilde{y}'(t)]_r = \left[\underline{y}'(t;r), \overline{y}'(t_0;r)\right], \ldots, \left[\tilde{y}^{(n-1)}(t)\right]_r = \left[\underline{y}^{(n-1)}(t;r), \overline{y}^{(n-1)}(t;r)\right],$$

$$[\tilde{y}(t_0)]_r = \left[\underline{y}(t_0;r), \overline{y}(t_0;r)\right], [\tilde{y}'(t_0)]_r = \left[\underline{y}'(t_0;r), \overline{y}'(t_0;r)\right], \ldots, \left[\tilde{y}^{(n-1)}(t_0)\right]_r = \left[\underline{y}^{(n-1)}(t_0;r), \overline{y}^{(n-1)}(t_0;r)\right],$$

where $\widetilde{w}(t)$ the is crisp or fuzzy inhomogeneous term such that

$$[\widetilde{w}(t)]_r = \left[\underline{w}(t;r), \overline{w}(t;r)\right].$$

Since $y^{(n)}(t) = f\left(t, y(t), y'(t), y''(t), \ldots, y^{(n-1)}(t)\right) + w(t)$. Let $\mathcal{Y}(t) = y(t), y'(t), y''(t), \ldots, y^{(n-1)}(t)$ such that

$$\widetilde{\mathcal{Y}}(t;r) = \left[\underline{\mathcal{Y}}(t;r), \overline{\mathcal{Y}}(t;r)\right] = \left[\underline{y}(t;r), \underline{y}'(t;r), \ldots, \underline{y}^{(n-1)}(t;r), \overline{y}(t;r), \overline{y}'(t;r), \ldots, \overline{y}^{(n-1)}(t;r)\right].$$

According to HAM described in Section 3, Eq.(4.1) can be written as follows:

$$y^{(n)}(t) = f\big(t, \mathcal{Y}(t)\big) + w(t), \tag{4.3}$$

then we can write the following equations

$$\begin{cases} \underline{y}^{(n)}(t;r) = \mathcal{N}\left[\underline{y}(t;r)\right] \\ \overline{y}^{(n)}(t;r) = \mathcal{N}\left[\overline{y}(t;r)\right]. \end{cases} \tag{4.4}$$

Also we can write Eq.(4.5) as follows

$$\begin{cases} \mathcal{N}\left[\underline{y}(t;r)\right] = 0 \\ \mathcal{N}\left[\overline{y}(t;r)\right] = 0, \end{cases} \tag{4.5}$$

$$\mathcal{F}\left(\left[\tilde{\emptyset}(t;p)\right]\right) = min\{\, \mathcal{N}[\underline{\emptyset}(t;p)]_r, \mathcal{N}[\overline{\emptyset}(t;p)]_r \colon \mu|\mu \in [\tilde{y}(t;r)]\},$$

$$\mathcal{G}\left(\left[\tilde{\emptyset}(t;p)\right]\right) = max\{\, \mathcal{N}[\underline{\emptyset}(t;p)]_r, \mathcal{N}[\overline{\emptyset}(t;p)]_r \colon \mu|\mu \in [\tilde{y}(t;r)]\},$$

where μ represent the membership function of Eq.(4.1). The zero-order deformation equation of Eq.(4.1) can be written as:

$$\begin{cases} (1-p)\underline{\mathcal{L}}_n\left[\underline{\emptyset}(t;p) - \underline{y}_0(t;r)\right] = p\hbar H(t)\, \mathcal{F}\left(\left[\tilde{\emptyset}(t;p)\right]\right) \\ (1-p)\overline{\mathcal{L}}_n\left[\overline{\emptyset}(t;p) - \overline{y}_0(t;r)\right] = p\hbar H(t)\mathcal{G}\left(\left[\tilde{\emptyset}(t;p)\right]\right). \end{cases} \tag{4.6}$$

As we discussed in Section 3, $p \in [0,1]$ is an embedding parameter, and $\hbar$ is nonzero convergence-control parameter. The $H(t)$ is an auxiliary functions while the operators

$$\underline{\mathcal{L}}_n = \frac{\partial^n\left[\underline{\emptyset}(t;p)\right]_r}{\partial t^n}$$

and

$$\overline{\mathcal{L}}_n = \frac{\partial^n\overline{\emptyset}[(t;p)]_r}{\partial t^n}$$

are auxiliary linear operators.

