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Fuzzy Dot Subalgebras and Fuzzy Dot Ideals of B-algebras

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Abstract

In this paper, the notions of fuzzy dot subalgebras, fuzzy normal dot subalgebras and fuzzy dot ideals of B-algebras are introduced and investigated some of their properties. The homomorphic image and inverse image of fuzzy dot subalgebras and fuzzy dot ideals are studied. Also introduced the notion of fuzzy relations on the family of fuzzy dot subalgebras and fuzzy dot ideals of B-algebras and investigated some related properties.

Keywords: B-algebra, fuzzy dot subalgebra, fuzzy normal dot subalgebra, fuzzy dot ideal, fuzzy ρ-product relation, left (resp, right) fuzzy relation

1 Introduction

The study of BCK/BCI-algebras [4] was initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Neggers and Kim [6, 7] introduced a new notion, called B-algebras which is related to several classes of algebras of interest such as BCK/BCI-algebras. Cho and Kim [3] discussed further relations between B-algebras and other topics especially quasigroups. Park and Kim [8] obtained that every quadratic B-algebra on a field X with $|X| \geq 3$ is a BCI-algebra. Jun et al. [5] fuzzyfied (normal) B-algebras and gave a characterization of a fuzzy B-algebras. Saeid [10] introduced the notion of fuzzy topological B-algebras. Using the notion of interval-valued fuzzy set, Saeid [11] introduced the concept of interval-valued fuzzy B-subalgebras of a B-algebra, and studied some of their properties. Some systems of axioms defining a B-algebra were given by Walendziak [18] with a proof of the independent of the axioms. In addition to it, Walendziak obtained a simplified axiomatization of commutative B-algebras. Senapati et al. [16, 12, 17] introduced fuzzy closed ideals of B-algebras and fuzzy subalgebras of B-algebras with respect to t-norm. Also, the authors [2, 13, 14, 15] done lot of works on BG-algebras which is a generalization of B-algebras. For the general development of B-algebras, the ideal theory and subalgebras play important role. To the best of our knowledge no works are available on fuzzy dot subalgebras and fuzzy dot ideals of B-algebras. For this reason we motivated to develop these theories for B-algebras.

In this paper, fuzzy dot subalgebras of B-algebras are defined and lot of properties are investigated. The notion of ρ-product relations on the family of all fuzzy dot subalgebras of a B-algebra are introduced and some related properties are investigated. The remainder of this article is structured as follows: Section 2 proceeds with a recapitulation of all required definitions and properties. In Section 3, the concepts and operations of fuzzy dot subalgebras are introduced and discussed their properties in details. In Section 4, some properties of fuzzy normal dot subalgebras are investigated. In Section 5, ρ-product relations on fuzzy dot subalgebras are given. In Section 6, the notion of fuzzy dot ideals of B-algebras are considered and investigated their properties in details. Finally, in Section 7, conclusion and scope for future research are given.

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2 Preliminaries

In this section, some elementary aspects that are necessary for this paper are included. A B-algebra is an important class of logical algebras introduced by Neggers and Kim [6] and was extensively investigated by several researchers. This algebra is defined as follows.

Definition 1 ([6]) *A non-empty set X with a constant 0 and a binary operation $*$ is said to be B-algebra if it satisfies the following axioms*

$B1$. $x * x = 0$;

$B2$. $x * 0 = x$;

$B3$. $(x * y) * z = x * (z * (0 * y))$, *for all* $x, y, z \in X$.

Example 1 ([6]) *Let X be the set of all real numbers except the negative integer $-n$. Define a binary operation $*$ on X by $x * y = n(x - y)/(n + y)$. Then $(X, *, 0)$ is a B-algebra.*

Lemma 1 ([3]) *If X is a B-algebra, then $0 * (0 * x) = x$ for all $x \in X$.*

Lemma 2 ([6]) *If X is a B-algebra, then $(x * y) * (0 * y) = x$ for all $x, y \in X$.*

A non-empty subset S of a B-algebra X is called a subalgebra [7] of X if $x * y \in S$ for any $x, y \in S$. A mapping $f : X \to Y$ of B-algebra is called a homomorphism [7] if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a B-homomorphism, then $f(0) = 0$. A non-empty subset N of a B-algebra X is said to be normal if $(x * a) * (y * b) \in N$ whenever $x * y \in N$ and $a * b \in N$. Note that any normal subset N of a B-algebra X is a subalgebra of X, but the converse need not be true [7]. A non-empty subset N of a B-algebra X is called a normal subalgebra of X if it is both a subalgebra and normal. A partial ordering "$\leq$" on X can be defined by $x \leq y$ if and only if $x * y = 0$.

