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Uncertain Random Graph and Uncertain Random Network

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Abstract

This paper proposes a concept of uncertain random graph in which some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure, and discusses the connectivity index of an uncertain random graph. In addition, this paper presents a concept uncertain random network in which some weights are random variables and others are uncertain variables, and obtains the shortest path distribution of an uncertain random network.

Keywords: uncertainty theory, uncertain random variable, uncertain random graph, uncertain random network

1 Introduction

We are usually in the state of indeterminacy. For dealing with indeterminacy phenomena, probability theory was developed by Kolmogorov [13] in 1933 for modeling frequencies, while uncertainty theory was founded by Liu [14] in 2007 for modeling belief degrees.

In many cases, uncertainty and randomness simultaneously appear in a complex system. In order to describe this phenomenon, chance theory was pioneered by Liu [21] in 2013 with the concepts of uncertain random variable, chance measure and chance distribution. Liu [21] also proposed the concepts of expected value and variance of uncertain random variables. As an important contribution to chance theory, Liu [22] presented an operational law of uncertain random variables. In addition, Yao and Gao [27] verified a law of large numbers for uncertain random variables, and Hou [11] investigated the distance between uncertain random variables. In order to model optimization problems with not only uncertainty but also randomness, uncertain random programming was founded by Liu [22] in 2013. As extensions, Zhou, Yang and Wang [30] proposed uncertain random multiobjective programming, Qin [25] proposed uncertain random goal programming, and Ke [12] proposed uncertain random multilevel programming. In order to deal with uncertain random systems, Liu and Ralescu [23] invented the tool of uncertain random risk analysis, and Wen and Kang [26] presented the tool of uncertain random reliability analysis.

Random graph was defined by Erdős and Rényi [3] in 1959 and independently by Gilbert [8] at nearly the same time. As an alternative, uncertain graph was proposed by Gao and Gao [5] via uncertainty theory. After that, the Euler index was discussed by Zhang and Peng [29], and the diameter was investigated by Gao, Yang and Li [7] for an uncertain graph.

Random network was first investigated by Frank and Hakimi [4] in 1965 for modeling communication network with random capacities. From then on, the random network was well developed and widely applied. As a breakthrough approach, uncertain network was first explored by Liu [16] for modeling project scheduling problem with uncertain duration times. Besides, the shortest path problem was investigated by Gao [6], the maximum flow problem was discussed by Han, Peng and Wang [9], the uncertain minimum cost flow problem was dealt with by Ding [2], and Chinese postman problem was explored by Zhang and Peng [28] for an uncertain random network.

This paper will assume that in a graph some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure, and define the concept of uncertain random graph. This

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paper will also assume some weights are random variables and others are uncertain variables, and initialize the concept of uncertain random network.

2 Preliminaries

This section will introduce some preliminary knowledge about uncertainty theory and chance theory, the former is a branch of mathematics for modeling belief degrees, and the latter is a methodology for modeling complex systems with not only uncertainty but also randomness.

2.1 Uncertainty Theory

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event. Liu [14] defined an *uncertain measure* by the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{1}$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an *uncertainty space*. Furthermore, Liu [15] defined a product uncertain measure by the fourth axiom:

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{2}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

An *uncertain variable* was defined by Liu [14] as a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B. In order to describe an uncertain variable in practice, the concept of *uncertainty distribution* was defined by Liu [14] as

$$\Phi(x) = \mathcal{M}\left\{\xi \le x\right\}, \quad \forall x \in \Re. \tag{3}$$

Peng and Iwamura [24] verified that a function $\Phi : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$. An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \quad \lim_{x\to+\infty} \Phi(x) = 1. \tag{4}$$

Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ [16]. It was also verified by Liu [18] that a function $\Phi^{-1}(\alpha) : (0, 1) \to \Re$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to α.