We can define the initial approximation $[\tilde{y}_0(t)]_r = \left[\underline{y}_0(t;r), \overline{y}_0(t;r)\right]$ of $\underline{y}(t;r), \overline{y}(t;r)$ as follows:

$$\underline{y}_0(t;r) = \underline{C}_1(r) + \underline{C}_2(r)t + \underline{C}_3(r)t^2 + \cdots + \underline{C}_n(r)t^{(n-1)}, \tag{4.7}$$

$$\overline{y}_0(t;r) = \overline{C}_1(r) + \overline{C}_2(r)t + \overline{C}_3(r)t^2 + \cdots + \overline{C}_n(r)t^{(n-1)}, \tag{4.8}$$

where for all $r \in [0,1]$, $\tilde{C}_1(r), \tilde{C}_2(r), \ldots$ are the constants can be determined easily from the initial conditions (5.2). When $p = 0$ and $p = 1$ it can be concluded that

$$\begin{cases} \left[\underline{\emptyset}(t;0)\right]_r = \underline{y}_0(t;r) \\ \left[\overline{\emptyset}(t;0)\right]_r = \overline{y}_0(t;r), \end{cases} \qquad \begin{cases} \left[\underline{\emptyset}(t;1)\right]_r = \underline{y}(t;r) \\ \left[\overline{\emptyset}(t;1)\right]_r = \overline{y}(t;r). \end{cases} \tag{4.9}$$

Hence, as p increases from 0 to 1, the solution $\left[\underline{\emptyset}(t;p)\right]_r$, $\left[\overline{\emptyset}(t;p)\right]_r$ varies from the initial guess $\underline{y}_0(t;r), \overline{y}_0(t;r)$ to the solution $\underline{y}(t;r), \overline{y}(t;r)$. By expanding $\left[\underline{\emptyset}(t;p)\right]_r$ and $\left[\overline{\emptyset}(t;p)\right]_r$ as a Taylor series with respect to p, we can obtain

$$\begin{cases} \left[\underline{\emptyset}(t;p)\right]_r = \underline{y}_0(t;r) + \sum_{m=1}^{\infty} \underline{y}_m(t;r)\, p^m \\ \left[\overline{\emptyset}(t;p)\right]_r = \overline{y}_0(t;r) + \sum_{m=1}^{\infty} \overline{y}_m(t;r)\, p^m, \end{cases} \tag{4.10}$$

where

$$\begin{cases} \underline{y}_m(t;r) = \frac{1}{m!}\frac{\partial\left[\underline{\emptyset}(t;p)\right]_r}{\partial p^m}\Big|_{p=0} \\ \overline{y}_m(t;r) = \frac{1}{m!}\frac{\left[\overline{\emptyset}(t;p)\right]_r}{\partial p^m}\Big|_{p=0}. \end{cases} \tag{4.11}$$

If auxiliary linear operator $\tilde{\mathcal{L}}_n$, the initial guesses $\underline{y}_0(t;r), \overline{y}_0(t;r)$, the convergence control parameter $\hbar$, and the auxiliary function $H(t)$, are properly chosen, then the series (4.10) converges to the exact solution at $p = 1$ and we have:

$$\begin{cases} \left[\underline{\emptyset}(t;1)\right]_r = \underline{y}_0(t;r) + \sum_{m=1}^{\infty} \underline{y}_m(t;r) \\ \left[\overline{\emptyset}(t;1)\right]_r = \overline{y}_0(t;r) + \sum_{m=1}^{\infty} \overline{y}_m(t;r). \end{cases} \tag{4.12}$$

The governing equations can be deduced from the zero-order deformation equations (4.6). We define the vectors

$$\begin{cases} \underline{\vec{y}}_i(t;r) = \{\underline{y}_0(t;r), \underline{y}_1(t;r), \dots, \underline{y}_m(t;r)\} \\ \overline{\vec{y}}_i(t;r) = \{\overline{y}_0(t;r), \overline{y}_1(t;r), \dots, \overline{y}_m(t;r)\}. \end{cases} \tag{4.13}$$

Now ddifferentiating (4.6) m times with respect to the embedding parameter p and then setting $p = 0$ and dividing them by $m!$, we have the m^{th}-order deformation equation

$$\begin{cases} \underline{\mathcal{L}}_n \left[\underline{y}_m(t;r) - \chi_m \underline{y}_{m-1}(t;r)\right] = \hbar \mathcal{R}_m(\underline{\vec{y}}_{m-1}(t;r)) \\ \overline{\mathcal{L}}_n \left[\overline{y}_m(t;r) - \chi_m \overline{y}_{m-1}(t;r)\right] = \hbar \mathcal{R}_m\left(\overline{\vec{y}}_{m-1}(t;r)\right), \end{cases} \tag{4.14}$$

where

$$\begin{cases} \mathcal{R}_m\left(\underline{\vec{y}}_{m-1}(t;r)\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\mathcal{F}([\tilde{\phi}(t;p)])}{\partial p^{m-1}}\Big|_{p=0} \\ \mathcal{R}_m\left(\overline{\vec{y}}_{m-1}(t;r)\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\mathcal{G}([\tilde{\phi}(t;p)])}{\partial p^{m-1}}\Big|_{p=0}. \end{cases} \tag{4.15}$$

The solution of the m^{th}-order deformation for $m \geq 1$ is:

$$\begin{cases} \underline{y}_m(t;r) = \chi_m \underline{y}_{m-1}(t;r) + \hbar \underline{\mathcal{L}}_n^{-1} \mathcal{R}_m(\underline{\vec{y}}_{m-1}(t;r)) \\ \overline{y}_m(t;r) = \chi_m \overline{y}_{m-1}(t;r) + \hbar \overline{\mathcal{L}}_n^{-1} \mathcal{R}_m\left(\overline{\vec{y}}_{m-1}(t;r)\right), \end{cases} \tag{4.16}$$