We now review some fuzzy logic concepts as follows:

Let X be the collection of objects denoted generally by x. Then a fuzzy set [19] A in X is defined as $A = \{< x, \mu_A(x) >: x \in X\}$, where $\mu_A(x)$ is called the membership value of x in A and $0 \leq \mu_A(x) \leq 1$. The complement of A is denoted by A^c and is given by $A^c = \{< x, \mu_A^c(x) >: x \in X\}$ where $\mu_A^c(x) = 1 - \mu_A(x)$. For any two fuzzy sets $A = \{< x, \mu_A(x) >: x \in X\}$ and $B = \{< x, \mu_B(x) >: x \in X\}$ in X, the following operations [19] are defined

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \ \ \forall x \in X, \quad A \cap B = \min\{\mu_A(x), \mu_B(x)\} \ \ \forall x \in X.$$

Let f be a mapping from the set X into the set Y. Let B be a fuzzy set in Y. Then the inverse image [9] of B, denoted by $f^{-1}(B)$ in X and is given by $f^{-1}(\mu_B)(x) = \mu_B(f(x))$.

Conversely, let A be a fuzzy set in X with membership function μ_A. Then the image [9] of A, denoted by $f(A)$ in Y and is given by

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise.} \end{cases}$$

A fuzzy relation μ on a set X is a fuzzy subset of $X \times X$, that is, a map $\mu : X \times X \to [0, 1]$.

Definition 2 ([5]) *A fuzzy set A in X is called a fuzzy subalgebra if it satisfies the inequality $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.*

Definition 3 ([12]) *A fuzzy set A in X is called a fuzzy ideal of X if it satisfies the inequality (i) $\mu_A(0) \geq \mu_A(x)$ and (ii) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in X$.*

3 Fuzzy Dot Subalgebras of B-algebras

In this section, fuzzy dot subalgebras of B-algebras are defined and some important properties are presented. In what follows, let $(X, *, 0)$ or simply X denote a B-algebra unless otherwise specified.

Definition 4 *Let A be a fuzzy set in a B-algebra X. Then A is called a fuzzy dot subalgebra of X if for all $x, y \in X$, $\mu_A(x * y) \geq \mu_A(x) \cdot \mu_A(y)$, where $\cdot$ denotes ordinary multiplication.*

Example 2 *Let* $X=\{0,1,2,3,4,5\}$ *be a set with the following Cayley table:*

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X,*,0)$ *is a B-algebra (see [7], Example 3.5). Define a fuzzy set* A *in* X *by* $\mu_A(1)=\mu_A(2)=0.7$ *and* $\mu_A(x)=0.5$ *for all* $x\in X\setminus\{1,2\}$*. Then* A *is a fuzzy dot subalgebra of* X*.*

Note that every fuzzy subalgebra of X *is a fuzzy dot subalgebra of* X*, but the converse is not true.*

In fact, the fuzzy dot subalgebra in above example is not a fuzzy subalgebra, since $\mu_A(1*1)=\mu_A(0)=0.5<0.7=\mu_A(1)=\min\{\mu_A(1),\mu_A(1)\}$*.*

Proposition 1 *Every fuzzy dot subalgebras* A *of* X *satisfies the inequality* $\mu_A(0)\geq(\mu_A(x))^2$ *for all* $x\in X$*.*

Proof: For all $x\in X$, we have $x*x=0$. Then $\mu_A(0)=\mu_A(x*x)\geq\mu_A(x)\cdot\mu_A(x)=(\mu_A(x))^2$. □

Theorem 1 *Let* A *be a fuzzy dot subalgebra of* X*. If there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty}(\mu_A(x_n))^2=1$*, then* $\mu_A(0)=1$*.*

Proof: By Proposition 1, $\mu_A(0)\geq(\mu_A(x))^2$ for all $x\in X$. Therefore, $\mu_A(0)\geq(\mu_A(x_n))^2$ for every positive integer n. Consider, $1\geq\mu_A(0)\geq\lim_{n\to\infty}(\mu_A(x_n))^2=1$. Hence, $\mu_A(0)=1$. □

Proposition 2 *If* A *is a fuzzy dot subalgebra of* X*, then* A^m *(*m *is any positive integer) is a fuzzy dot subalgebra of* X*.*

Proof: For any $x\in X$, A^m is a fuzzy set on X defined by $A^m(x)=\mu_A^m(x)$, where m is any positive integer. Let A is a fuzzy dot subalgebra of X. Then $\mu_A(x*y)\geq\mu_A(x)\cdot\mu_A(y)$ for all $x,y\in X$. We have

$$\mu_A^m(x*y)=[\mu_A(x*y)]^m\geq[\mu_A(x)\cdot\mu_A(y)]^m=[\mu_A(x)]^m\cdot[\mu_A(y)]^m=\mu_A^m(x)\cdot\mu_A^m(y).$$

Consequently, A^m is a fuzzy dot subalgebra of X. □

The intersection of two fuzzy dot subalgebras is also a fuzzy dot subalgebra, which is proved in the following theorem.

Theorem 2 *Let* A_1 *and* A_2 *be two fuzzy dot subalgebras of* X*. Then* $A_1\cap A_2$ *is a fuzzy dot subalgebra of* X*.*

Proof: Let $x,y\in A_1\cap A_2$. Then $x,y\in A_1$ and A_2. Now,

$$\begin{aligned}\mu_{A_1\cap A_2}(x*y) &= \min\{\mu_{A_1}(x*y),\mu_{A_2}(x*y)\}\\ &\geq \min\{\mu_{A_1}(x)\cdot\mu_{A_1}(y),\mu_{A_2}(x)\cdot\mu_{A_2}(y)\}\\ &= (\min\{\mu_{A_1}(x),\mu_{A_2}(x)\})\cdot(\min\{\mu_{A_1}(y),\mu_{A_2}(y)\})\\ &= \mu_{A_1\cap A_2}(x)\cdot\mu_{A_1\cap A_2}(y).\end{aligned}$$

Hence, $A_1\cap A_2$ is a fuzzy dot subalgebra of X. □

The above theorem can be generalized as follows.

