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Ranking Fuzzy Random Variables Based on New Fuzzy Stochastic Orders

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Abstract

The stochastic ordering of random variables is extended to the cases where the available data are imprecise quantities, rather than crisp. To do this, using some elements of fuzzy set theory, we suggest the fuzzy reversed hazard rate and fuzzy mean inactivity time functions and apply them to construct some new fuzzy stochastic orders for ranking fuzzy random variables. In addition, we study the relations between the proposed fuzzy stochastic orders.

Keywords: fuzzy random variable, likelihood ratio, mean inactivity time, reversed hazard rate, stochastic order

1 Introduction

Stochastic ordering of random variables is a one of the most applicable topics in statistics and probability. Various concepts of stochastic comparison between random variables have been defined and studied in the literature, because of their usefulness in modelling for reliability, economics applications and as mathematical tools for proving important results in applied probability [24]. Mann and Whitney [18] introduced the concept of stochastic order and Brinbaum [6] used their approach in the study of Peakedness. Van Zwet [27] investigated some other important orders and then, the notation of the ordering distributions studied by Lehmann [16] and also Bickel and Lehmann [5] (for more detailes, see Marshall and Olkin [19]). Shaked and Shanthikumar presented a comprehensive report about the stochastic orders and the various studies that have been done by researchers in this field (see, [24], section 1.E).

However, in practice, there are some situations in which, due to uncertainty in underlying system, the classical random variables are not suitable tools for modelling the desired quantities. In such situations, if we use the conventional methods for comparing the random variables, then we confront with some difficulties and need to provide the new techniques. Different approaches and theories have been proposed for treating uncertainty and imprecision during the past decades, among them the fuzzy set theory have a key role and several researchers have concentrated on applying this theory to various fields [33], especially in probability and statistics [2, 3, 7, 25, 28].

As authors known, even though there are many researchers pay attention to ranking the fuzzy numbers [8, 9, 14, 29, 30], but there are only a few researches for comparing the fuzzy random variables. Piriyakumar and Renganathan [21] provided a fuzzy analogue of the stochastic orderings based on the fuzzy random variables introduced by Kwakernaak [15]. To rank the fuzzy random variables, a joint extension of interval comparison and random variable comparison method is proposed by Aiche and Dubois [1]. Recently, Zarei et al. proposed some new fuzzy stochastic orders for fuzzy random variables and also fuzzy order statistics [31].

The present paper aims to develop some important stochastic orders for fuzzy random variables, based on extending the classical approach in stochastic orderings. The proposed approach has some novelty among them is that, this approach uses some concepts of fuzzy lifetime data in stochastic orderings under fuzzy environment. Meanwhile, we develop some popular stochastic orders for ranking fuzzy random variables based on the concept of ranking system for real valued interval.

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This paper is organized as follows: In Section 2, we recalls some concepts of fuzzy numbers, fuzzy random variables, and also fuzzy cumulative distribution function. In Section 3, we introduce the fuzzy likelihood ratio and fuzzy proportional likelihood ratio orders for comparing fuzzy random variables. Also, the relation between these orders was studied. In Section 4, we introduce a method to provide two stochastic orders to compare fuzzy random variables based on two important function in reliability theory. First, in Subsection 4.1, we proposed the reversed hazard rate order for fuzzy random variable and also fuzzy order statistics based on the concept of fuzzy reversed hazard rate function. Then, the fuzzy mean inactivity order was investigated using an extension of the concept of mean inactivity time function for imprecise observations in Subsection 4.2. Finally, we will study some properties of fuzzy mean inactivity time order and connections with fuzzy reversed hazard rate order. In Section 5, we compare our method with some other works. A brief conclusion is provided in Section 6.

2 Preliminaries

In the following, we recalls some definitions and results related to fuzzy numbers, fuzzy random variables and fuzzy distribution function.

2.1 Fuzzy Numbers

Let $\mathcal{X}$ be a universal set and $S_X = \{x \in \mathcal{X} : f(x;\theta) > 0\}$ be the support of X. A fuzzy subset (briefly, a fuzzy set) $\tilde{x}$ of S_X is defined by its membership function $\mu_{\tilde{x}} : S_X \to [0,1]$. We denote the α-cuts of $\tilde{x}$ by $\tilde{x}_\alpha = \{x : \mu_{\tilde{x}}(x) \geq \alpha\}$, and $\tilde{x}_0$ is the closure of the set $\{x : \mu_{\tilde{x}}(x) > 0\}$.

The fuzzy set $\tilde{x}$ is called a normal fuzzy set if there exists $x \in S_X$ such that $\mu_{\tilde{x}}(x) = 1$, and called convex fuzzy sets if $\mu_{\tilde{x}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{x}}(x), \mu_{\tilde{x}}(y)\}$ for every $x, y \in S_X$ and $\lambda \in [0,1]$. The fuzzy set $\tilde{x}$ is called fuzzy number if it is a normal and convex fuzzy set and its α-cuts are bounded for all $\alpha \in [0,1]$. In addition, if $\tilde{x}$ is a fuzzy number and the support of its membership function $\mu_{\tilde{x}}$ is compact, then we called $\tilde{x}$ as a bounded fuzzy number.

If $\tilde{x}$ is a closed and bounded fuzzy number with $x^L_\alpha = \inf\{x : x \in \tilde{x}_\alpha\}$ and $x^U_\alpha = \sup\{x : x \in \tilde{x}_\alpha\}$ and its membership function be strictly increasing on the interval $[x^L_\alpha, x^L_1]$ and strictly decreasing on the interval $[x^U_1, x^U_\alpha]$, then $\tilde{x}$ is called a canonical fuzzy number.

2.2 Fuzzy Random Variables

The fuzzy number $\tilde{x}$ with membership function $\mu_{\tilde{x}}(r)$ can be induced by any real number $x \in S_X$ such that $\mu_{\tilde{x}}(x) = 1$ and $\mu_{\tilde{x}}(r) < 1$ for $r \neq x$. We denote the set of all fuzzy real numbers induced by real number $x \in S_X$ by $\mathbf{F}(S_X)$.