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 0, \end{cases}$$

where $\underline{\mathcal{L}}_n^{-1} = \int\int \cdots \int \underline{[.]}_r \, dtdt \cdots dt$ and $\overline{\mathcal{L}}_n^{-1} = \int\int \cdots \int \overline{[.]}_r \, dtdt \cdots dt$ are the inverse fuzzy integral operators, $\forall r \in [0,1]$. The approximate solution series the generated from Eq.(4.16) of HAM is given by

$$\tilde{y}_m(t;r;\hbar) = \sum_{m=1}^{\infty} \tilde{y}_m(t;r;\hbar). \tag{4.17}$$

As we can see from Eq.(4.17) the solution series contain the convergence-control parameter $\hbar$ the accuracy of the HAM depend on the selected the right value of this parameter that will discuss later in the next section. Finally the exact solution of Eq.(4.1) after selected the value of $\hbar$ may be obtained by

$$\tilde{Y}(t;r) = \lim_{m \to 1} \sum_{m=1}^{\infty} \tilde{y}_m(t;r). \tag{4.18}$$

Finally the absolute error $[\tilde{E}]_r = [\underline{E}, \overline{E}]_r$ of HAM for the solution of Eq.(4.1) is given by

$$\tilde{E}(t,\hbar;r) = \left|\tilde{y}_m(t;r;\hbar) - \tilde{Y}(t;r)\right|. \tag{4.19}$$

For the values of the convergence-control parameter, Liao in [26] suggested to investigate the convergence of some special quantities.

5 Convergence of HAM for Solving n^{th}-Order FIVP

The convergence of the approximate solution of Eq.(4.1) depends on the convergence-control parameter $\hbar$. There are several ways to determine the value of $\hbar$ to give as much accuracy for a certain order of HAM approximate series solution as possible. One way is to define the square residual error [25]. If we substitute (4.13) in the original Eq.(4.1) we have

$$\begin{cases} \underline{re}(t;r;\hbar) = \underline{y}_m^{(n)}(t;r;\hbar) - \mathcal{F}\left(t; \tilde{\mathcal{Y}}_m(t;r;\hbar)\right) - \underline{w}(t;r) \\ \overline{re}(t;r;\hbar) = \overline{y}_m^{(n)}(t;r;\hbar) - \mathcal{G}\left(t; \tilde{\mathcal{Y}}_m(t;r;\hbar)\right) - \overline{w}(t;r). \end{cases} \tag{5.1}$$

We check that the corresponding squared residual fuzzy integrated in the whole region $t \in [t_0, T]$ such that for all $r \in [0,1]$ we have

$$\begin{cases} \underline{Sq}(\hbar;r) = \int_{t_0}^{T} \left[\underline{re}(t;\tau;\hbar)\right]^2 d\tau \\ \overline{Sq}(\hbar;r) = \int_{t_0}^{T} [\overline{re}(t;r;\hbar)]^2 d\tau. \end{cases} \tag{5.2}$$

As has been highlighted by other researchers the aim is to find a region of $\hbar$ in which $\widetilde{Sq}(\hbar;r)$ tend to zero as the order of approximation increases. This would then enable us to obtain, the optimal value of $\hbar$ corresponding to the minimum of the residual error of the Eq.(4.1). The optimal values of $\hbar$ as mentioned in [26] for all of the cases considered are obtained by minimizing (4.2) using the Mathematica package. To estimate the best value of $\hbar$ for the approximate solution $\tilde{y}_m(t;r;\hbar)$ for all $r \in [0,1]$, if there exist $c_0 \in [t_0, T]$ by plotting these quantities $\tilde{y}'_m(c_0;r;\hbar)$,

$\tilde{y}''_m(c_0; r; \hbar)$ or $\tilde{y}'''_m(c_0; r; \hbar)$ for $-2 < \hbar < 0$ such that the best values $\hbar \in \mathbb{C}$, where $\mathbb{C}$ can give the best values of $\hbar$ these plotted curves are called by $\hbar$-curves [20]. We seek to find the valid region $\mathbb{C}$ of $\hbar$, which is known to correspond to the line segment nearly parallel to the horizontal axis [34]. In our problems we need to plot the $\hbar$-curve for each r-level but for simplicity we just need to fix any value of $r \in [0,1]$ say r_0, and then we can plot $\tilde{y}'_m(c_0; r_0; \hbar)$, $\tilde{y}''_m(c_0; r_0; \hbar)$ or even $\tilde{y}'''_m(c_0; r_0; \hbar)$ for the lower and the upper bound of the approximate solution of Eq.(4.1) and apply this value of $\hbar$ for each r-level solutions. Note this way is applicable also for high order FIVP and we can show this in the next section.

6 Numerical Experiments

In this section, two numerical examples are presented to illustrate of the capability of HAM and applied to FIVP.