Theorem 3 *Let* $\{A_i|i=1,2,3,\ldots\}$ *be a family of fuzzy dot subalgebras of* X*. Then* $\bigcap A_i$ *is also a fuzzy dot subalgebra of* X*, where* $\bigcap A_i=\min\mu_{A_i}(x)$*.*

As is well known, the characteristic function of a set is a special fuzzy set. Suppose A is a non-empty subset of X. By χ_A we denote the characteristic function of A, that is,

$$\chi_A(x)=\begin{cases}1, & \text{if } x\in A\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 4 *Let* A *be a non empty subset of* X*. Then* A *is a subalgebra of* X *if and only if* χ_A *is a fuzzy dot subalgebra of* X*.*

Proof: Let A is a subalgebra of X and $x, y \in A$. Then $x * y \in A$. Then we have

$$\chi_A(x * y) = 1 \geq \chi_A(x) \cdot \chi_A(y).$$

If $x \in A$ and $y \notin A$ (or $x \notin A$ and $y \in A$), then we get $\chi_A(x) = 1$ or $\chi_A(y) = 0$. Therefore,

$$\chi_A(x * y) \geq \chi_A(x) \cdot \chi_A(y) = 1 \cdot 0 = 0.$$

Conversely, assume that χ_A is a fuzzy dot subalgebra of X and let $x, y \in A$. Then

$$\chi_A(x * y) \geq \chi_A(x) \cdot \chi_A(y) = 1 \cdot 1 = 1$$

and so $x * y \in A$. This completes the proof. □

Theorem 5 *Let $f : X \to Y$ be a homomorphism of B-algebras. If $B = \{< x, \mu_B(x) >: x \in Y\}$ is a fuzzy dot subalgebra of Y, then the pre-image $f^{-1}(B) = \{< x, f^{-1}(\mu_B(x)) >: x \in X\}$ of B under f is a fuzzy dot subalgebra of X.*

Proof: Assume that B is a fuzzy dot subalgebra of Y and let $x, y \in X$. Then

$$f^{-1}(\mu_B)(x * y) = \mu_B(f(x * y)) = \mu_B(f(x) * f(y)) \geq \mu_B(f(x)) \cdot \mu_B(f(y)) = f^{-1}(\mu_B)(x) \cdot f^{-1}(\mu_B)(y).$$

Therefore, $f^{-1}(B) = \{< x, f^{-1}(\mu_B(x)) >: x \in X\}$ is a fuzzy dot subalgebra of X. □

Theorem 6 *Let $f : X \to Y$ be a homomorphism from a B-algebra X onto a B-algebra Y. If A is a fuzzy dot subalgebra of X, then the image $f(A) = \{< x, f_{\sup}(\mu_A)(x) >: x \in Y\}$ of A under f is a fuzzy dot subalgebra of Y.*

Proof: Let A be a fuzzy subalgebra of X and let $y_1, y_2 \in Y$. Noticing that, $\{x_1 * x_2 : x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X : x \in f^{-1}(y_1 * y_2)\}$. We have

$$\begin{aligned}
f_{\sup}(\mu_A)(y_1 * y_2) &= \sup\{\mu_A(x) : x \in f^{-1}(y_1 * y_2)\} \\
&\geq \sup\{\mu_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \ \ and \ \ x_2 \in f^{-1}(y_2)\} \\
&\geq \sup\{(\mu_A(x_1)) \cdot (\mu_A(x_2)) : x_1 \in f^{-1}(y_1) \ \ and \ \ x_2 \in f^{-1}(y_2)\} \\
&= (\sup\{\mu_A(x_1) : x_1 \in f^{-1}(y_1)\}) \cdot (\sup\{\mu_A(x_2) : x_2 \in f^{-1}(y_2)\}) \\
&= (f_{\sup}(\mu_A)(y_1)) \cdot (f_{\sup}(\mu_A)(y_2)).
\end{aligned}$$

Hence, $f(A) = \{< x, f_{\sup}(\mu_A)(x) >: x \in Y\}$ is a fuzzy dot subalgebra of Y. □

Theorem 7 ([1]) *A fuzzy set A of a B-algebra X is a fuzzy subalgebra of X if and only if for every $t \in [0, 1]$, a non-empty level subset $U(\mu_A : t) = \{x \in X : \mu_A(x) \geq t\}$ is a subalgebra of X.*

Remark 1: If A is a fuzzy dot subalgebra of X, then $U(\mu_A : t)$ need not be a subalgebra of X. In Example 2, A is a fuzzy dot subalgebra of X but $U(\mu_A : 0.7) = \{x \in X : \mu_A(x) \geq 0.7\} = \{1, 2\}$ is not a subalgebra of X since $1 * 1 = 0 \notin U(\mu_A : 0.7)$.

