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Linear Combinations of T2 Fuzzy Variables

Yankui Liu*, Xuejie Bai
College of Management, Hebei University, Baoding 071002, Hebei, China

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Abstract

Type-2 (T2) fuzzy variable is a measurable map from a fuzzy possibility space to Euclidean space. This note discusses fuzzy arithmetic about the linear combinations of common T2 fuzzy variables. Under appropriate assumptions, we obtain some useful results about the representation of secondary possibility distributions, which have potential applications when we apply T2 fuzzy theory to practical decision making problems.

Keywords: T2 fuzzy variable, fuzzy possibility space, fuzzy arithmetic, linear combination

1 Introduction

Type-2 fuzzy set was first proposed by Zadeh [11, 12] to overcome the difficulty of determining the crisp membership function of a fuzzy set. Recently, fuzzy possibility theory was developed by Liu and Liu [6] with the intention of adopting a variable-based approach to handling type-2 fuzziness. Fuzzy possibility measure, type-2 fuzzy variable, type-2 possibility distribution and secondary possibility distribution are fundamental concepts in fuzzy possibility theory. This theory has now been further developed by a number of researchers [1, 3, 5, 10]. Besides, many interesting applications of T2 fuzzy theory can be found in the literature, such as portfolio optimization [2, 9], data envelopment analysis [7, 8], and transportation problem [4]. In the application areas mentioned above, the linear combinations of T2 fuzzy variables are often used to build practical optimization models, but their secondary possibility distributions haven't been well-established so far. In this note, we resolve this issue for common T2 fuzzy variables, including T2 normal, gamma, trapezoidal and triangular fuzzy variables.

2 Sum of T2 Fuzzy Variables

In this section, we adopt the concepts and notations documented in [5] and the references therein.

If we denote $\mu(x) = \exp\{-(x-\mu)^2/(2\sigma^2)\}, x \in \Re$, then a type-2 fuzzy variable $\tilde{\eta}$ is called normal if its secondary possibility distribution $\tilde{\mu}_{\tilde{\eta}}(x)$ is the following regular triangular fuzzy variable $(\mu(x) - \theta_l \min\{1-\mu(x), \mu(x)\}, \mu(x), \mu(x) + \theta_r \min\{1-\mu(x), \mu(x)\})$ for any $x \in \Re$, where $\mu \in \Re, \sigma > 0$, and $\theta_l, \theta_r \in [0,1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\eta}$ takes the value x. For simplicity, we denote the type-2 normal fuzzy variable $\tilde{\eta}$ with the above second possibility distribution by $\tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$, whose principle possibility distribution is denoted as $n(\mu, \sigma^2)$.

The first result is about the linear combination of T2 normal fuzzy variables, which is stated as:

Theorem 1. *Let $\tilde{\eta}_i = \tilde{n}(\mu_i, \sigma_i^2; \theta_{il}, \theta_{ir})$ be a T2 normal fuzzy variable defined on a fuzzy possibility space $(\Gamma_i, \mathcal{A}_i, \tilde{\text{Pos}}_i)$ for each positive integer $i \le n$. Suppose the principle possibility distributions $n(\mu_i, \sigma_i^2)$'s are mutually independent, and x_i's are real numbers. Then $\tilde{\eta} = \sum_{i=1}^n x_i \tilde{\eta}_i$ is the T2 normal fuzzy variable $\tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$, where the parameters $\mu = \sum_{i=1}^n x_i \mu_i$, $\sigma = \sum_{i=1}^n x_i \sigma_i$, $\theta_l = \max_{1 \le i \le n} \theta_{il}$ and $\theta_r = \min_{1 \le i \le n} \theta_{ir}$.*

Proof. Since the principle possibility distributions $n(\mu_i, \sigma_i^2)$'s are mutually independent, for any nonzero real numbers x_i's, the linear combination $\tilde{\eta} = \sum_{i=1}^n x_i \tilde{\eta}_i$ has principle possibility distribution $n(\mu, \sigma^2)$ with

*Corresponding author. Email: yliu@hbu.edu.cn (Y. Liu).