Example 6.1 Consider the second-order fuzzy nonlinear differential equation:

$$\begin{aligned} &y''(t) = -(y'(t))^2, \quad 0 \le t \le 0.1 \\ &y(0) = (r, 2-r), y'(0) = (1+r, 3-r) \\ &\forall r \in [0,1]. \end{aligned} \tag{6.1}$$

It can be verified that the exact solution is:

$$\underline{Y}(t;r) = \ln[(e^r + e^r r)t + e^r]\ ,\ \overline{Y}(t;r) = \ln[(3e^{2-r} - e^{2-r} r)t + e^{2-r}].$$

The initials approximation for this problem is taken as follows:

$$\begin{cases} \underline{y}_0(t;r) = r + (1+r)t \\ \overline{y}_0(t;r) = (2-r) + (3-r)t. \end{cases} \tag{6.2}$$

The linear operators of Eq.(6.1) are

$$\underline{\mathcal{L}}_2 = \frac{\partial^2 [\underline{\emptyset}(t;p)]_r}{\partial t^2}$$

and

$$\overline{\mathcal{L}}_2 = \frac{\partial^2 \overline{\emptyset}[(t;p)]_r}{\partial t^2}.$$

According to Section 3, we construct the zeroth-order and m^{th}-order deformation equations for $m \ge 1$ of Eq.(6.1) as follows:

$$\begin{cases} \underline{y}_m(t;r) = \chi_m \underline{y}_{m-1}(t;r) + \hbar \underline{\mathcal{L}}_2^{-1} \mathcal{R}_m(\underline{\vec{y}}_{m-1}(t;r)) \\ \overline{y}_m(t;r) = \chi_m \overline{y}_{m-1}(t;r) + \hbar \overline{\mathcal{L}}_2^{-1} \mathcal{R}_m\left(\overline{\vec{y}}_{m-1}(t;r)\right), \end{cases} \tag{6.3}$$

$$\underline{y}_m(0;r) = 0, \overline{y}_m(0;r) = 0, \underline{y}'_m(0;r) = 0, \overline{y}'_m(0;r) = 0,$$

where

$$\begin{cases} \mathcal{R}_m\left(\underline{\vec{y}}_{m-1}(t;r)\right) = \underline{y}''_{m-1}(t;r) + \sum_{j=0}^{m-1} \underline{y}'_j(t;r) \underline{y}'_{m-1-j}(t;r) \\ \mathcal{R}_m\left(\overline{\vec{y}}_{m-1}(t;r)\right) = \overline{y}''_{m-1}(t;r) + \sum_{j=0}^{m-1} \overline{y}'_j(t;r) \overline{y}'_{m-1-j}(t;r) \end{cases} \tag{6.4}$$

The components of homotopy series solution in (4.9) are obtained for the lower and the upper bound as follows:

$$\begin{aligned} &\underline{y}_0(t;\hbar;r) = r + (1+r)t, \\ &\overline{y}_0(t;\hbar;r) = 2 - r + (3-r)t, \\ &\underline{y}_1(t;\hbar;r) = \left(\frac{t^2}{2} + rt^2 + \frac{r^2 t^2}{2}\right)\hbar, \\ &\overline{y}_1(t;\hbar;r) = \frac{1}{2}(-3+r)^2 t^2 \hbar, \\ &\underline{y}_2(t;\hbar;r) = (\frac{t^2}{2}\left(\frac{t^2}{2} + rt^2 + \frac{r^2 t^2}{2}\right)\hbar + \frac{1}{6}(1+r)^2 t^2 (3 + 2(1+r)t)\hbar^2, \\ &\overline{y}_2(t;\hbar;r) = \frac{1}{2}(-3+r)^2 t^2 \hbar - \frac{1}{6}(-3+r)^2 t^2 (-3 + 2(-3+r)t)\hbar^2, \\ &\vdots \end{aligned}$$

It is to be noted that the series solution (4.17) depends upon t, r and the convergent control-parameter $\hbar$. According to [19] the convergent control-parameter $\hbar$ can be employed to adjust the convergence region of the homotopy analysis solution. Towards this end, we have plotted the $\hbar$-curve for the $\tilde{y}'(0.1;\hbar;1)$ and $\tilde{y}''(0;\hbar;1)$ by 12th-order approximation of the HAM. In our case for fuzzy differential equation, we must plot the $\hbar$-curve for each value of

$\hbar$, $0 \le r \le 1$ for the lower and the upper bound of Eq.(6.1). For a simple illustration we plotted the $\hbar$-curve when $r = 1$ to select the values of $\hbar$.

