Journal of Uncertain Systems
Vol.7, No.4, pp.316-320, 2013
Online at: www.jus.org.uk

The Treewidth of the Product of Special Structure Graph

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Received 16 January 2013; Revised 27 March 2013

Abstract

The treewidth of the graph is to extend G to be a chordal supergraph $H(G \subseteq H)$ such that the cardinality of maximum cliques is minimized. It is an important parameter in theory of graph minors and VLSI layout designs, data structure, etc. The treewidth problem of general graph is shown to be NP-complete. As to the special structure graph, if its upper bound and lower bound are determined, the treewidth of the graph can be determined. With this method, this paper studies the treewidth formula of the product of 2-tree and a partial k-tree.

Keywords: product graph, treewidth, partial k-tree

1 Introduction

The notion of treewidth was introduced by Robertson and Seymour in theory of graph minors in the 1980's [9, 10]. The same parameter was rose from different application backgrounds such as VLSI layout designs, data structure, sparse matrix computation, coding theory, mathematical model in molecular biology, numerical analysis and interconnection networks, etc.[11, 2]. In a word, the treewidth has many interpretations and applications. From graph-theoretic perspective, the treewidth problem of the graph is to extend G to be a chordal supergraph $H(G \subseteq H)$ such that the cardinality of maximum cliques is minimized. From algebra-theoretic perspective, the treewidth problem corresponds to the wavefront of matrix in sloving a system of liner equations by the Gauss elimination. The treewidth problem was studied from the following aspects: the algorithmic theory and the computational complexity, the results of special graphs etc. In 1987, S. Arnborg, D. Corneil and A. Proskurowski proved that the treewidth problem of general graph was NP-complete [1]. In 1995, T.Klocks and D. Kratsch proved the treewidth of chordal bipartite graphs was solvable in polynomial [5], so the researchers began to study the results of special structure graphs. As to the special structure graph, the workers have got some studying results in Reference [13, 6, 12, 7, 14, 8, 4, 16, 15], these results are still fewer than the other labeling problems. For some of the known special structure graphs, we can use to decompose and reduce theorem for its treewidth, and if its upper bound and lower bound are known, the treewidth of the graph can be determined. In Reference [14], J. Yuan studied treewidth of the product of a tree and a partial k-tree under some given conditions. In Reference [4], A. Feng studied treewidth of the product of K_3 and a partial k-tree under some conditions. K_3 is the simple 2-tree. In this paper, we extend this result and determine the formula of treewidth of the product of 2-tree and partial k-tree.

The paper is organized as follows. In Section 2, we give some definitions and lemmas of the treewidth, which play important roles in studying the treewidth of special graphs. In Section 3, we discuss the upper and lower bound for the treewidth of the product graph of 2-tree and partial k-tree. Finally, we determined the important result of this paper. Graphs considered in this paper are finite and simple [3], for a graph $G, V(G)$ and $E(G)$ denote the sets of its vertices and edges respectively.

2 Preliminaries

Treewidth has many interpretations in the literature, we prefer the following three definitions [9, 10, 11, 2]:

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A simple definition of treewidth is the one in terms of K-tree, which can be recursively defined as follows:

(1) the complete graph K_{k+1} is a k-tree;

(2) If G is a k-tree, then the graph obtained from G by joining a new vertex to all vertices of a subgraph K_k in G is also k-tree.

Based on the definition, we can construct 2-tree: the complete graph K_3 is the simplest 2-tree; the graph obtained from K_3 by joining a new vertex to all vertices of a subgraph K_2 in K_3 is also 2-tree.

Definition 1 *The treewidth of graph G, denoted by $TW(G)$, is the minimum integer k such that G is a subgraph of a k-tree. That is*

$$TW(G) = min\{k : G \subseteq H, H \ is \ a \ k - tree\}.$$

A graph is called a chordal graph if every cycle of length greater than three has a chord. Clearly, a k-tree is a connected chordal graph such that its clique number (i.e., the maximum size of a clique) is k+1.

If G is a graph such that $TW(G) = k$, we call G a partial k-tree.

By Reference [10] the treewidth of a graph can be defined equivalently by the following way.