The *expected value* of an uncertain variable ξ was defined by Liu [14] as the following form,

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\}\mathrm{d}x \tag{5}$$

provided that at least one of the two integrals is finite. If ξ has an uncertainty distribution Φ, then the expected value may be calculated by

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x, \tag{6}$$

or equivalently,

$$E[\xi]=\int_{-\infty}^{+\infty}x\mathrm{d}\Phi(x). \tag{7}$$

If Φ is also regular, then

$$E[\xi]=\int_0^1\Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{8}$$

The independence of uncertain variables was defined by Liu [15]. The uncertain variables $\xi_1,\xi_2,\ldots,\xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^n(\xi_i\in B_i)\right\}=\bigwedge_{i=1}^n\mathcal{M}\left\{\xi_i\in B_i\right\} \tag{9}$$

for any Borel sets $B_1,B_2,\ldots,B_n$. More generally, the independence of uncertain vectors was given by Liu [19].

Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. Assume the function $f(x_1,x_2,\cdots,x_n)$ is strictly increasing with respect to $x_1,x_2,\ldots,x_m$ and strictly decreasing with respect to $x_{m+1},x_{m+2},\ldots,x_n$. Liu [16] showed that $\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)). \tag{10}$$

In addition, Liu and Ha [20] proved that the uncertain variable ξ has an expected value

$$E[\xi]=\int_0^1 f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{11}$$

2.2 Chance Theory

Let $(\Gamma,\mathcal{L},\mathcal{M})$ be an uncertainty space and let $(\Omega,\mathcal{A},\Pr)$ be a probability space. Then the product $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ is called a *chance space*. Essentially, it is another triplet,

$$(\Gamma\times\Omega,\mathcal{L}\times\mathcal{A},\mathcal{M}\times\Pr) \tag{12}$$

where $\Gamma\times\Omega$ is the universal set, $\mathcal{L}\times\mathcal{A}$ is the product σ-algebra, and $\mathcal{M}\times\Pr$ is the product measure.

The universal set $\Gamma\times\Omega$ is clearly the set of all ordered pairs of the form (γ,ω), where $\gamma\in\Gamma$ and $\omega\in\Omega$. That is,

$$\Gamma\times\Omega=\left\{(\gamma,\omega)\,|\,\gamma\in\Gamma,\omega\in\Omega\right\}. \tag{13}$$

The product σ-algebra $\mathcal{L}\times\mathcal{A}$ is the smallest σ-algebra containing measurable rectangles of the form $\Lambda\times A$, where $\Lambda\in\mathcal{L}$ and $A\in\mathcal{A}$. Any element Θ in $\mathcal{L}\times\mathcal{A}$ is called an *event* in the chance space. Then the chance measure of Θ was defined by Liu [21] as

$$\mathrm{Ch}\{\Theta\}=\int_0^1\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta\}\geq x\}\mathrm{d}x. \tag{14}$$

Liu [21] proved that the chance measure $\mathrm{Ch}\{\Theta\}$ is a monotone increasing function of Θ and

$$\mathrm{Ch}\{\Lambda\times A\}=\mathcal{M}\{\Lambda\}\times\Pr\{A\} \tag{15}$$

for any $\Lambda\in\mathcal{L}$ and any $A\in\mathcal{A}$. Especially, it holds that

$$\mathrm{Ch}\{\emptyset\}=0,\quad \mathrm{Ch}\{\Gamma\times\Omega\}=1. \tag{16}$$

Liu [21] also proved that the chance measure is self-dual. That is, for any event Θ, we have

$$\mathrm{Ch}\{\Theta\}+\mathrm{Ch}\{\Theta^c\}=1. \tag{17}$$

In addition, Hou [10] verified that the chance measure is subadditive. That is, for any countable sequence of events $\Theta_1, \Theta_2, \ldots$, we have

$$\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty}\Theta_i\right\} \le \sum_{i=1}^{\infty}\mathrm{Ch}\{\Theta_i\}. \tag{18}$$

An uncertain random variable was defined by Liu [21] as a function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B.