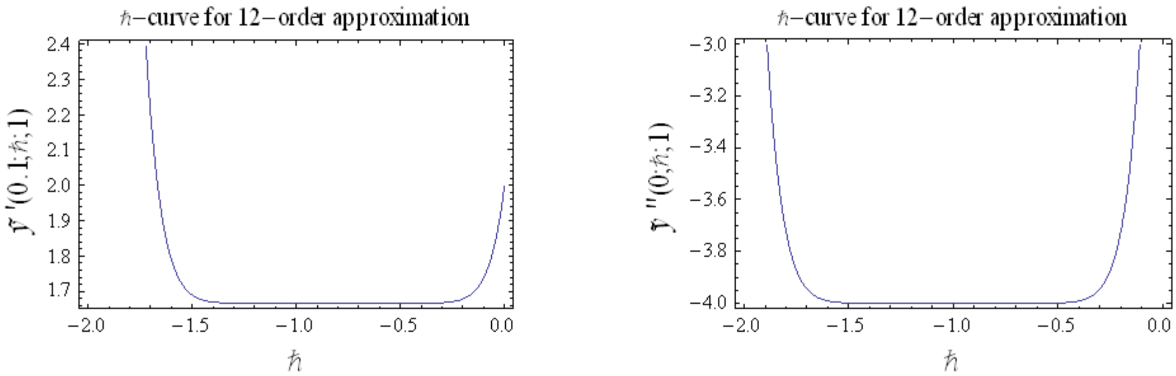


Figure 1: The $\hbar$-curve for approximate solution of Eq.(6.1) given by 12^{th}-order HAM approximate solution when $H(t) = 1$

According to these curves, it is easy to discover the valid region of $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis [20]. From Figure 1, it is clear that the HAM series solution is convergent when $-1.55 \le \hbar \le -0.55$. In Table 1, we have tabulated the absolute errors $[\underline{E}]_r$ and $[\overline{E}]_r$ between the approximate solutions $\underline{y}(0.1;\hbar;r)$ and $\overline{y}(0.1;\hbar;r)$ obtained by using 12^{th}-order HAM series solution and the exact solutions. According to Section 4 the best value of $\hbar$ are $\hbar = -1$ and $\hbar = -0.9$. The errors are very small and thus it can be concluded that HAM approximates the exact solution with a high-order of accuracy. In this table we have shown the errors for $\hbar = -1$ and $\hbar = -0.9$.

Table 1: Absolute errors of Eq.(6.1) given by 12^{th}-order HAM approximate solution

r	$[\underline{E}]_r, \hbar = -1$	$[\underline{E}]_r, \hbar = -0.9$	$[\overline{E}]_r, \hbar = -1$	$[\overline{E}]_r, \hbar = -0.9$
0	5.82867×10^{-16}	0	2.66965×10^{-9}	1.88471×10^{-12}
0.25	1.45994×10^{-14}	5.55112×10^{-17}	8.04252×10^{-10}	2.78666×10^{-13}
0.5	1.82854×10^{-13}	0	2.15786×10^{-10}	3.04201×10^{-14}
0.75	1.55131×10^{-12}	0	5.03153×10^{-11}	2.22045×10^{-15}
1	9.86278×10^{-12}	0	9.86278×10^{-12}	0

The figures show the absolute errors $\tilde{E}(t;\hbar;r)$ for $r \in [0,1]$ for the upper and the lower solutions at $t = 0.1$.

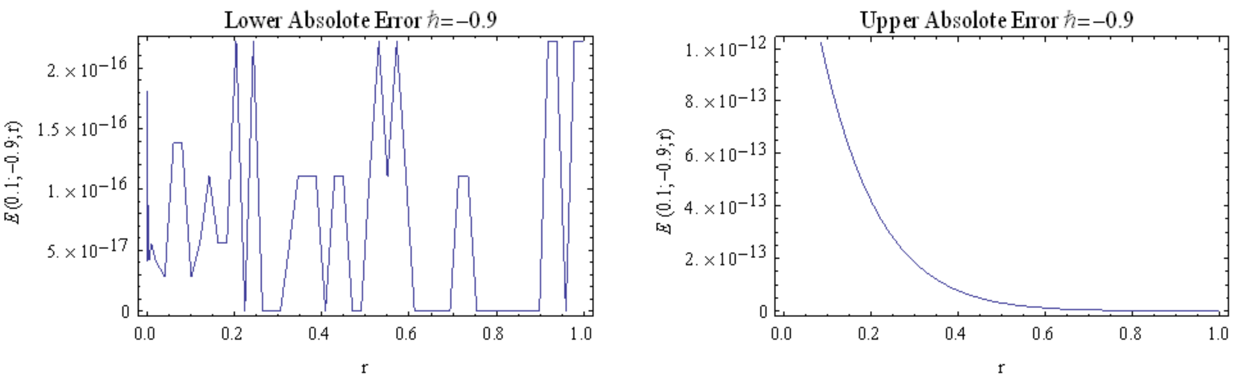


Figure 2: Absolute errors of 12^{th}-order HAM approximate solution and the exact solution of Eq.(6.1) at $\hbar = -0.9$

The following figure the shows the solution obtained by using HAM with 12-terms solution series and the exact solution for various $0 \le r \le 1$.