parameters $\mu = \sum_{i=1}^n x_i\mu_i$, and $\sigma = \sum_{i=1}^n x_i\sigma_i$. Furthermore, for any $z \in \Re$, there exist real numbers z_i's such that $z = \sum_{i=1}^n z_i$ and $\tilde{\mathrm{Pos}}\{\tilde{\eta} = z\} = \min_{i=1}^n \tilde{\mathrm{Pos}}_i\{\tilde{\eta}_i = z_i\}$. Therefore, by the independence of regular fuzzy variables $\tilde{\mathrm{Pos}}_i\{\tilde{\eta}_i = z_i\}$'s, we have $\theta_l = \max_{1\le i\le n}\theta_{il}$ and $\theta_r = \min_{1\le i\le n}\theta_{ir}$. The proof of theorem is complete. □

If we introduce the following three real-valued functions $r_1(x;\theta_l) = r_2(x) - \theta_l \min\{1 - r_2(x), r_2(x)\}$, $r_2(x) = (x/(\lambda r))^r \exp(r - x/\lambda)$, and $r_3(x;\theta_r) = r_2(x) + \theta_r \min\{1 - r_2(x), r_2(x)\}$, then a type-2 fuzzy variable $\tilde{\zeta}$ is called gamma if its secondary possibility distribution $\tilde{\mu}_{\tilde{\zeta}}(x)$ is the regular triangular fuzzy variable $(r_1(x;\theta_l), r_2(x), r_3(x;\theta_r))$ for any $x \in \Re^+$, where $\lambda > 0$ (r is treated as a fixed positive constant), and $\theta_l, \theta_r \in [0,1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\zeta}$ takes the value x. For simplicity, we denote the type-2 gamma fuzzy variable $\tilde{\zeta}$ with the above second possibility distribution by $\tilde{\gamma}(\lambda;\theta_l,\theta_r)$, whose principle possibility distribution is denoted as $\gamma(\lambda)$.

Theorem 2. *Let $\tilde{\zeta}_i = \tilde{\gamma}(\lambda_i;\theta_{il},\theta_{ir})$ be a T2 gamma fuzzy variable defined on a fuzzy possibility space $(\Gamma_i, \mathcal{A}_i, \tilde{\mathrm{Pos}}_i)$ for each positive integer $i \le n$. Suppose the principle possibility distributions $\gamma(\lambda_i)$'s are mutually independent, and x_i's are nonnegative real numbers. Then $\tilde{\zeta} = \sum_{i=1}^n x_i\tilde{\zeta}_i$ is the T2 gamma fuzzy variable $\tilde{\gamma}(\lambda;\theta_l,\theta_r)$, where the parameters $\lambda = \sum_{i=1}^n x_i\lambda_i$, $\theta_l = \max_{1\le i\le n}\theta_{il}$ and $\theta_r = \min_{1\le i\le n}\theta_{ir}$.*

Proof. For each positive integer $i \le n$, let ζ_i be the gamma fuzzy variable associated with principle possibility distribution $\gamma(\lambda_i)$. By supposition, ζ_i's are mutually independent gamma fuzzy variables. Therefore, for any $(x_1, x_2, \ldots, x_n) \in \Re_n^+$, $\sum_{i=1}^n x_i\zeta_i$ is the gamma fuzzy variable $\gamma(\lambda)$ with $\lambda = \sum_{i=1}^n x_i\lambda_i$. It is evident that $\sum_{i=1}^n x_i\zeta_i$ is the principle possibility distribution of $\tilde{\zeta}$. As a consequence, for any $z \in \Re$, there exist real numbers z_i's such that $z = \sum_{i=1}^n z_i$ and $\tilde{\mathrm{Pos}}\{\tilde{\zeta} = z\} = \min_{i=1}^n \tilde{\mathrm{Pos}}_i\{\tilde{\zeta}_i = z_i\}$. By the independence of regular fuzzy variables $\tilde{\mathrm{Pos}}_i\{\tilde{\zeta}_i = z_i\}$'s, we have $\theta_l = \max_{1\le i\le n}\theta_{il}$ and $\theta_r = \min_{1\le i\le n}\theta_{ir}$, which complete the proof of theorem. □