Let ξ be an uncertain random variable on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, and let B be a Borel set. Liu [21] showed that $\{\xi \in B\}$ is an uncertain random event with chance measure

$$\mathrm{Ch}\{\xi \in B\} = \int_0^1 \Pr\left\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid \xi(\gamma, \omega) \in B\} \ge x\right\} \mathrm{d}x. \tag{19}$$

Liu [21] also proved that $\mathrm{Ch}\{\xi \in B\}$ is a monotone increasing function of B and

$$\mathrm{Ch}\{\xi \in \emptyset\} = 0, \quad \mathrm{Ch}\{\xi \in \Re\} = 1. \tag{20}$$

Furthermore, for any Borel set B, we have

$$\mathrm{Ch}\{\xi \in B\} + \mathrm{Ch}\{\xi \in B^c\} = 1. \tag{21}$$

The chance distribution of an uncertain random variable ξ was defined by Liu [21] as

$$\Phi(x) = \mathrm{Ch}\{\xi \le x\} \tag{22}$$

for any $x \in \Re$. A sufficient and necessary condition for chance distribution was verified by Liu [21]. That is, a function $\Phi : \Re \to [0, 1]$ is a chance distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$. The chance inversion theorem [21] says that if ξ is an uncertain random variable with continuous chance distribution Φ, then for any real number x, we have

$$\mathrm{Ch}\{\xi \le x\} = \Phi(x), \quad \mathrm{Ch}\{\xi \ge x\} = 1 - \Phi(x). \tag{23}$$

Assume $\eta_1, \eta_2, \cdots, \eta_m$ are independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and $\tau_1, \tau_2, \ldots, \tau_n$ are independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$, respectively. Liu [22] proved that the uncertain random variable

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \tag{24}$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \tag{25}$$

where $F(x; y_1, \cdots, y_m)$ is the uncertainty distribution of uncertain variable $f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)$ for any real numbers $y_1, \ldots, y_m$.

Assume $\eta_1, \eta_2, \ldots, \eta_m$ are independent Boolean random variables, i.e.,

$$\eta_i = \begin{cases} 1 \text{ with probability measure } a_i \\ 0 \text{ with probability measure } 1 - a_i \end{cases} \tag{26}$$

for $i = 1, 2, \ldots, m$, and $\tau_1, \tau_2, \ldots, \tau_n$ are independent Boolean uncertain variables, i.e.,

$$\tau_j = \begin{cases} 1 \text{ with uncertain measure } b_j \\ 0 \text{ with uncertain measure } 1 - b_j \end{cases} \tag{27}$$

for $j = 1, 2, \ldots, n$. When f is a Boolean function (not necessarily monotone), Liu [22] showed that

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \tag{28}$$

is a Boolean uncertain random variable such that

$$\mathrm{Ch}\{\xi = 1\} = \sum_{(x_1,\ldots,x_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(x_i)\right) f^*(x_1, \cdots, x_m) \tag{29}$$

where

$$f^*(x_1,\cdots,x_m) = \begin{cases} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \text{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=0} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \text{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) \ge 0.5, \end{cases} \tag{30}$$

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (i = 1, 2, \ldots, m), \tag{31}$$

$$\nu_j(y_j) = \begin{cases} b_j, & \text{if } y_j = 1 \\ 1 - b_j, & \text{if } y_j = 0 \end{cases} \quad (j = 1, 2, \ldots, n). \tag{32}$$

In order to measure the size of an uncertain random variable ξ, an expected value was defined by Liu [21] as

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^{0} \mathrm{Ch}\{\xi \le x\}\mathrm{d}x \tag{33}$$

provided that at least one of the two integrals is finite. When ξ is an uncertain random variable with chance distribution Φ, Liu [21] showed that

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x, \tag{34}$$

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \tag{35}$$

If the chance distribution Φ is regular, then

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{36}$$

Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and let $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$, respectively. When $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a strictly increasing function or a strictly decreasing function with respect to $\tau_1, \ldots, \tau_n$, Liu [22] proved that the uncertain random variable

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \tag{37}$$

has an expected value

$$E[\xi] = \int_{\Re^m} \int_0^1 f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha))\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m).$$

3 Uncertain Random Graph

In classic graph theory, the edges and vertices are all deterministic, either exist or not. However, in practical applications, some indeterminacy factors will no doubt appear in graphs. Thus it is reasonable to assume that in a graph some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure. In order to model this problem, let us introduce the concept of *uncertain random graph* by using chance theory.