The relation $\sim$ on $\mathbf{F}(S_X)$ define as $\tilde{x}_1 \sim \tilde{x}_2$ if and only if $\tilde{x}_1$ and $\tilde{x}_2$ are induced by the same real number x. Then $\sim$ is an equivalence relation, which induce the equivalence classes $[\tilde{x}] = \{\tilde{a} : \tilde{a} \sim \tilde{x}\}$. The set $(\mathbf{F}(S_X)/\sim)$ called a fuzzy real number system. In practice, we take only one element $\tilde{x}$ from each equivalence class $[\tilde{x}]$ to form the fuzzy real number system $(\mathbf{F}(S_X)/\sim)$. If the fuzzy real number system $(\mathbf{F}(S_X)/\sim)$ consists all of the canonical fuzzy real numbers, then we call $(\mathbf{F}(S_X)/\sim)$ as the canonical fuzzy real number system.

Let X be a random variable with support S_X and $\mathcal{F}(S_X)$ is the set of all canonical fuzzy numbers induce the real numbers in S_X. A fuzzy random variable is a function $\tilde{X} : \Omega \to \mathcal{F}(S_X)$ where for all $\alpha \in [0,1]$

$$\{(\omega, x) : \omega \in \Omega, \quad x \in \tilde{X}_\alpha(\omega)\} \in \mathcal{F} \times \mathcal{B}.$$

Noting that $\mathcal{F}(S_X)$ is the support of the fuzzy random variable $\tilde{X}$ and hence, each α-cut set of $\tilde{X}$ depends on the random variable X. In other words, the fuzzy random variable $\tilde{X}$ is induced by X [15].

Puri and Ralescu [22] provided another definition for fuzzy random variable.

Definition 2.1 *Let $\mathcal{F}(\mathcal{R})$ be a canonical fuzzy real number system. Then $\tilde{X}$ is a Fr.v. if and only if $\tilde{X}^L_\alpha$ and $\tilde{X}^U_\alpha$ are ordinary random variables for all $\alpha \in [0,1]$.*

In this paper, we using this definition for fuzzy random variable and to simplicity denoted it by Fr.v.

Let X be a random variable with cumulative distribution function $F(x)$ and $\tilde{X}$ be a Fr.v. induced by

X. The fuzzy function $\tilde{F}(\tilde{x})$ is said a fuzzy cumulative distribution function of the Fr.v. $\tilde{X}$, whenever its membership function is given by

$$\mu_{\tilde{F}(\tilde{x})}(y) = \sup_{0\leq\alpha\leq 1} \alpha I_{[[\tilde{F}(\tilde{x})]^L_\alpha, [\tilde{F}(\tilde{x})]^U_\alpha]}(y),$$

where

$$[\tilde{F}(\tilde{x})]^L_\alpha = \min\left\{\inf_{\alpha\leq\beta\leq 1}\{F(\tilde{x}^L_\beta)\}, \inf_{\alpha\leq\beta\leq 1}\{F(\tilde{x}^U_\beta)\}\right\},$$

$$[\tilde{F}(\tilde{x})]^U_\alpha = \max\left\{\sup_{\alpha\leq\beta\leq 1}\{F(\tilde{x}^L_\beta)\}, \sup_{\alpha\leq\beta\leq 1}\{F(\tilde{x}^U_\beta)\}\right\}.$$

The interval $[[\tilde{F}(\tilde{x})]^L_\alpha, [\tilde{F}(\tilde{x})]^U_\alpha]$ will contain all of the cumulative distribution functions of $\tilde{X}^L_\beta$ and $\tilde{X}^U_\beta$ for $\beta \geq \alpha$. We note that $\tilde{x}^L_\alpha \leq \tilde{x}^L_1 \leq \tilde{x}^U_1 \leq \tilde{x}^U_\alpha$, thus

$$[[\tilde{F}(\tilde{x})]^L_\alpha, [\tilde{F}(\tilde{x})]^U_\alpha] = [F(\tilde{x}^L_\alpha), F(\tilde{x}^U_\alpha)]. \tag{1}$$

Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two real intervals. Dobuis [10] was investigated a ranking system for such real valued interval as follows:

$$[\underline{a}, \overline{a}] >_1 [\underline{b}, \overline{b}] \Leftrightarrow \underline{a} > \overline{b}, \qquad [\underline{a}, \overline{a}] >_2 [\underline{b}, \overline{b}] \Leftrightarrow \underline{a} > \underline{b},$$
$$[\underline{a}, \overline{a}] >_3 [\underline{b}, \overline{b}] \Leftrightarrow \overline{a} > \overline{b}, \qquad [\underline{a}, \overline{a}] >_4 [\underline{b}, \overline{b}] \Leftrightarrow \overline{a} > \underline{b}.$$

Through this paper, we use the concept of this ranking system, to provide the fuzzy stochastic orders.

3 Fuzzy Likelihood Ratio and Fuzzy Proportional Likelihood Ratio Orders

Likelihood ratio ordering [11, 16] prevalent in stochastic processes is very useful in extending bounds an approximation for performance measures of stochastic systems. In the following definition, we extend the likelihood ratio order to the case when the available observations are modelled as fuzzy random variables, rather than crisp version's.

Definition 3.1 *Let $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s with fuzzy density functions $\tilde{f}$ and $\tilde{g}$, respectively. We propose four relations to compare $\tilde{X}$ and $\tilde{Y}$ as follows*

(i) $\tilde{X} \preceq^{lr}_1 \tilde{Y}$ *if* $\dfrac{[\tilde{g}(y)]^L_\alpha}{[\tilde{f}(y)]^U_\alpha} \geq \dfrac{[\tilde{g}(x)]^L_\alpha}{[\tilde{f}(x)]^U_\alpha}, \quad \forall x \leq y, \quad \alpha \in [0,1].$

(ii) $\tilde{X} \preceq^{lr}_2 \tilde{Y}$ *if* $\dfrac{[\tilde{g}(y)]^L_\alpha}{[\tilde{f}(y)]^L_\alpha} \geq \dfrac{[\tilde{g}(x)]^L_\alpha}{[\tilde{f}(x)]^L_\alpha}, \quad \forall x \leq y, \quad \alpha \in [0,1].$

(iii) $\tilde{X} \preceq^{lr}_3 \tilde{Y}$ *if* $\dfrac{[\tilde{g}(y)]^U_\alpha}{[\tilde{f}(y)]^U_\alpha} \geq \dfrac{[\tilde{g}(x)]^U_\alpha}{[\tilde{f}(x)]^U_\alpha}, \quad \forall x \leq y, \quad \alpha \in [0,1].$

(iv) $\tilde{X} \preceq^{lr}_4 \tilde{Y}$ *if* $\dfrac{[\tilde{g}(y)]^U_\alpha}{[\tilde{f}(y)]^L_\alpha} \geq \dfrac{[\tilde{g}(x)]^U_\alpha}{[\tilde{f}(x)]^L_\alpha}, \quad \forall x \leq y, \quad \alpha \in [0,1].$

We present the following example as an evidence of Definition 3.1.