Theorem 8 *Let A be a fuzzy dot subalgebra of X. Then $U(\mu_A : 1) = \{x \in X : \mu_A(x) = 1\}$ is either empty or is a subalgebra of X.*

Proof: Assume that $U(\mu_A : 1) \neq \phi$. Obviously, $0 \in U(\mu_A : 1)$. If $x, y \in U(\mu_A : 1)$, then $\mu_A(x * y) \geq \mu_A(x) \cdot \mu_A(y) = 1$. Hence, $\mu_A(x * y) = 1$, which implies that $x * y \in U(\mu_A : 1)$. Consequently, $U(\mu_A : 1)$ is a subalgebra of X. □

Definition 5 *Let $A = \{< x, \mu_A(x) >: x \in X\}$ and $B = \{< x, \mu_B(x) >: x \in X\}$ be two fuzzy sets in X. The Cartesian product $A \times B : X \times X \to [0, 1]$ is defined by $(\mu_A \times \mu_B)(x, y) = \mu_A(x) \cdot \mu_B(y)$ for all $x, y \in X$.*

Proposition 3 *Let A and B be two fuzzy dot subalgebras of X, then $A \times B$ is a fuzzy dot subalgebra of $X \times X$.*

Proof: Let (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then

$$\begin{aligned}
(\mu_A \times \mu_B)((x_1, y_1) * (x_2, y_2)) &= (\mu_A \times \mu_B)(x_1 * x_2, y_1 * y_2) \\
&= \mu_A(x_1 * x_2) \cdot \mu_B(y_1 * y_2) \\
&\geq ((\mu_A(x_1) \cdot (\mu_A(x_2)) \cdot ((\mu_B(y_1) \cdot (\mu_B(y_2)) \\
&= ((\mu_A(x_1) \cdot (\mu_B(y_1)) \cdot ((\mu_A(x_2) \cdot (\mu_B(y_2)) \\
&= (\mu_A \times \mu_B)(x_1, y_1) \cdot (\mu_A \times \mu_B)(x_2, y_2).
\end{aligned}$$

Hence, $A \times B$ is a fuzzy dot subalgebra of $X \times X$. □

4 Fuzzy Normal Dot Subalgebras of B-algebras

In this section, fuzzy normal dot subalgebras of B-algebras are defined and discussed the relationship between fuzzy normal dot subalgebra, fuzzy normal subalgebra and fuzzy dot subalgebra of B-algebras.

Definition 6 *Let A be a fuzzy set in a B-algebra X. Then A is called a fuzzy normal dot subalgebra of X if $\mu_A((x*a)*(x*b)) \geq \mu_A(x*y)\cdot\mu_A(a*b)$ for all $x, y \in X$.*

Example 3 *Define a fuzzy set A in X by $\mu_A(0) = 0.8$, $\mu_A(1) = \mu_A(2) = 0.7$ and $\mu_A(3) = \mu_A(4) = \mu_A(5) = 0.3$ for all $x \in X$ in Example 2. Then A is a fuzzy normal dot subalgebra of X.*

Note that every fuzzy normal dot subalgebra of X is a fuzzy normal subalgebra of X, but the converse is not true.

In fact, if we define a fuzzy set A in Example 2 by $\mu_A(0) = 0.8$, $\mu_A(1) = 0.5$, $\mu_A(2) = 0.6$ and $\mu_A(3) = \mu_A(4) = \mu_A(5) = 0.7$ for all $x \in X$, then it is a fuzzy normal dot subalgebra but not a fuzzy normal subalgebra since $\mu_A(1) = \mu_A(0*2) = \mu_A((0*0)*(0*1)) = 0.5 < \min\{\mu_A(0*0), \mu_A(0*1)\} = \min\{\mu_A(0), \mu_A(2)\} = 0.6$.

Theorem 9 *Every fuzzy normal dot subalgebra is a fuzzy dot subalgebra.*

Proof: Let A be a fuzzy normal dot subalgebra and $x, y \in X$. Then

$$\mu_A(x*y) = \mu_A((x*y)*(0*0)) \geq \mu_A(x*0)\cdot\mu_A(y*0) = \mu_A(x)\cdot\mu_A(y).$$

Consequently, A be a fuzzy dot subalgebra. □

Remark 2: The converse of the above theorem is not true in general. Let A be a set in Example 2 such that $\mu_A(2) = 0.46$, $\mu_A(0) = \mu_A(3) = 0.7$ and $\mu_A(3) = \mu_A(4) = \mu_A(5) = 0.6$ for all $x \in X$. Then it a fuzzy dot subalgebra but not a normal dot subalgebra since $\mu_A((2*5)*(4*1)) = \mu_A(2) = 0.46 < 0.49 = \mu_A(2*4)\cdot\mu_A(5*1)$.