We say a graph is of order n if it has n vertices labeled by $1, 2, \ldots, n$. In this section, we assume the graph is always of order n, and has a collection of vertices,

$$\mathcal{V} = \{1, 2, \ldots, n\}. \tag{38}$$

Let us define two collections of edges,

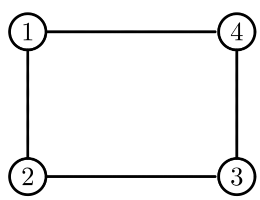
$$\mathcal{U} = \{(i,j) \mid 1 \le i < j \le n \text{ and } (i,j) \text{ are uncertain edges}\}, \tag{39}$$

$$\mathcal{R} = \{(i,j) \mid 1 \le i < j \le n \text{ and } (i,j) \text{ are random edges}\}. \tag{40}$$

Note that all deterministic edges are regarded as special uncertain ones. Then $\mathcal{U} \cup \mathcal{R} = \{(i,j) \mid 1 \le i < j \le n\}$ that contains $n(n-1)/2$ edges. We will call

$$\mathcal{T} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} \tag{41}$$

an *uncertain random adjacency matrix* if α_{ij} represent the truth values in uncertain measure or probability measure that the edges between vertices i and j exist, $i, j = 1, 2, \ldots, n$, respectively. Note that $\alpha_{ii} = 0$ for $i = 1, 2, \ldots, n$, and $\mathcal{T}$ is a symmetric matrix, i.e., $\alpha_{ij} = \alpha_{ji}$ for $i, j = 1, 2, \ldots, n$. See Figure 1.



$$\begin{pmatrix} 0 & 0.8 & 0 & 0.5 \\ 0.8 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.3 \\ 0.5 & 0 & 0.3 & 0 \end{pmatrix}$$

Figure 1: An uncertain random graph

Definition 1 *Assume $\mathcal{V}$ is the collection of vertices, $\mathcal{U}$ is the collection of uncertain edges, $\mathcal{R}$ is the collection of random edges, and $\mathcal{T}$ is the uncertain random adjacency matrix. Then the quartette $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ is said to be an uncertain random graph.*

Please note that the uncertain random graph becomes a *random graph* [3, 8] if the collection $\mathcal{U}$ of uncertain edges vanishes; and becomes an *uncertain graph* [5] if the collection $\mathcal{R}$ of random edges vanishes.

In order to deal with uncertain random graph, let us introduce some symbols. Write

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \tag{42}$$

and

$$\mathbb{X} = \left\{ X \;\middle|\; \begin{array}{l} x_{ij} = 0 \text{ or } 1, \ \text{ if } (i,j) \in \mathcal{R} \\ x_{ij} = 0, \ \text{ if } (i,j) \in \mathcal{U} \\ x_{ij} = x_{ji},\ i, j = 1, 2, \ldots, n \\ x_{ii} = 0,\ i = 1, 2, \ldots, n \end{array} \right\}. \tag{43}$$

For each given matrix

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}, \tag{44}$$

the *extension class* of Y is defined by

$$Y^* = \left\{ X \mid \begin{array}{c} x_{ij} = y_{ij}, \ \text{if } (i,j) \in \mathcal{R} \\ x_{ij} = 0 \text{ or } 1, \ \text{if } (i,j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \ i,j = 1,2,\ldots,n \\ x_{ii} = 0, \ i = 1,2,\ldots,n \end{array} \right\}. \tag{45}$$