Example 3.2 *Let $X(\lambda_i)$, i=1,2 be two Exponential random variables with the following density function*

$$f_{\lambda_i}(x) = \lambda_i e^{-\lambda_i x}, \qquad x > 0,\ \lambda_i > 0, \qquad i = 1, 2.$$

In addition, suppose that $\tilde{X}(\lambda_i)$, i=1,2, are two triangular Fr.v.'s induced by $X(\lambda_i)$ with the following membership functions

$$\mu_{\tilde{X}(\lambda_i)}(t) = \begin{cases} \frac{t-X(\lambda_i)+a}{a} & X(\lambda_i) - a \leq t < X(\lambda_i) \\ \frac{X(\lambda_i)+b-t}{b} & X(\lambda_i) \leq t \leq X(\lambda_i) + b, \end{cases} \qquad a < b, \quad i = 1, 2.$$

The α-cuts of $\tilde{X}(\lambda_1)$ is

$$[\tilde{X}(\lambda_1)]_\alpha = [X(\lambda_1) - a(1-\alpha), X(\lambda_1) + b(1-\alpha)]$$

and also the α-cuts of $\tilde{X}(\lambda_2)$ is

$$[\tilde{X}(\lambda_2)]_\alpha = [X(\lambda_2) - a(1-\alpha), X(\lambda_2) + b(1-\alpha)].$$

In the following, we denote the $\tilde{f}_1(\tilde{x})$ and $\tilde{f}_2(\tilde{x})$ as fuzzy density functions of $\tilde{X}(\lambda_1)$ and $\tilde{X}(\lambda_2)$, respectively. Therefore, the α-cuts of $\tilde{f}_1(\tilde{x})$ and $\tilde{f}_2(\tilde{x})$ are given by (for $x > a(1-\alpha)$)

$$[\tilde{f}_1(\tilde{x})]_\alpha = [\lambda_1 \exp\{-\lambda_1[x - a(1-\alpha)]\}, \lambda_1 \exp\{-\lambda_1[x + b(1-\alpha)]\}],$$

$$[\tilde{f}_2(\tilde{x})]_\alpha = [\lambda_2 \exp\{-\lambda_2[x - a(1-\alpha)]\}, \lambda_2 \exp\{-\lambda_2[x + b(1-\alpha)]\}].$$

Therefore, $\tilde{X}(\lambda_1) \preceq_1^{lr} \tilde{X}(\lambda_2)$ if and only if

$$\begin{aligned}
&\lambda_2 \exp\{-\lambda_2[y - a(1-\alpha)]\}\lambda_1 \exp\{-\lambda_1[x + b(1-\alpha)]\} \\
&\geq \lambda_2 \exp\{-\lambda_2[x - a(1-\alpha)]\}\lambda_1 \exp\{-\lambda_1[y + b(1-\alpha)]\} \\
&\Leftrightarrow (y - x)(\lambda_1 - \lambda_2) \geq 0 \\
&\Leftrightarrow \lambda_1 \geq \lambda_2.
\end{aligned}$$

Note 3.3 *It should be mentioned that, if the fuzzy random variables reduce to the crisp random variables then, for every $\alpha \in [0, 1]$,*

$$\frac{[\tilde{g}(.)]^L_\alpha}{[\tilde{f}(.)]^U_\alpha} = \frac{[\tilde{g}(.)]^L_\alpha}{[\tilde{f}(.)]^L_\alpha} = \frac{[\tilde{g}(.)]^U_\alpha}{[\tilde{f}(.)]^U_\alpha} = \frac{[\tilde{g}(.)]^U_\alpha}{[\tilde{f}(.)]^L_\alpha} = \frac{g(.)}{f(.)}$$

which is the classical likelihood ratio order.

Note 3.4 *The order index $\preceq_1$ is the strongest and $\preceq_4$ is the weakest index in two previous fuzzy stochastic orders and also for fuzzy stochastic orders will propose in the next section. Hence, the following relations are holds (see [31] for more details):*

(I) $\tilde{X} \preceq_1 \tilde{Y} \overset{(a)}{\Longrightarrow} \tilde{X} \preceq_2 \tilde{Y} \overset{(b)}{\Longrightarrow} \tilde{X} \preceq_4 \tilde{Y},$

(II) $\tilde{X} \preceq_1 \tilde{Y} \overset{(c)}{\Longrightarrow} \tilde{X} \preceq_3 \tilde{Y} \overset{(d)}{\Longrightarrow} \tilde{X} \preceq_4 \tilde{Y}.$

The concept of proportional likelihood ratio order, which is closely related to the usual likelihood ratio order, was defined by Ramos Romero and Sordo Diaz [23]. The proportional likelihood ratio for Fr.v.'s, as presented in the following definition, can be investigated in a manner similar to the fuzzy likelihood ratio order.

Definition 3.5 *Suppose that $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s with fuzzy density functions $\tilde{f}$ and $\tilde{g}$, respectively. We define the following indexes to compare these Fr.v.'s*

(i) $\tilde{X} \preceq_1^{plr} \tilde{Y}$ *if* $\dfrac{[\tilde{g}(\lambda y)]^L_\alpha}{[\tilde{f}(y)]^U_\alpha} \geq \dfrac{[\tilde{g}(\lambda x)]^L_\alpha}{[\tilde{f}(x)]^U_\alpha}, \ \forall x \leq y, \ \ \lambda \in (0, 1], \ \ \alpha \in [0, 1].$

(ii) $\tilde{X} \preceq_2^{plr} \tilde{Y}$ *if* $\dfrac{[\tilde{g}(\lambda y)]^L_\alpha}{[\tilde{f}(y)]^L_\alpha} \geq \dfrac{[\tilde{g}(\lambda x)]^L_\alpha}{[\tilde{f}(x)]^L_\alpha}, \ \forall x \leq y, \ \ \lambda \in (0, 1], \ \ \alpha \in [0, 1].$

(iii) $\tilde{X} \preceq_3^{plr} \tilde{Y}$ *if* $\dfrac{[\tilde{g}(\lambda y)]^U_\alpha}{[\tilde{f}(y)]^U_\alpha} \geq \dfrac{[\tilde{g}(\lambda x)]^U_\alpha}{[\tilde{f}(x)]^U_\alpha}, \ \forall x \leq y, \ \ \lambda \in (0, 1], \ \ \alpha \in [0, 1].$