Let $r_i \in \Re, i = 1,2,3,4$ with $r_1 < r_2 \le r_3 < r_4$, and $\mu(x)$ be the following function

$$\mu(x) = \begin{cases} \frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, r_2] \\ 1, & \text{if } x \in (r_2, r_3] \\ \frac{r_4-x}{r_4-r_3}, & \text{if } x \in (r_3, r_4]. \end{cases} \tag{1}$$

Then a type-2 fuzzy variable $\tilde{\xi}$ is called trapezoidal if its secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ is the regular triangular fuzzy variable $(\mu(x) - \theta_l \min\{1-\mu(x), \mu(x)\}, \mu(x), \mu(x) + \theta_r \min\{1-\mu(x), \mu(x)\})$ for any $x \in [r_1, r_4]$, where $\theta_l, \theta_r \in [0,1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\xi}$ takes the value x. For simplicity, we denote the type-2 trapezoidal fuzzy variable $\tilde{\xi}$ with the above second possibility distribution by $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4;\theta_l,\theta_r)$, whose principle possibility distribution is the function $\mu(x)$ defined in Eq. (1).

For the linear combination of T2 trapezoidal fuzzy variables, we have:

Theorem 3. *Let $\tilde{\xi}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \tilde{r}_{i3}, \tilde{r}_{i4};\theta_{il},\theta_{ir})$ be a T2 trapezoidal fuzzy variable defined on a fuzzy possibility space $(\Gamma_i, \mathcal{A}_i, \tilde{\mathrm{Pos}}_i)$ for each positive integer $i \le n$. Suppose the principle possibility distributions of $\tilde{\xi}_i$'s are mutually independent, and x_i's are real numbers. Then $\tilde{\xi} = \sum_{i=1}^n x_i\tilde{\xi}_i$ is the T2 trapezoidal fuzzy variable $(r_1(x), r_2(x), r_3(x), r_4(x);\theta_l,\theta_r)$, where the parameters $\theta_l = \max_{1\le i\le n}\theta_{il}$, $\theta_r = \min_{1\le i\le n}\theta_{ir}$, and*

$$r_1(x) = \sum_{i=1}^n \left(x_i^+ r_{i1} - x_i^- r_{i4}\right), \quad r_2(x) = \sum_{i=1}^n \left(x_i^+ r_{i2} - x_i^- r_{i3}\right), \tag{2}$$

$$r_3(x) = \sum_{i=1}^n \left(x_i^+ r_{i3} - x_i^- r_{i2}\right), \quad r_4(x) = \sum_{i=1}^n \left(x_i^+ r_{i4} - x_i^- r_{i1}\right) \tag{3}$$

with $x_i^+ = \max\{x_i, 0\}$, and $x_i^- = \max\{-x_i, 0\}$.

Proof. The proof of theorem is similar to that of Theorem 1. □

As a corollary of Theorem 3, we have the following result about the linear combination of T2 triangular fuzzy variables.

Corollary 1. *Let $\tilde{\xi}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \tilde{r}_{i3}; \theta_{il}, \theta_{ir})$ be a T2 triangular fuzzy variable defined on a fuzzy possibility space $(\Gamma_i, \mathcal{A}_i, \tilde{\text{Pos}}_i)$ for each positive integer $i \le n$. Suppose the principle possibility distributions of $\tilde{\xi}_i$'s are mutually independent, and x_i's are real numbers. Then $\tilde{\xi} = \sum_{i=1}^{n} x_i \tilde{\xi}_i$ is the T2 triangular fuzzy variable $(r_1(x), r_2(x), r_3(x); \theta_l, \theta_r)$, where the parameters $\theta_l = \max_{1\le i\le n} \theta_{il}$, $\theta_r = \min_{1\le i\le n} \theta_{ir}$, and*

$$r_1(x) = \sum_{i=1}^{n} \left(x_i^+ r_{i1} - x_i^- r_{i3}\right), \quad r_2(x) = \sum_{i=1}^{n} x_i r_{i2}, \quad r_3(x) = \sum_{i=1}^{n} \left(x_i^+ r_{i3} - x_i^- r_{i1}\right) \tag{4}$$

with $x_i^+ = \max\{x_i, 0\}$, and $x_i^- = \max\{-x_i, 0\}$.

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