Connectivity Index of Uncertain Random Graph

An uncertain random graph is connected for some realizations of uncertain and random edges, and disconnected for some other realizations. In order to show how likely an uncertain random graph is connected, a connectivity index of an uncertain random graph is defined as the chance measure that the uncertain random graph is connected. Let $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. It is easy to prove that the connectivity index is

$$\rho = \sum_{Y \in \mathbb{X}} \left(\prod_{(i,j) \in \mathcal{R}} \nu_{ij}(Y) \right) f^*(Y) \tag{46}$$

where

$$f^*(Y) = \begin{cases} \sup\limits_{X \in Y^*, f(X)=1} \min\limits_{(i,j) \in \mathcal{U}} \nu_{ij}(X), & \text{if } \sup\limits_{X \in Y^*, f(X)=1} \min\limits_{(i,j) \in \mathcal{U}} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{X \in Y^*, f(X)=0} \min\limits_{(i,j) \in \mathcal{U}} \nu_{ij}(X), & \text{if } \sup\limits_{X \in Y^*, f(X)=1} \min\limits_{(i,j) \in \mathcal{U}} \nu_{ij}(X) \ge 0.5, \end{cases}$$

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases} \quad (i,j) \in \mathcal{U}, \tag{47}$$

$$f(X) = \begin{cases} 1, & \text{if } I + X + X^2 + \cdots + X^{n-1} > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{48}$$

$\mathbb{X}$ is the class of matrixes satisfying (43), and Y^* is the extension class of Y satisfying (45).

Remark 1: If the uncertain random graph becomes a random graph, then the connectivity index is

$$\rho = \sum_{X \in \mathbb{X}} \left(\prod_{1 \le i < j \le n} \nu_{ij}(X) \right) f(X) \tag{49}$$

where

$$\mathbb{X} = \left\{ X \mid \begin{array}{c} x_{ij} = 0 \text{ or } 1, \ i,j = 1,2,\ldots,n \\ x_{ij} = x_{ji}, \ i,j = 1,2,\ldots,n \\ x_{ii} = 0, \ i = 1,2,\ldots,n \end{array} \right\}. \tag{50}$$

Remark 2: ([5]) If the uncertain random graph becomes an uncertain graph, then the connectivity index is

$$\rho = \begin{cases} \sup\limits_{X \in \mathbb{X}, f(X)=1} \min\limits_{1 \le i < j \le n} \nu_{ij}(X), & \text{if } \sup\limits_{X \in \mathbb{X}, f(X)=1} \min\limits_{1 \le i < j \le n} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{X \in \mathbb{X}, f(X)=0} \min\limits_{1 \le i < j \le n} \nu_{ij}(X), & \text{if } \sup\limits_{X \in \mathbb{X}, f(X)=1} \min\limits_{1 \le i < j \le n} \nu_{ij}(X) \ge 0.5 \end{cases}$$

where $\mathbb{X}$ becomes

$$\mathbb{X} = \left\{ X \mid \begin{array}{l} x_{ij} = 0 \text{ or } 1,\ i, j = 1, 2, \ldots, n \\ x_{ij} = x_{ji},\ i, j = 1, 2, \ldots, n \\ x_{ii} = 0,\ i = 1, 2, \ldots, n \end{array} \right\}. \tag{51}$$

4 Uncertain Random Network

The term *network* is a synonym for a weighted graph, where the weights may be understood as cost, distance or time consumed. In this section, we assume the uncertain random network is always of order n, and has a collection of nodes,

$$\mathcal{N} = \{1, 2, \ldots, n\} \tag{52}$$

where "1" is always the source node, and "n" is always the destination node. Let us define two collections of arcs,

$$\mathcal{U} = \{(i, j) \mid (i, j) \text{ are uncertain arcs}\}, \tag{53}$$

$$\mathcal{R} = \{(i, j) \mid (i, j) \text{ are random arcs}\}. \tag{54}$$

Note that all deterministic arcs are regarded as special uncertain ones. Let w_{ij} denote the weights of arcs (i, j), $(i, j) \in \mathcal{U} \cup \mathcal{R}$, respectively. Then w_{ij} are uncertain variables if $(i, j) \in \mathcal{U}$, and random variables if $(i, j) \in \mathcal{R}$. Write

$$\mathcal{W} = \{w_{ij} \mid (i, j) \in \mathcal{U} \cup \mathcal{R}\}. \tag{55}$$

Definition 2 *Assume $\mathcal{N}$ is the collection of nodes, $\mathcal{U}$ is the collection of uncertain arcs, $\mathcal{R}$ is the collection of random arcs, and $\mathcal{W}$ is the collection of uncertain and random weights. Then the quartette $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ is said to be an uncertain random network.*

Please note that the uncertain random network becomes a *random network* [4] if all weights are random variables; and becomes an *uncertain network* [16] if all weights are uncertain variables.