(iv) $\tilde{X} \preceq_4^{plr} \tilde{Y}$ *if* $\dfrac{[\tilde{g}(\lambda y)]^U_\alpha}{[\tilde{f}(y)]^L_\alpha} \geq \dfrac{[\tilde{g}(\lambda x)]^U_\alpha}{[\tilde{f}(x)]^L_\alpha}, \ \forall x \leq y, \ \ \lambda \in (0, 1], \ \ \alpha \in [0, 1].$

Example 3.6 *Let $X(\theta_i)$, i=1,2, be two Lindely random variables with the following density function*

$$f_{\theta_i}(x) = \frac{\theta_i + 1}{\theta_i} e^{-\theta_i x}, \qquad x > 0,\ \theta_i > 0, \qquad i = 1, 2.$$

In addition, suppose that $\tilde{X}(\theta_i)$, i=1,2, are two triangular Fr.v.'s induced by $X(\theta_i)$ with the following membership functions

$$\mu_{\tilde{X}(\theta_i)}(t) = \begin{cases} \frac{t - X(\theta_i) + a}{a} & X(\theta_i) - a \le t < X(\theta_i) \\ \frac{X(\theta_i) + b - t}{b} & X(\theta_i) \le t \le X(\theta_i) + b, \end{cases} \qquad a < b, \quad i = 1, 2.$$

Now, we can say that $\tilde{X}(\theta_1) \preceq_1^{plr} \tilde{X}(\theta_2)$ if and only if

$$\begin{aligned} &\exp\{-\lambda\theta_2[y - a(1-\alpha)]\} \exp\{-\theta_1[x + b(1-\alpha)]\} \\ &\ge \exp\{-\lambda\theta_2[x - a(1-\alpha)]\} \exp\{-\theta_1[y + b(1-\alpha)]\} \\ &\Leftrightarrow (y - x)(\theta_1 - \lambda\theta_2) \ge 0 \\ &\Leftrightarrow \lambda\theta_2 \le \theta_1. \end{aligned}$$

The following result characterizes the fuzzy proportional likelihood ratio order by means of the fuzzy likelihood ratio order.

Theorem 3.7 *The two Fr.v.'s $\tilde{X}$ and $\tilde{Y}$ satisfy $\tilde{X} \preceq_i^{plr} \tilde{Y}$ if and only if $\tilde{X} \preceq_i^{lr} a\tilde{Y}$ for all $a > 1$ $(i = 1, 2, 3, 4)$.*

Proof. Suppose that $\tilde{X} \preceq_1^{plr} \tilde{Y}$. Thus, we have that

$$\begin{aligned} \frac{[\tilde{g}_{aY}(y)]_\alpha^L}{[\tilde{f}(y)]_\alpha^U} &= \frac{\frac{1}{a}[\tilde{g}_Y(\frac{y}{a})]_\alpha^L}{[\tilde{f}(y)]_\alpha^U} \\ (\lambda = \frac{1}{a}) \qquad &= \frac{\lambda[\tilde{g}_Y(\lambda y)]_\alpha^L}{[\tilde{f}(y)]_\alpha^U} \\ (\textit{since } \tilde{X} \preceq_1^{plr} \tilde{Y}) \qquad &\ge \frac{\lambda[\tilde{g}_Y(\lambda x)]_\alpha^L}{[\tilde{f}(x)]_\alpha^U} \\ &= \frac{[\tilde{g}_{aY}(x)]_\alpha^L}{[\tilde{f}(x)]_\alpha^U} \end{aligned}$$

which implies that $\tilde{X} \preceq_1^{lr} a\tilde{Y}$. The similar proof are holds for another ranking indexes.

4 Fuzzy Reversed Hazard Rate and Fuzzy Mean Inactivity Time Orders

In this section, we provide a novel approach to fuzzy stochastic orderings based on extending the definitions of reversed hazard rate and mean inactivity time functions, which are two important functions in lifetime data, in fuzzy environment.

4.1 Fuzzy Reversed Hazard Rate Order

In the literature, the function $r(t) = f(t)/F(t)$ is known as the reversed hazard rate function which has been found to be quite useful in the Forensic science, estimating the survival function for left censored data, modelling and analyzing right truncated data and in other areas of applied probability and statistics. There are quite cases in applications where the reversed hazard rate order (see [24] for more details) appears actually as a natural condition which implies useful inequalities, and which leads to optimal structures. In addition, it is worthwhile to mention that there are many situations in economics and risk theory where it is useful to make comparisons between two distributions based on reversed hazard rate order due to naturally arising this order in these fields [20].

Now, let us consider the situations which we cannot use the classical method to analyze our problem as mentioned earlier in Section 1.

Definition 4.1 *Let X be a non-negative random variable with density function $f(x)$ and cumulative distribution function $F(x)$ respectively, and $\tilde{X}$ be a Fr.v. induced by X. The fuzzy function $\tilde{r}_{\tilde{F}}(\tilde{x})$ (for simplicity $\tilde{r}(\tilde{x})$) is said a fuzzy reversed hazard rate of Fr.v. $\tilde{X}$, whenever its membership function is given by*

$$\mu_{\tilde{r}}(y) = \sup_{0\leq\alpha\leq 1} \alpha I_{[[\tilde{r}(\tilde{x})]^L_\alpha, [\tilde{r}(\tilde{x})]^U_\alpha]}(y),$$

where

$$[\tilde{r}(\tilde{x})]^L_\alpha = \min\left\{ \inf_{\alpha\leq\beta\leq 1}\{r(x) : x = \tilde{x}^L_\beta\}, \inf_{\alpha\leq\beta\leq 1}\{r(x) : x = \tilde{x}^U_\beta\}\right\},$$
$$[\tilde{r}(\tilde{x})]^U_\alpha = \max\left\{ \sup_{\alpha\leq\beta\leq 1}\{r(x) : x = \tilde{x}^L_\beta\}, \sup_{\alpha\leq\beta\leq 1}\{r(x) : x = \tilde{x}^U_\beta\}\right\},$$

such that the interval $[[\tilde{r}(\tilde{x})]^L_\alpha, [\tilde{r}(\tilde{x})]^U_\alpha]$ will contain all of the reversed hazard rate of each $\tilde{x}^L_\beta$ and $\tilde{x}^U_\beta$ for $\beta \geq \alpha$.