Definition 7 *A fuzzy set A in X is called a fuzzy normal dot B-algebra if it is a fuzzy dot B-algebra which is fuzzy normal.*

Proposition 4 *If a fuzzy set A in X is a fuzzy normal dot B-algebra with $\mu_A(0) = 1$, then $\mu_A(x*y) = \mu_A(y*x)$ for all $x, y \in X$.*

Proof: Let $x, y \in X$. Then by $(B1)$ and $(B2)$,

$$\mu_A(x*y) = \mu_A((x*y)*(x*x)) \geq \mu_A(x*x)\cdot\mu_A(y*x) = \mu_A(0)\cdot\mu_A(y*x) = \mu_A(y*x).$$

Interchanging x and y, we obtain $\mu_A(y*x) \geq \mu_A(x*y)$, which proves the proposition. □

The intersection of two fuzzy normal dot subalgebras is also a fuzzy normal dot subalgebra since every fuzzy normal dot subalgebra is a fuzzy normal B-algebra. This can be generalized as follows.

Theorem 10 *Let $\{A_i | i = 1, 2, 3, 4, \ldots\}$ be a family of fuzzy normal dot subalgebra of X. Then $\bigcap A_i$ is also a fuzzy normal dot subalgebra of X, where $\bigcap A_i = \min \mu_{A_i}(x)$.*

5 Fuzzy ρ-product Relation of B-algebra

In this section, strongest fuzzy ρ-relation and fuzzy ρ-product relation of B-algebras are defined and presented some of its properties.

Definition 8 *Let ρ be a fuzzy subset of X. The strongest fuzzy ρ-relation on X is the fuzzy subset μ_ρ of $X \times X$ given by $\mu_\rho(x, y) = \rho(x)\cdot\rho(y)$ for all $x, y \in X$.*

Theorem 11 *Let μ_ρ be the strongest fuzzy ρ-relation on X, where ρ is a subset of X. If ρ is a fuzzy dot subalgebra of X, then μ_ρ is a fuzzy dot subalgebra of $X \times X$.*

Proof: Suppose that ρ is a fuzzy dot subalgebra of X. For any (x_1, y_1) and $(x_2, y_2) \in X \times X$, we have

$$\begin{aligned}\mu_\rho((x_1, y_1)*(x_2, y_2)) &= \mu_\rho(x_1*x_2, y_1*y_2) = \rho(x_1*x_2)\cdot\rho(y_1*y_2)\\ &\geq (\rho(x_1)\cdot\rho(x_2))\cdot(\rho(y_1).\rho(y_2)) = (\rho(x_1)\cdot\rho(y_1))\cdot(\rho(x_2)\cdot\rho(y_2))\\ &= \mu_\rho(x_1, y_1)\mu_\rho(x_2, y_2).\end{aligned}$$

Hence, μ_ρ is a fuzzy dot subalgebra of $X \times X$. □

Definition 9 *Let ρ be a fuzzy subset of X. A fuzzy relation μ on X is called a fuzzy ρ-product relation $\mu(x,y) \geq \rho(x) \cdot \rho(y)$ for all $x, y \in X$.*

Definition 10 *Let ρ be a fuzzy subset of X. A fuzzy relation μ on X is called a left fuzzy relation $\mu(x,y) = \rho(x)$ for all $x, y \in X$.*

Similarly, we can define a right fuzzy relation on ρ. Note that a left (respectively, right) fuzzy relation on ρ is a ρ-product relation.

Theorem 12 *Let μ be a left fuzzy relation on a fuzzy subset ρ of X. If μ is a fuzzy dot subalgebra of $X \times X$, then ρ is a fuzzy dot subalgebra of X.*

Proof: Suppose that a left fuzzy relation μ on ρ is a fuzzy dot subalgebra of $X \times X$. Then

$$\rho(x_1 * x_2) = \mu(x_1 * x_2, y_1 * y_2) = \mu((x_1, y_1) * (x_2, y_2)) \geq \mu(x_1, y_1) \cdot \mu(x_2, y_2) = \rho(x_1) \cdot \rho(x_2)$$

for all $x_1, x_2, y_1, y_2 \in X$. Hence, ρ is a fuzzy dot subalgebra of a B-algebra X. □

Theorem 13 *Let μ be a fuzzy relation on X satisfying the inequality $\mu(x,y) \leq \mu(x,0)$ for all $x, y \in X$. Given $s \in X$, let ρ_s be a fuzzy subset of X defined by $\rho_s(x) = \mu(x,s)$ for all $x \in X$. If μ is a fuzzy dot subalgebra of $X \times X$, then ρ_s is a fuzzy dot subalgebra of X for all $s \in X$.*

Proof: Let $x, y, s \in X$. Then

$$\begin{aligned} \rho_s(x * y) &= \mu(x * y, s) = \mu(x * y, s * 0) = \mu((x,s) * (y,0)) \\ &\geq \mu(x,s) \cdot \mu(y,0) \geq \mu(x,s) \cdot \mu(y,s) = \rho_s(x) \cdot \rho_s(y). \end{aligned}$$