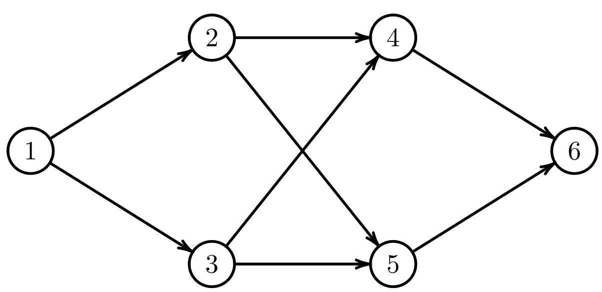


Figure 2: An uncertain random network

Figure 2 shows an uncertain random network $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ of order 6 in which

$$\mathcal{N} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}, \tag{56}$$

$$\mathcal{U} = \{(1, 2),\ (1, 3),\ (2, 4),\ (2, 5),\ (3, 4),\ (3, 5)\}, \tag{57}$$

$$\mathcal{R} = \{(4, 6),\ (5, 6)\}, \tag{58}$$

$$\mathcal{W} = \{w_{12},\ w_{13},\ w_{24},\ w_{25},\ w_{34},\ w_{35},\ w_{46},\ w_{56}\}. \tag{59}$$

Shortest Path Distribution of Uncertain Random Network

Consider an uncertain random network $(\mathcal{N},\mathcal{U},\mathcal{R},\mathcal{W})$. Assume the uncertain weights w_{ij} have uncertainty distributions Υ_{ij} for $(i,j)\in\mathcal{U}$, and the random weights w_{ij} have probability distributions Ψ_{ij} for $(i,j)\in\mathcal{R}$, respectively. Then the shortest path length from a source node to a destination node has a chance distribution

$$\Phi(x)=\int_0^{+\infty}\cdots\int_0^{+\infty}F(x;y_{ij},(i,j)\in\mathcal{R})\prod_{(i,j)\in\mathcal{R}}\mathrm{d}\Psi_{ij}(y_{ij}) \tag{60}$$

where $F(x;y_{ij},(i,j)\in\mathcal{R})$ is determined by its inverse uncertainty distribution

$$F^{-1}(\alpha;y_{ij},(i,j)\in\mathcal{R})=f(c_{ij},(i,j)\in\mathcal{U}\cup\mathcal{R}), \tag{61}$$

$$c_{ij}=\begin{cases}\Upsilon_{ij}^{-1}(\alpha), & \text{if } (i,j)\in\mathcal{U}\\ y_{ij}, & \text{if } (i,j)\in\mathcal{R},\end{cases} \tag{62}$$

and f may be calculated by the Dijkstra algorithm [1] for each given α.

Remark 3: If the uncertain random network becomes a random network, then the probability distribution of shortest path length is

$$\Phi(x)=\int_{f(y_{ij},(i,j)\in\mathcal{R})\le x}\prod_{(i,j)\in\mathcal{R}}\mathrm{d}\Psi_{ij}(y_{ij}). \tag{63}$$

Remark 4: ([6]) If the uncertain random network becomes an uncertain network, then the inverse uncertainty distribution of shortest path length is

$$\Phi^{-1}(\alpha)=f(\Upsilon_{ij}^{-1}(\alpha),(i,j)\in\mathcal{U}). \tag{64}$$

5 Conclusion

This paper proposed a concept of uncertain random graph in which some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure, and discussed the connectivity index of an uncertain random graph. In addition, this paper presented a concept uncertain random network in which some weights are random variables and others are uncertain variables, and obtained the shortest path distribution of an uncertain random network.

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