Definition 4.2 *Let $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s with fuzzy reversed hazard rate functions $\tilde{r}(t)$ and $\tilde{q}(t)$, respectively. Then, We compare $\tilde{X}$ and $\tilde{Y}$ in the sense of fuzzy reversed hazard rate as follows*

(i) $\tilde{X} \preceq^{rh}_1 \tilde{Y}$ *if* $[\tilde{r}(t)]^U_\alpha \leq [\tilde{q}(t)]^L_\alpha, \ \forall t \geq 0, \ \forall \alpha \in [0,1].$

(ii) $\tilde{X} \preceq^{rh}_2 \tilde{Y}$ *if* $(\tilde{r}(t))^L_\alpha \leq [\tilde{q}(t)]^L_\alpha, \ \forall t \geq 0, \ \forall \alpha \in [0,1].$

(iii) $\tilde{X} \preceq^{rh}_3 \tilde{Y}$ *if* $[\tilde{r}(t)]^U_\alpha \leq [\tilde{q}(t)]^U_\alpha, \ \forall t \geq 0, \ \forall \alpha \in [0,1].$

(iv) $\tilde{X} \preceq^{rh}_4 \tilde{Y}$ *if* $[\tilde{r}(t)]^L_\alpha \leq [\tilde{q}(t)]^U_\alpha, \ \forall t \geq 0, \ \forall \alpha \in [0,1].$

Lemma 4.3 *Suppose that $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s with fuzzy cumulative distribution functions $\tilde{F}$ and $\tilde{G}$, respectively. Then,*

(a) $\tilde{X} \preceq^{rh}_1 \tilde{Y}$ *if and only if* $[\tilde{G}(u)]^L_\alpha / [\tilde{F}(u)]^U_\alpha$ *is increasing in* $u > 0$.

(b) $\tilde{X} \preceq^{rh}_2 \tilde{Y}$ *if and only if* $[\tilde{G}(u)]^L_\alpha / [\tilde{F}(u)]^L_\alpha$ *is increasing in* $u > 0$.

(c) $\tilde{X} \preceq^{rh}_3 \tilde{Y}$ *if and only if* $[\tilde{G}(u)]^U_\alpha / [\tilde{F}(u)]^U_\alpha$ *is increasing in* $u > 0$.

(d) $\tilde{X} \preceq^{rh}_4 \tilde{Y}$ *if and only if* $[\tilde{G}(u)]^U_\alpha / [\tilde{F}(u)]^L_\alpha$ *is increasing in* $u > 0$.

The next theorem gives the relationship between the fuzzy likelihood ratio and fuzzy reversed hazard rate orders.

Theorem 4.4 *Suppose that $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s with fuzzy cumulative distribution functions $\tilde{F}$ and $\tilde{G}$, and also fuzzy reversed hazard rate functions $\tilde{r}(t)$ and $\tilde{q}(t)$, respectively. The fuzzy likelihood ratio ordering is stronger than the fuzzy reversed hazard rate ordering.*

Proof. Suppose that $\tilde{X} \preceq^{lr}_1 \tilde{Y}$. Then, for all $x \leq y$, we can write

$$\begin{aligned} \left[\frac{1}{\tilde{r}(y)}\right]^U_\alpha = \frac{[\tilde{F}(y)]^U_\alpha}{[\tilde{f}(y)]^U_\alpha} &= \int_0^y \frac{[\tilde{f}(x)]^U_\alpha}{[\tilde{f}(y)]^U_\alpha} dx \\ (By\ Definition\ 3.1) \quad &\geq \int_0^y \frac{[\tilde{g}(x)]^L_\alpha}{[\tilde{g}(y)]^L_\alpha} dx = \left[\frac{1}{\tilde{q}(y)}\right]^L_\alpha. \end{aligned}$$

Now, regarding to Definition 4.2, proof is complete.

Suppose that $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n$ and $\tilde{Y}_1, \tilde{Y}_2, ..., \tilde{Y}_n$ are two independent fuzzy random samples of size n, induced by $X_1, X_2, ..., X_n$ (with cumulative distribution functions F) and $Y_1, Y_2, ..., Y_n$ (with cumulative distribution functions G), respectively. We denote $\tilde{r}_k(\tilde{x})$ as the fuzzy reversed hazard rate of the *k-th* fuzzy order statistic $\tilde{X}_{k:n}$, as the following

$$\mu_{\tilde{r}_k(\tilde{x})}(y) = \sup_{0\le\alpha\le1} \alpha I_{[[\tilde{r}_k(\tilde{x})]^L_\alpha, [\tilde{r}_k(\tilde{x})]^U_\alpha]}(y), \tag{2}$$

where

$$[\tilde{r}_k(\tilde{x})]^L_\alpha = \min[\quad \inf_{\alpha\le\beta\le1} \left\{ \frac{\frac{n!}{(k-1)!(n-k)!}\frac{f(x)}{F(x)}}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}[\frac{\bar{F}(x)}{F(x)}]^{k-j}} : x = \tilde{x}^L_\beta \right\},$$
$$\inf_{\alpha\le\beta\le1} \left\{ \frac{\frac{n!}{(k-1)!(n-k)!}\frac{f(x)}{F(x)}}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}[\frac{\bar{F}(x)}{F(x)}]^{k-j}} : x = \tilde{x}^U_\beta \right\}],$$

$$[\tilde{r}_k(\tilde{x})]^U_\alpha = \max[\quad \sup_{\alpha\le\beta\le1} \left\{ \frac{\frac{n!}{(k-1)!(n-k)!}\frac{f(x)}{F(x)}}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}[\frac{\bar{F}(x)}{F(x)}]^{k-j}} : x = \tilde{x}^L_\beta \right\},$$
$$\sup_{\alpha\le\beta\le1} \left\{ \frac{\frac{n!}{(k-1)!(n-k)!}\frac{f(x)}{F(x)}}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}[\frac{\bar{F}(x)}{F(x)}]^{k-j}} : x = \tilde{x}^U_\beta \right\}].$$

Theorem 4.5 *Suppose that $\tilde{X}_i \preceq^{rh}_1 \tilde{Y}_i$ for all $i = 1, 2, ..., n$ and*

$$\inf_{\alpha\le\beta\le1} \left\{ \frac{g(x)}{G(x)} : x = \tilde{x}^L_\beta \right\} \ge \sup_{\alpha\le\beta\le1} \left\{ \frac{f(x)}{F(x)} : x = \tilde{x}^U_\beta \right\}, \tag{3}$$

then $\tilde{X}_{i:n} \preceq^{rh}_1 \tilde{Y}_{i:n}$.