Therefore, ρ_s is a fuzzy dot subalgebra of X. □

Theorem 14 *Let μ be a fuzzy relation on X and let ρ_μ be a fuzzy subset of X given by $\rho_\mu(x) = \inf_{y \in X} \mu(x,y) \cdot \mu(y,x)$ for all $x \in X$. If μ is a fuzzy dot subalgebra of $X \times X$ satisfying the equality $\mu(x,0) = 1 = \mu(0,x)$ for all $x \in X$, then ρ_μ is a fuzzy dot subalgebra of X.*

Proof: Let $x, y, z \in X$, we have

$$\begin{aligned} \mu(x * y, z) &= \mu(x * y, z * 0) = \mu((x,z) * (y,0)) \geq \mu(x,z) \cdot \mu(y,0) = \mu(x,z), \\ \mu(z, x * y) &= \mu(z * 0, x * y) = \mu((z,x) * (0,y)) \geq \mu(z,x) \cdot \mu(0,y) = \mu(z,x). \end{aligned}$$

It follows that

$$\mu(x * y, z) \cdot \mu(z, x * y) \quad \geq \quad \mu(x,z) \cdot \mu(z,x) \geq (\mu(x,z) \cdot \mu(z,x)) \cdot (\mu(y,z) \cdot \mu(z,y))$$

so that

$$\rho_\mu(x * y) \quad = \quad \inf_{z \in X} \mu(x * y, z) \cdot \mu(z, x * y) \geq (\inf_{z \in X} \mu(x,z) \cdot \mu(z,x)) \cdot (\inf_{z \in X} \mu(y,z) \cdot \mu(z,y)) = \rho_\mu(x) \cdot \rho_\mu(y).$$

Therefore, ρ_μ is a fuzzy dot subalgebra of X. □

6 Fuzzy Dot Ideals of B-algebras

In this section, fuzzy dot ideals of B-algebras are defined and studied some of its results.

Definition 11 *Let A be a fuzzy set in a B-algebra X. Then A is called a fuzzy dot ideal of X if it satisfies:*
(B4) $\mu_A(0) \geq \mu_A(x)$,
(B5) $\mu_A(x) \geq \mu_A(x * y) \cdot \mu_A(y)$
for all $x, y \in X$.

Example 4 *Let $X=\{0,1,2,3\}$ be a set with the following Cayley table:*

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X,*,0)$ is a B-algebra. Define a fuzzy set A in X by $\mu_A(0)=\mu_A(1)=0.8$, $\mu_A(2)=0.6$ and $\mu_A(3)=0.5$ for all $x\in X$. Then A is a fuzzy dot ideal of X.

Note that every fuzzy ideal of X is a fuzzy dot ideal of X, but the converse is not true.

In fact, the fuzzy dot ideal in above example is not a fuzzy ideal, since $0.5=\mu_A(3)\ngeq\min\{0.6,0.8\}=\min\{\mu_A(3*1),\mu_A(1)\}$.

Theorem 15 *If A is a fuzzy ideal of X such that $\mu_A(0*x)\ge\mu_A(x)$ for all $x\in X$. Then A is a fuzzy dot ideal of X.*

Proof: Let $x,y\in X$. Then $(x*y)*(0*y)=x$ by Lemma 2. We have $\mu_A(x)=\mu_A((x*y)*(0*y))\ge\mu_A(x*y)\cdot\mu_A(0*y)\ge\mu_A(x*y)\cdot\mu_A(y)$. Hence, A is a fuzzy dot ideal of X. □

Proposition 5 *Let A be a fuzzy dot ideal of X. If $x\le y$ in X, then $\mu_A(x)\ge\mu_A(0)\cdot\mu_A(y)$ for all $x,y\in X$.*

Proof: Let $x,y\in X$ such that $x\le y$. Then $x*y=0$, and thus $\mu_A(x)\ge\mu_A(x*y)\cdot\mu_A(y)=\mu_A(0)\cdot\mu_A(y)$. This completes the proof. □

Proposition 6 *Let A be a fuzzy dot ideal of X. If the inequality $x*y\le z$ holds in X, then $\mu_A(x)\ge\mu_A(0)\cdot\mu_A(y)\cdot\mu_A(z)$ for all $x,y,z\in X$.*

Proof: Let $x,y,z\in X$ such that $x*y\le z$. Then $(x*y)*z=0$, and thus $\mu_A(x)\ge\mu_A(x*y)\cdot\mu_A(y)\ge\mu_A((x*y)*z)\cdot\mu_A(z)\cdot\mu_A(y)=\mu_A(0)\cdot\mu_A(z)\cdot\mu_A(y)=\mu_A(0)\cdot\mu_A(y)\cdot\mu_A(z)$. This completes the proof. □

Proposition 7 *If A is a fuzzy dot ideal of X, then A^m (m is any positive integer) is a fuzzy dot ideal of X.*

Proof: For any $x\in X$, A^m is a fuzzy set in X defined by $A^m(x)=\mu_A^m(x)$, where m is any positive integer. Let A be a fuzzy dot ideal of X. Then $\mu_A(0)\ge\mu_A(x)$ and $\mu_A(x)\ge\mu_A(x*y)\cdot\mu_A(y)$ for all $x,y\in X$. We have $\mu_A^m(0)=[\mu_A(0)]^m\ge[\mu_A(x)]^m=\mu_A^m(x)$ and $\mu_A^m(x)=[\mu_A(x)]^m\ge[\mu_A(x*y)\cdot\mu_A(y)]^m=[\mu_A(x*y)]^m\cdot[\mu_A(y)]^m=\mu_A^m(x*y)\cdot\mu_A^m(y)$. Hence, A^m is a fuzzy dot ideal of X. □