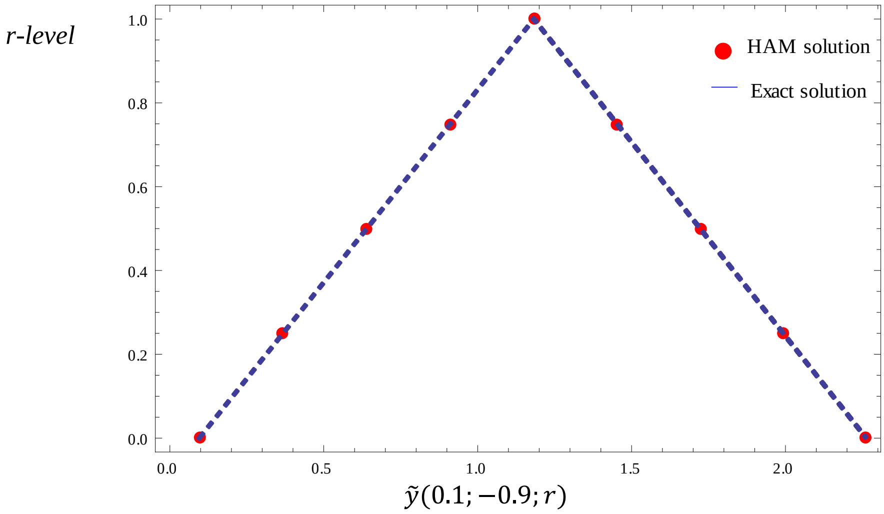


Figure 3: Approximate HAM series solutions the exact and of Eq.(6.1) at t = 0.1

From Figure 3, it can be seen that both exact and HAM approximate solution with 12^{th}-order series solution at t=0.1 and for all $0 \le r \le 1$ satisfy the fuzzy numbers properties in Section 2 by taking the triangle shape.

Example 6.2 Consider the following third-order fuzzy linear differential equation [1]:

$$\begin{aligned} & y'''(t) = 2y''(t) + 3y'(t), \quad 0 \le t \le 1, \\ & y(0) = (3+r, 5-r), \quad y'(0) = (r-3, -1-r), \\ & y'(0) = (r+8, 10-r), \qquad \forall r \in [0,1]. \end{aligned} \tag{6.5}$$

It can be verified from [1] the exact solution of Eq.(6.5) is

$$\underline{Y}(t;r) = \frac{4r-5}{3} + \left(\frac{17-2r}{4}\right)e^{-t} + \left(\frac{1}{6}r + \frac{5}{12}\right)e^{3t},$$
$$\overline{Y}(t;r) = \frac{3-4r}{3} + \left(\frac{13+2r}{4}\right)e^{-t} + \left(\frac{3}{4} - \frac{1}{6}r\right)e^{3t}.$$

We choose the initial approximation as:

$$\begin{cases} \underline{y}_0(t;r) = (3+r) + (r-3)t + (4+\frac{r}{2})t^2 \\ \overline{y}_0(t;r) = (5-r) + (-1-r)t + \left(5 - \frac{r}{2}\right)t^2. \end{cases}$$

The linear operators of Eq.(6.5) are

$$\underline{\mathcal{L}}_3 = \frac{\partial^3 [\underline{\emptyset}(t;p)]_r}{\partial t^3}$$

and

$$\overline{\mathcal{L}}_3 = \frac{\partial^3 \overline{\emptyset}[(t;p)]_r}{\partial t^3}.$$

According to Section 3, we construct the zeroth-order and m^{th}-order deformation equations for $m \ge 1$ of Eq.(6.5) as follows:

$$\begin{cases} \underline{y}_m(t;r) = \chi_m \underline{y}_{m-1}(t;r) + \hbar \underline{\mathcal{L}}_3^{-1} \mathcal{R}_m(\vec{\underline{y}}_{m-1}(t;r)) \\ \overline{y}_m(t;r) = \chi_m \overline{y}_{m-1}(t;r) + \hbar \overline{\mathcal{L}}_3^{-1} \mathcal{R}_m\left(\vec{\overline{y}}_{m-1}(t;r)\right), \end{cases} \tag{6.6}$$

$$\underline{y}_m(0;r) = 0, \quad \overline{y}_m(0;r) = 0, \quad \underline{y}'_m(0;r) = 0, \quad \overline{y}'_m(0;r) = 0, \quad \underline{y}''_m(0;r) = 0, \quad \overline{y}''_m(0;r) = 0,$$

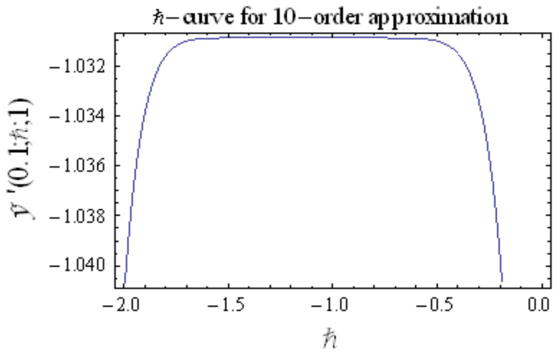
where

$$\begin{cases} \mathcal{R}_m\left(\vec{\underline{y}}_{m-1}(t;r)\right) = \underline{y}'''_{m-1}(t;r) - 2\underline{y}''_{m-1}(t;r) - 3\underline{y}'_{m-1}(t;r) \\ \mathcal{R}_m\left(\vec{\overline{y}}_{m-1}(t;r)\right) = \overline{y}'''_{m-1}(t;r) - 2\overline{y}''_{m-1}(t;r) - 3\overline{y}'_{m-1}(t;r). \end{cases} \tag{6.7}$$