Proof. We can prove easily that the function $\left[\sum_{j=k}^n \frac{n!}{j!(n-j)!}(\frac{1-x}{x})^{k-j}\right]^{-1}$ is nondecreasing function in x. Since $\tilde{X}_i \preceq^{rhr}_1 \tilde{Y}_i$, we can conclude that

$$\frac{1}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}\left(\frac{1-F(\tilde{x}^L_\alpha)}{F(\tilde{x}^L_\alpha)}\right)^{k-j}} \ge \frac{1}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}\left(\frac{1-G(\tilde{x}^U_\alpha)}{G(\tilde{x}^U_\alpha)}\right)^{k-j}}.$$

Let us suppose that

$$A = \{\frac{n!}{j!(n-j)!}\frac{f(x)}{F(x)} : x = \tilde{x}^U_\beta\}, \quad B = \{\frac{1}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}\left(\frac{1-F(\tilde{x}^L_\alpha)}{F(\tilde{x}^L_\alpha)}\right)^{k-j}} : x = \tilde{x}^U_\beta\},$$
$$C = \{\frac{n!}{j!(n-j)!}\frac{g(x)}{G(x)} : x = \tilde{x}^L_\beta\}, \quad D = \{\frac{1}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}\left(\frac{1-G(\tilde{x}^L_\alpha)}{G(\tilde{x}^L_\alpha)}\right)^{k-j}} : x = \tilde{x}^L_\beta\}.$$

Now, by using the infimum and supremum property and inequality 3, we have

$$\inf_{\alpha\le\beta\le1} \left\{ \frac{\frac{n!}{(r-1)!(n-r)!}\frac{g(x)}{G(x)}}{\sum_{j=k}^n \frac{n!}{j!(n-j)!}[\frac{\bar{G}(x)}{G(x)}]^{k-j}} : x = \tilde{x}^L_\beta \right\} \ge \sup_{\alpha\le\beta\le1} \left\{ \frac{\frac{n!}{(r-1)!(n-r)!}\frac{f(x)}{F(x)}}{\sum_{j=1}^{r-1} \frac{n!}{j!(n-j)!}[\frac{\bar{F}(x)}{F(x)}]^{k-j}} : x = \tilde{x}^U_\beta \right\},$$

or equivalently $[\tilde{r}_k(\tilde{x})]^L_\alpha \ge [\tilde{r}_k(\tilde{x})]^U_\alpha$ and hence, $\tilde{X}_{i:n} \preceq^{rh}_1 \tilde{Y}_{i:n}$.

Remark 4.6 *Fuzzy proportional reversed hazard rate order can be defined as in a manner similar to the Definition 3.5. Also, the relation between this order and fuzzy reversed hazard rate order can be established.*

4.2 Fuzzy Mean Inactivity Time Order

Let T be a continuous random variable denoting the lifetime of a device or a system and assume that the system has failed sometime before $t > 0$. In the real world, the underlying systems often are not monitored continuously due to some problems. In such situations, one might be interested in getting inference more about the history of the system. Consider the conditional random variable $t - T|T < t$. The mean inactivity time (mit) of the component, can be defined as

$$m(t) = E(t - T|T \leq t) = \frac{\int_0^t F(x)dx}{F(t)}.$$

Recently, various concepts of stochastic comparisons between random variables have been defined entitled mean inactivity time order (in some literature, reversed mean residual order) based on this definition [13]. However, in some situations, we can not determine the time elapsed from the failure due to human errors, machine errors, or some other unexpected conditions.

In the following definition, we provide the fuzzy mean inactivity time and also related fuzzy stochastic order, to modelling these mentioned situations.

Definition 4.7 *Let X be a random variable with density function $f(x)$ and cumulative distribution function $F(x)$, and $\tilde{X}$ be a Fr.v. induced by X. The fuzzy function $\tilde{m}_{\tilde{F}}(\tilde{x})$ (for simplicity $\tilde{m}(\tilde{x})$) is said a fuzzy mean inactivity time of Fr.v. $\tilde{X}$, whenever its membership function is given by*

$$\mu_{\tilde{m}}(y) = \sup_{0 \leq \alpha \leq 1} \alpha I_{[[\tilde{m}(\tilde{x})]^L_\alpha, [\tilde{m}(\tilde{x})]^U_\alpha]}(y),$$

where

$$[\tilde{m}(\tilde{x})]^L_\alpha = \min\left\{ \inf_{\alpha \leq \beta \leq 1} \{m(x) : x = \tilde{x}^L_\beta\}, \inf_{\alpha \leq \beta \leq 1} \{m(x) : x = \tilde{x}^U_\beta\} \right\},$$

$$[\tilde{m}(\tilde{x})]^U_\alpha = \max\left\{ \sup_{\alpha \leq \beta \leq 1} \{m(x) : x = \tilde{x}^L_\beta\}, \sup_{\alpha \leq \beta \leq 1} \{m(x) : x = \tilde{x}^U_\beta\} \right\}.$$

Now, using this definition, we investigate the fuzzy mean inactivity time order as follows.

Definition 4.8 *Suppose that $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s with fuzzy mean inactivity time functions $\tilde{l}$ and $\tilde{m}$, respectively. Then, we define $\tilde{X}$ is smaller than $\tilde{Y}$ in the fuzzy mean inactivity time order as follows*

(i) $\tilde{X} \preceq_1^{mit} \tilde{Y}$ *if* $[\tilde{l}(t)]^L_\alpha \geq [\tilde{m}(t)]^U_\alpha, \ \forall t \geq 0, \ \forall \alpha \in [0, 1].$

(ii) $\tilde{X} \preceq_2^{mit} \tilde{Y}$ *if* $[\tilde{l}(t)]^L_\alpha \geq [\tilde{m}(t)]^L_\alpha, \ \forall t \geq 0, \ \forall \alpha \in [0, 1].$

(iii) $\tilde{X} \preceq_3^{mit} \tilde{Y}$ *if* $[\tilde{l}(t)]^U_\alpha \geq [\tilde{m}(t)]^U_\alpha, \ \forall t \geq 0, \ \forall \alpha \in [0, 1].$

(iv) $\tilde{X} \preceq_4^{mit} \tilde{Y}$ *if* $[\tilde{l}(t)]^U_\alpha \geq [\tilde{m}(t)]^L_\alpha, \ \forall t \geq 0, \ \forall \alpha \in [0, 1].$