Proposition 8 *If A and A^c are both fuzzy dot ideals of X, then A is constant function.*

Proof: Let A and A^c be both fuzzy dot ideals of X. Then $\mu_A(0)\ge\mu_A(x)$ and $\mu_A^c(0)\ge\mu_A^c(x)\Rightarrow 1-\mu_A(0)\ge 1-\mu_A(x)\Rightarrow\mu_A(0)\le\mu_A(x)$ for all $x\in X$. Therefore, $\mu_A(0)=\mu_A(x)$. Hence, A is a constant function. □

The intersection of two fuzzy dot ideals is also a fuzzy dot ideal, which is proved in the following theorem.

Theorem 16 *If A_1 and A_2 be two fuzzy dot ideals of X, then $A_1\cap A_2$ is also a fuzzy dot ideal of X.*

Proof: Let A_1 and A_2 be two fuzzy dot ideals of X. Then for any $x\in X$, $\mu_{A_1}(0)\ge\mu_{A_1}(x)$ and $\mu_{A_2}(0)\ge\mu_{A_2}(x)$. Now $\mu_{A_1\cap A_2}(0)=\min\{\mu_{A_1}(0),\mu_{A_2}(0)\}\ge\min\{\mu_{A_1}(x),\mu_{A_2}(x)\}=\mu_{A_1\cap A_2}(x)$. Again for any $x,y\in X$, we have

$$\begin{aligned}\mu_{A_1\cap A_2}(x) &= \min\{\mu_{A_1}(x),\mu_{A_2}(x)\}\ge\min\{\mu_{A_1}(x*y)\cdot\mu_{A_1}(y)\mu_{A_2}(x*y)\cdot\mu_{A_2}(y)\}\\ &= (\min\{\mu_{A_1}(x*y),\mu_{A_2}(x*y)\})\cdot(\min\{\mu_{A_1}(y),\mu_{A_2}(y)\})=\mu_{A_1\cap A_2}(x*y)\cdot\mu_{A_1\cap A_2}(y).\end{aligned}$$

Hence, $A_1\cap A_2$ is a fuzzy dot ideal of X. □

The above theorem can be generalized as follows.

Theorem 17 *Let $\{A_i|i=1,2,3,\ldots\}$ be a family of fuzzy dot ideals of X. Then $\bigcap A_i$ is also a fuzzy dot ideal of X, where $\bigcap A_i=\min\mu_{A_i}(x)$.*

Proposition 9 *Let A and B be two fuzzy dot ideals of X, then $A\times B$ is a fuzzy dot ideal of $X\times X$.*

Proof: Let (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then

$$\begin{aligned}(\mu_A \times \mu_B)(x_1, y_1) &= \mu_A(x_1) \cdot \mu_B(y_1) \\ &\geq (\mu_A(x_1 * x_2) \cdot \mu_A(x_2)) \cdot (\mu_B(y_1 * y_2) \cdot \mu_B(y_2)) \\ &= (\mu_A(x_1 * x_2) \cdot \mu_B(y_1 * y_2)) \cdot (\mu_A(x_2) \cdot \mu_B(y_2)) \\ &= (\mu_A \times \mu_B)(x_1 * x_2, y_1 * y_2) \cdot (\mu_A \times \mu_B)(x_2, y_2) \\ &= (\mu_A \times \mu_B)((x_1, y_1) * (x_2, y_2)) \cdot (\mu_A \times \mu_B)(x_2, y_2).\end{aligned}$$

Hence, $A \times B$ is a fuzzy dot ideal of $X \times X$. □

Theorem 18 ([12]) *A fuzzy set A of a B-algebra X is a fuzzy ideal of X if and only if for every $t \in [0, 1]$, a non-empty level subset $U(\mu_A : t) = \{x \in X : \mu_A(x) \geq t\}$ is an ideal of X.*

Remark 3: If A is a fuzzy dot ideal of X, then $U(\mu_A : t)$ need not be an ideal of X. In Example 4, A is a fuzzy dot ideal of X but $U(\mu_A : 0.6) = \{x \in X : \mu_A(x) \geq 0.6\} = \{0, 1, 2\}$ is not an ideal of X since $1 * 2 = 3 \notin U(\mu_A : 0.6)$.