We can obtain the components of HAM series solution (4.9) for the lower and the upper bound as follows:

$$\underline{y}_0(t;\hbar;r) = (0.5r+4)t^2 + (r-3)t + r + 3,$$
$$\overline{y}_0(t;\hbar;r) = (5-0.5r)t^2 + (-r-1)t - r + 5,$$
$$\underline{y}_1(t;\hbar;r) = t^3\hbar(r(-0.125t-0.83333) - 1.t - 1.16666)$$
$$\overline{y}_1(t;\hbar;r) = t^3\hbar(r(0.125t+0.83333) - 1.25t - 2.83333)$$
$$\begin{aligned}\underline{y}_2(t;\hbar;r) = {} & t^3\hbar^2(0.0125r(t-1.45497)(t+4)(t+11.45497)\\ & +0.1(t-1.57773)(t+1.20839)(t+6.119344199813407))\\ & +t^3\hbar(r(-0.125t-0.83333)-1.t-1.16666)\end{aligned}$$
$$\begin{aligned}\overline{y}_2(t;\hbar;r) = {} & t^3\hbar^2(0.125(t-1.52101)(t+2.22582)(t+6.69518)\\ & -0.0125r(t-1.45497)(t+4)(t+11.45497))\\ & +t^3\hbar(r(0.125t+0.83333)-1.25t-2.83333)\end{aligned}$$
$$\vdots$$

As in Example (6.1), it is be noted that the series solution (4.17) depends upon t, r and the convergent control-parameter $\hbar$. According to [19], the convergent control-parameter $\hbar$ can be employed to adjust the convergence region of the homotopy analysis solution. Towards this end, we have plotted the $\hbar$-curve for the $\tilde{y}'(0.1;\hbar;1)$ and $\tilde{y}''(0.1;\hbar;1)$ by 10^{th}-order approximation of the HAM.



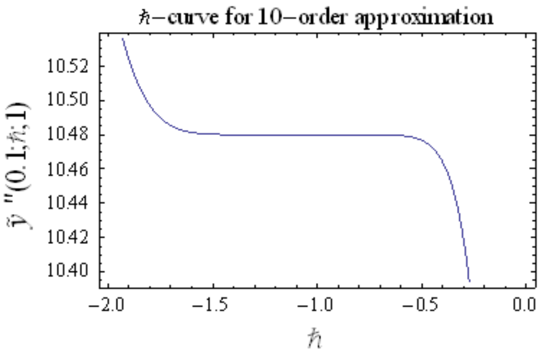


Figure 4: The $\hbar$-curve for approximate solution of Eq.(6.5) given by 10^{th}-order HAM approximate solution when $H(t) = 1$

According to these curves, it is easy to find the valid region of $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis [20]. From Figure 4, it is clear that the HAM series solution is convergent when $-1.7 \leq \hbar \leq -0.6$. In Tables 2 and 3, we have tabulated the absolute errors $[\underline{E}]_r$ and $[\overline{E}]_r$ between the approximate solutions $\underline{y}(1;\hbar;r)$ and $\overline{y}(1;\hbar;r)$ obtained by using 12^{th}-order HAM series solution and the exact solutions. According to Section 4 the best values of $\hbar$ are $\hbar = -1$, $\hbar = 0 - 1.145$ and $\hbar = -1.2$. The errors are very small and thus it can be concluded that HAM approximates the exact solution with a high-order of accuracy. In this table we have shown the errors for $\hbar = -1$, $\hbar = 0 - 1.145$ and $\hbar = -1.2$.

Table 2: Absolute error of Eq.(6.5) given by10^{th}-order HAM approximate solution for $\underline{y}(1;\hbar;r)$

r	$[\underline{E}]_r, \hbar = -1$	$[\underline{E}]_r, \hbar = -1.145$	$[\underline{E}]_r, \hbar = -1.2$
0	4.4584×10^{-6}	4.85223×10^{-9}	5.01324×10^{-8}
0.5	5.8067×10^{-6}	7.03786×10^{-9}	6.98496×10^{-8}
1	7.15500×10^{-6}	9.22349×10^{-9}	8.95668×10^{-8}

Table 3: Absolute error of Eq.(6.5) given by10^{th}-order HAM approximate solution for $\overline{y}(1;\hbar;r)$

r	$[\overline{E}]_r, \hbar = -1$	$[\overline{E}]_r, \hbar = -1.145$	$[\overline{E}]_r, \hbar = -1.2$
0	9.85159×10^{-6}	1.35941×10^{-8}	1.29005×10^{-7}
0.5	8.50329×10^{-6}	1.14082×10^{-8}	1.09289×10^{-7}
1	7.15500×10^{-6}	9.22349×10^{-9}	8.95668×10^{-8}

The following figure show the absolute errors $\tilde{E}(t;\hbar;r)$ for all $r \in [0,1]$ for the upper and the lower solutions at $t = 1$.