Example 4.9 *Let X be a $uniform$ random variable defined on interval $(0, \theta)$, i.e.,*

$$f_\theta(x) = \frac{1}{\theta}, \quad 0 < x < \theta, \quad \theta > 0.$$

Suppose that $\tilde{X}$ and $\tilde{Y}$ are two triangular Fr.v.'s induced by X with the following membership functions

$$\mu_{\tilde{X}}(t) = \begin{cases} \frac{t-X+a_1}{a_1} & X - a_1 \leq t < X \\ \frac{X+b_1-t}{b_1} & X \leq t \leq X + b_1, \end{cases} \qquad \mu_{\tilde{Y}}(t) = \begin{cases} \frac{t-X+a_2}{a_2} & X - a_2 \leq t < X \\ \frac{X+b_2-t}{b_2} & X \leq t \leq X + b_2. \end{cases}$$

It can be checked that $m(t) = t/2$. Hence

$$[\tilde{l}(t)]^L_\alpha = \frac{t - a_1(1-\alpha)}{2} \geq \frac{t - a_2(1-\alpha)}{2} = [\tilde{m}(t)]^L_\alpha \quad \Leftrightarrow \quad a_1 \leq a_2.$$

In the following lemma, we present the equivalent condition for fuzzy mean inactivity time order.

Lemma 4.10 *Suppose that $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s with fuzzy cumulative distribution functions $\tilde{F}$ and $\tilde{G}$, respectively. Then,*

(a) $\tilde{X} \preceq_1^{mit} \tilde{Y}$ *if and only if* $\int_0^t [\tilde{F}(u)]_\alpha^L \, du / \int_0^t [\tilde{G}(u)]_\alpha^U \, du$ *is decreasing in* $u > 0$.

(b) $\tilde{X} \preceq_2^{mit} \tilde{Y}$ *if and only if* $\int_0^t [\tilde{F}(u)]_\alpha^L \, du / \int_0^t [\tilde{G}(u)]_\alpha^L \, du$ *is decreasing in* $u > 0$.

(c) $\tilde{X} \preceq_3^{mit} \tilde{Y}$ *if and only if* $\int_0^t [\tilde{F}(u)]_\alpha^U \, du / \int_0^t [\tilde{G}(u)]_\alpha^U \, du$ *is decreasing in* $u > 0$.

(d) $\tilde{X} \preceq_4^{mit} \tilde{Y}$ *if and only if* $\int_0^t [\tilde{F}(u)]_\alpha^U \, du / \int_0^t [\tilde{G}(u)]_\alpha^L \, du$ *is decreasing in* $u > 0$.

The non-negative and measurable function $h(x, y)$ is said to be totally positivity of order 2 (*TP2*) on $x \in A \subseteq \mathcal{R}$ and $y \in B \subseteq \mathcal{R}$ if for every $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

$$h(x_1, y_1) \times h(x_2, y_2) - h(x_1, y_2) \times h(x_2, y_1) \geq 0, \quad \forall x_1 \leq x_2, y_1 \leq y_2.$$

Let $h(x, y)$ is *TP2* function and $f(x), g(y)$ are two non-negative measurable functions. Then, $k(x, y) = f(x) \times g(y) \times h(x, y)$ is *TP2* function. In addition, the production of two *TP2* functions $h(x, y)$ and $k(x, y)$ is also *TP2* function (for more details see [12]).

The following theorem gives the relationship between the fuzzy mean inactivity time and fuzzy reversed hazard rate orders.

Theorem 4.11 *Suppose that $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s with fuzzy cumulative distribution functions $\tilde{F}$ and $\tilde{G}$, respectively.*

(a) If $\tilde{X} \preceq_1^{rh} \tilde{Y}$, then $\tilde{X} \preceq_4^{mit} \tilde{Y}$.

(b) If $\tilde{X} \preceq_2^{rh} \tilde{Y}$, then $\tilde{X} \preceq_2^{mit} \tilde{Y}$.

(c) If $\tilde{X} \preceq_3^{rh} \tilde{Y}$, then $\tilde{X} \preceq_3^{mit} \tilde{Y}$.

(d) If $\tilde{X} \preceq_4^{rh} \tilde{Y}$, then $\tilde{X} \preceq_1^{mit} \tilde{Y}$.

Proof. Suppose that $\tilde{X} \preceq_1^{rh} \tilde{Y}$. Based on Lemma 4.3, we know that $[\tilde{F_2}(u)]_\alpha^L / [\tilde{F_1}(u)]_\alpha^U$ is increasing in $u > 0$ and hence

$$[\tilde{F_1}(u_2)]_\alpha^U \, [\tilde{F_2}(u_1)]_\alpha^L \leq [\tilde{F_1}(u_1)]_\alpha^U \, [\tilde{F_2}(u_2)]_\alpha^L, \quad 0 < u_1 \leq u_2.$$

By defining

$$K(i, u) = \begin{cases} [\tilde{F_1}(u)]_\alpha^L & i = 1 \\ [\tilde{F_2}(u)]_\alpha^U & i = 2, \end{cases} \qquad L(u, t) = I_{(0,t]}(u); \; u, t > 0,$$

and recently discussion, we can easily see that the function $K(i, u)$ for $i = 1, 2$ is *TP2*. In other hand, since the function $L(u, t)$ is *TP2* function in $u, t > 0$, we have that the functions

$$M(i, t) = \int_0^\infty K(i, u) \; L(u, t) \; du, \quad i = 1, 2$$

are also *TP2*.
Thus, we can conclude that

$$\frac{\int_0^t [\tilde{F_1}(u)]_\alpha^U \; du}{\int_0^t [\tilde{F_2}(u)]_\alpha^L \; du}$$

is decreasing in $u > 0$. Now, using Lemma 4.10, the proof of part (a) is complete.

The proof of other parts are similar to that of the part (a). We omit it here.

The following example shows that, the converse of Theorem 4.11 in general is not true.

Example 4.12 *Consider the triangular Fr.v.'s $\tilde{X}$ and $\tilde{Y}$ which illustrated in Example 4.9. Just as saw in that example, $\tilde{X} \preceq_2^{mit} \tilde{Y}$ if and only if $a_1 \leq a_2$. With due attention to this condition and the fact that the reversed hazard rate of random variable X is $r(t) = 1/t$, we have that*

$$[\tilde{r}(t)]_\alpha^L = \frac{1}{t - a_1(1-\alpha)} \leq \frac{1}{t - a_2(1-\alpha)} = [\tilde{q}(t)]_\alpha^L$$

if and only if $a_1 \geq a_2$ which contradicted with the condition $a_1 \leq a_2$.