Theorem 19 *Let A be a fuzzy dot ideal of X. Then $U(\mu_A : 1) = \{x \in X : \mu_A(x) = 1\}$ is either empty or an ideal of X.*

Proof: Assume that $U(\mu_A : 1) \neq \phi$. Obviously, $0 \in U(\mu_A : 1)$. Let $x, y \in X$ such that $x*y$ and $y \in U(\mu_A : 1)$. Then $\mu_A(x * y) = 1 = \mu_A(y)$. It follows that $\mu_A(x) \geq \mu_A(x * y) \cdot \mu_A(y) = 1$. So, $\mu_A(x) = 1$ i.e., $x \in U(\mu_A : 1)$. Consequently, $U(\mu_A : 1)$ is an ideal of X. □

Theorem 20 *Let $f : X \to Y$ be a homomorphism of B-algebras. If B is a fuzzy dot ideal of Y, then the pre-image $f^{-1}(B) = f^{-1}(\mu_B)$ of B under f is a fuzzy dot ideal of X.*

Proof: Assume that B is a fuzzy dot ideal of Y. For all $x \in X$, $f^{-1}(\mu_B)(x) = \mu_B(f(x)) \leq \mu_B(0) = \mu_B(f(0)) = f^{-1}(\mu_B)(0)$. Again let $x, y \in X$. Then

$$\begin{aligned}f^{-1}(\mu_B)(x) &= \mu_B(f(x)) \geq \mu_B(f(x) * f(y)) \cdot \mu_B(f(y)) \\ &\geq \mu_B(f(x * y)) \cdot \mu_B(f(y)) = f^{-1}(\mu_B)(x * y) \cdot f^{-1}(\mu_B)(y).\end{aligned}$$

Therefore, $f^{-1}(B)$ is a fuzzy dot ideal of X. □

Theorem 21 *Let $f : X \to Y$ be an epimorphism of B-algebras. Then B is a fuzzy dot ideal of Y, if $f^{-1}(B)$ of B under f in X is a fuzzy dot ideal of X.*

Proof: For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$. Then $\mu_B(x) = \mu_B(f(a)) = f^{-1}(\mu_B)(a) \leq f^{-1}(\mu_B)(0) = \mu_B(f(0)) = \mu_B(0)$. Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. Thus

$$\begin{aligned}\mu_B(x) &= \mu_B(f(a)) = f^{-1}(\mu_B)(a) \geq f^{-1}(\mu_B)(a * b) \cdot f^{-1}(\mu_B)(b) \\ &= \mu_B(f(a * b)) \cdot \mu_B(f(b)) = \mu_B(f(a) * f(b)) \cdot \mu_B(f(b)) = \mu_B(x * y) \cdot \mu_B(y).\end{aligned}$$

Then B is a fuzzy dot ideal of Y. □

Theorem 22 *Let μ_ρ be the strongest fuzzy ρ-relation on X, where ρ is a subset of X. Then ρ is a fuzzy dot ideal of X if and only if μ_ρ is a fuzzy dot subalgebra of $X \times X$.*

Proof: Suppose that ρ is a fuzzy dot ideal of X. For any $x, y \in X$, we have $\mu_\rho(0, 0) = \rho(0) \cdot \rho(0) \geq \rho(x) \cdot \rho(y) = \mu_\rho(x, y)$. Let (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then

$$\begin{aligned}\mu_\rho(x_1, y_1) &= \rho(x_1) \cdot \rho(y_1) \\ &= (\rho(x_1 * x_2) \cdot \rho(x_2)) \cdot (\rho(y_1 * y_2) \cdot \rho(y_2)) \\ &= (\rho(x_1 * x_2) \cdot \rho(y_1 * y_2)) \cdot (\rho(x_2) \cdot \rho(y_2)) \\ &= \mu_\rho(x_1 * x_2, y_1 * y_2) \cdot \mu_\rho(x_2, y_2) \\ &= \mu_\rho((x_1, y_1) * (x_2, y_2)) \cdot \mu_\rho(x_2, y_2)\end{aligned}$$

Hence, μ_ρ is a fuzzy dot ideal of $X \times X$.

Conversely, assume that μ_ρ is a fuzzy dot ideal of $X \times X$. By applying (B4), we get $(\rho(0))^2 = \mu_\rho(0, 0) \geq \mu_\rho(x, x) = (\rho(x))^2$ and so $\rho(0) \geq \rho(x)$ for all $x \in X$. Next we have $(\rho(x))^2 = \mu_\rho(x, x) \geq \mu_\rho((x, x) * (y, y)) \cdot \mu_\rho(y, y) = \mu_\rho((x*y, x*y) \cdot \mu_\rho(y, y) = (\rho(x*y) \cdot \rho(y))^2$, which implies that $\rho(x) \geq \rho(x*y) \cdot \rho(y)$ for all $x, y \in X$. Therefore, ρ is a fuzzy dot ideal of X. □

7 Conclusions

In the present paper, the notions of fuzzy dot subalgebras, fuzzy normal dot subalgebras and fuzzy dot ideals of B-algebras has been introduced and some important properties of it are also studied. We have shown that the Cartesian product of any two fuzzy dot subalgebras is a fuzzy dot subalgebra. We have proved that the strongest fuzzy ρ-relation is a fuzzy dot subalgebra if the fuzzy subset is a fuzzy dot subalgebra. Also, we have proved that the strongest fuzzy ρ-relation is a fuzzy dot ideal if and only if the fuzzy subset is a fuzzy dot ideal. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BG-algebras, BF-algebras, lattices and Lie algebras. It is our hope that this work would other foundations for further study of the theory of B-algebras.

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