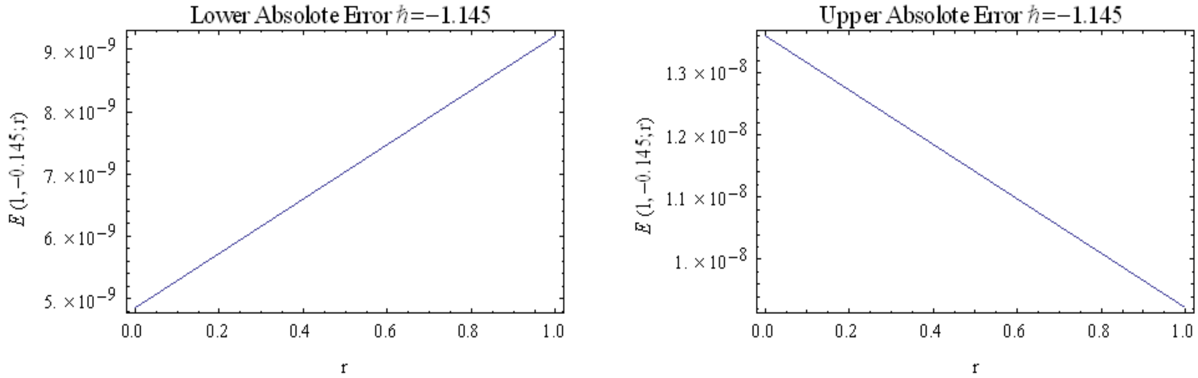


Figure 5: Absolute of 10^{th}-order HAM approximate solution and the exact solution of Eq.(6.5) at $\hbar = -1.145$

From the above figures one can note that the errors of the lower bound solution increase as r gets closer to 1. The upper bound solution decrease as r gets closer to 1. In the following figure, we show the solution obtained by using HAM with 10-terms solution series and the exact solution for various $0 \leq r \leq 1$.

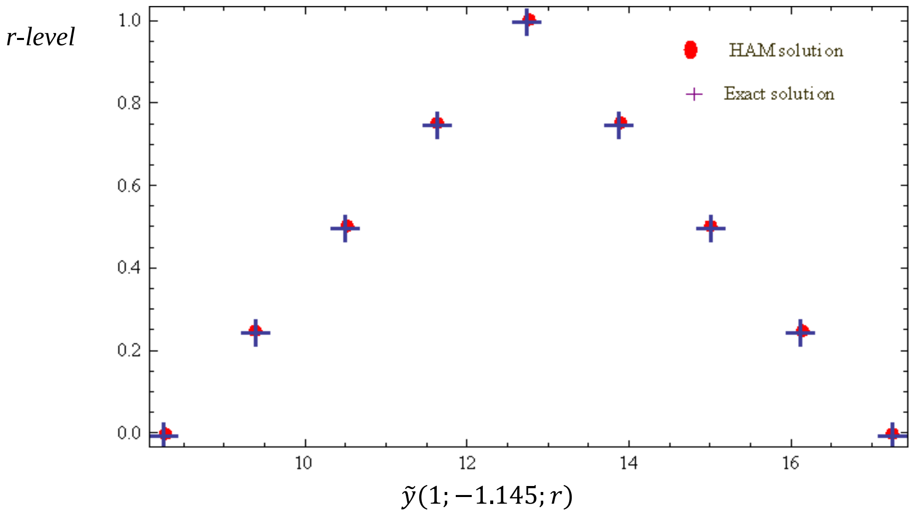


Figure 6: Approximate HAM series solutions the exact and of Eq.(6.5) at t =1

From Figure 6 one can see that both exact and HAM approximate solution with 10^{th}-order series solution at t =1 and for all $0 \leq r \leq 1$ satisfy the fuzzy numbers properties in Section 2 by taking the triangle shape.

According to Definition 1 of convex fuzzy number, one can see that the results in Figures 3 and 6 satisfy the fuzzy number conditions (i) to (iv) in Section 2 and approach to one solution for the upper and the lower bound solutions of Eq.(6.1) and Eq.(6.5).

7 Conclusion

In this paper, the Homotopy Analysis Method (HAM) has been successfully introduced and applied to solve high order ($\geq$ 2) fuzzy initial value problems to obtain an approximate solution. The problem was solved directly without it first being reduced to a first order system. The HAM solution contains the convergence control parameter $\hbar$ which provides a simple way to adjust and control the convergence region of the resulting infinite approximate solution series. The obtained results in this paper show that the HAM is a capable and accurate method for solving high order fuzzy initial value problems involving ordinary differential equations directly. Also the results of the problems in this paper obtained by HAM are satisfying the properties of fuzzy numbers related soultion.

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