In the following theorem, we state the necessary condition which leads to the converse of Theorem 4.11 is holds.

Theorem 4.13 *Let $\tilde{X}$ and $\tilde{Y}$ are two Fr.v.'s have a common fuzzy mean past life $\tilde{l}$ and $\tilde{m}$, respectively. Then,*

(a) If $[\tilde{l}(t)]_\alpha^U/[\tilde{m}(t)]_\alpha^L$ is increasing in $t > 0$, then $\tilde{X} \preceq_1^{rh} \tilde{Y}$ if and only if $\tilde{X} \preceq_4^{mit} \tilde{Y}$.

(b) If $[\tilde{l}(t)]_\alpha^L/[\tilde{m}(t)]_\alpha^L$ is increasing in $t > 0$, then $\tilde{X} \preceq_2^{rh} \tilde{Y}$ if and only if $\tilde{X} \preceq_2^{mit} \tilde{Y}$.

(c) If $[\tilde{l}(t)]_\alpha^U/[\tilde{m}(t)]_\alpha^U$ is increasing in $t > 0$, then $\tilde{X} \preceq_3^{rh} \tilde{Y}$ if and only if $\tilde{X} \preceq_3^{mit} \tilde{Y}$.

(d) If $[\tilde{l}(t)]_\alpha^L/[\tilde{m}(t)]_\alpha^U$ is increasing in $t > 0$, then $\tilde{X} \preceq_4^{rh} \tilde{Y}$ if and only if $\tilde{X} \preceq_1^{mit} \tilde{Y}$.

Proof. Suppose that $\tilde{r}$ and $\tilde{q}$ are fuzzy reversed hazard rate functions for $\tilde{X}$ and $\tilde{Y}$, respectively. Let us consider that $\tilde{X} \preceq_1^{mit} \tilde{Y}$, i.e., $[\tilde{l}(t)]_\alpha^U \geq [\tilde{m}(t)]_\alpha^L$ for all $t > 0$. In addition, since $[\tilde{l}(t)]_\alpha^U/[\tilde{m}(t)]_\alpha^L$ is increasing in $t > 0$, we have that

$$\frac{\frac{d}{dt}\{[\tilde{l}(t)]_\alpha^U\}}{[\tilde{l}(t)]_\alpha^U} \geq \frac{\frac{d}{dt}\{[\tilde{m}(t)]_\alpha^L\}}{[\tilde{m}(t)]_\alpha^L}$$

Hence, we can conclude that

$$[\tilde{r}(t)]_\alpha^U = \frac{1}{[\tilde{l}(t)]_\alpha^U} - \frac{\frac{d}{dt}\{[\tilde{l}(t)]_\alpha^U\}}{[\tilde{l}(t)]_\alpha^U} \leq \frac{1}{[\tilde{m}(t)]_\alpha^L} - \frac{\frac{d}{dt}\{[\tilde{m}(t)]_\alpha^L\}}{[\tilde{m}(t)]_\alpha^L} = [\tilde{q}(t)]_\alpha^L,$$

Now, by using Definition 4.2, the proof of part (a) is completed.

5 A Comparison Study

In this section, we compare our method with other well known methods proposed for ranking random variables in the fuzzy environment.

5.1 Comparison with Piriyakumar and Renganathan's Approach

Piriyakumar and Renganathan [22] studied the problem of stochastic ordering for fuzzy random variables based on Kwakernaak's definition [15]. They first provided usual stochastic order for this fuzzy random variables. Then, they introduced the notion of "probability with fuzzy state" (profust) life time and the profust hazard rate function and have presented a stochastic ordering application to the profust life time having a monotone profust hazard rate function.
Our proposed approach has the following advantages over Piriyakumar and Renganathan's method.

- Each proposed fuzzy stochastic order in our proposed approach, contrary to the classical version's and Piriyakumar and Renganathan, do not leads to a binary decision concern with ranking random variable. In fact, by using our approach in fuzzy stochastic orderings, a decision maker have a four choice to compare fuzzy random variables (see Note 3.4).

- We use the Puri and Ralescu's definition for fuzzy random variables instead of Kwakernaak's definition which is modelling existing uncertainty in underlying systems in simplest way rather than Kwakernaak's definition.
- Our proposed fuzzy stochastic orders were defined based on the α-cuts of fuzzy functions in order to give permission to us compare the fuzzy random variables in a general setting, while using the fuzzy stochastic orders were investigated by Piriyakumar and Renganathan, one can just compare the especial kind of fuzzy random variables.

5.2 Comparison with Aiche and Dubois's Approach

Aiche and Dubois [10] proposed a direct extension of stochastic dominance to fuzzy random variables, whose values are fuzzy intervals of the L-R type. They calculate the probability that a random interval is greater than another in the sense of prescribed relations between intervals. Thereafter, based on these extensions and the valued relations between L-R fuzzy intervals, proposed a direct extension of stochastic interval dominance to fuzzy random variables of type L-R.

Although we also use the interval comparison for proposed stochastic orders in order to ranking fuzzy random variables, similar to Aiche and Dubois' approach, but our approach have some nuances with their approach. First, they used the fuzzy random variables which introduced by Kwakernaak. Second, we provide some additional fuzzy stochastic orders especially concern with reliability theory that certainly have an widely application to system comparison in practical situations.

6 Conclusion

The fuzzy sets theory has successfully applied to ranking fuzzy random variables based on new fuzzy stochastic orders in this paper. To do this, the associated likelihood ratio, proportional likelihood ratio, reversed hazard rate, and also mean inactivity time orders were extended and proposed the unified indexes to ranking of fuzzy random variables.

By using the proposed fuzzy stochastic orders, we may arrive at a deeper inference in situations where due to insufficiency in the information, one can not use the classical random variable to modelling of underlying systems. On the other hand, intuitionistic fuzzy set theory (in some literature: vague set theory) and credibility theory are proposed to investigate the uncertainty in a more realistic manners [4, 17, 26, 32]. Therefor, regarding to our proposed approach in ranking fuzzy random variables in this paper, it is naturally realized that the intuitionistic fuzzy sets theory and credibility theory can also impose upon some known techniques of stochastic ordering of random variables in the future research.

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