Definition 2 *A tree-decomposition of a graph G is a pair (T, X) where T is a tree and $X = \{X_i : i \in I\}$ is a family of subsets of $V(G)$, with the following properties:*

(i) $\cup X_i = V(G)$;

(ii) For $uv \in E(G)$, there exists $i \in I$ such that $\{u, v\} \subseteq X_i$;

(iii) For $i, j, k \in I$, if j is on the path of T between i and k, then $X_i \cap X_k \subseteq X_j$.
The treewidth of G is defined by

$$TW(G) = \min_{(T,X)} \max_i |X_i| - 1.$$

It is easy to see the following equivalent version of treewidth:

$$TW(G) = \min\{\omega(H) - 1 : G \subseteq H, V(G) = V(H), H \ is \ a \ chordal \ graph\},$$

where $\omega(H)$ is the cardinality of a maximum clique of G.

So the treewidth problem turns into choral expansion problem. Reference [2] defines forward bandwidth based on the frontier branch viewpoint, and proved that the forward bandwidth is equal to the treewidth. Another equivalent version of treewidth is as follows:

Definition 3 *Let G be a simple graph, $S \subseteq V(G)$. Let*

$$N_G(S) = \{u \in V(G) \setminus S : \exists V \in S, such \ that \ uv \in E(G)\}$$

is a neighbor set of G. A labeling of G is a bijection $f : V(G) \to \{1, 2, \cdots, n\}$, which $n = |V(G)|$. For a given labeling f of G, and for $1 \leq k \leq n$, set

$$S_k(G, f) = \{u \in V(G) : 1 \leq f(u) \leq k\}.$$

$\hat{S}_k(G, f)$ is defined as the vertex set of the frontier branch of the induced subgraph $G[S_k(G, f)]$ which contains $f^{-1}(k)$ branch. In this labeling f, the forward bandwidth of graph G is defined as

$$B_1^*(G, f) = \max |N(\hat{S}_k(G, f))|.$$

The forward bandwidth of graph is defined as

$$B^*(G, f) = min\{B_1^*(G, f) : f \ is \ the \ lableing\} = \min_{k \in f} \max_{1 \leq i \leq n} |N(\hat{S}_k(G, f))|.$$

These definitions play different roles in studying the treewidth of special graph.

Definition 4 *Let G and H be two graphs. The product $G \times H$ of G and H is defined as follows:*

(1) $V(G \times H) = \{(x, y) : x \in V(G), y \in V(H))\}$;

(2) (x_1, y_1) and (x_2, y_2) are adjacent in $G \times H$ if and only if $x_1 = x_2$ and $y_1y_2 \in E(H)$ or $y_1 = y_2$ and $x_1x_2 \in E(G)$.

In order to study the treewidth of the product of 2-tree and partial tree, we need the following lemmas.

Lemma 1 ([11]) *Let (T, X) with $|V(T)| \geq 2$ be G a tree-decomposition. For each $t \in V(T)$, let G_t be a connected subgraph of G with $V(G_t) \cap X_t = \phi$. Then there exist $t, t' \in V(T)$, adjacent in T, such that $X_t \cap X_{t'}$ separates $V(G_t)$ and $V(G_{t'})$ in G.*

Lemma 2 ([11]) *Let G and H be two graphs, if H is isomorphic to a minor of G, then*

$$TW(H) \leq TW(G).$$

Lemma 3 ([6]) *Suppose that G_1 and G_2 are two graphs, then*

$$TW(G_1 \times G_2) \leq min\{|V(G_1)| \times TW(G_2), |V(G_2)| \times TW(G_1)\}.$$

Lemma 4 ([14]) *Suppose $TW(G) \leq k$. Let (T, X) is a tree decomposition of G such that $TW(G, T, X) \leq k$, and for $t, t' \in V(T)$, then if $t \neq t', |X_t \cap X_{t'}| < min\{|X_t)|, |X_{t'}|\}$.*

Lemma 5 (Menger's Theorem [3]) *Let G be a graph, let s and t be two vertices, and $k \in N$, then there are k edge-disjoint $s - t -$ paths if only if after deleting any $k - 1$ edges t is still reachable from s.*

3 Main Results

In this section, we first study the upper and lower bound for the treewidth of the product graph of 2-tree and partial k-tree.

Lemma 6 *Suppose that G is 2-tree with $|V(G)| = m \geq 4, H$ is a k-connected partial k-tree with $|V(H)| = n \geq 4$. If there is $S \subseteq V(H)$ such that $|S| = k$, and both S and $V(H) \setminus S$ are vertex cuts of H, then*

$$TW(G \times H) \leq min\{2n, mk\}.$$

Proof: It follows from Lemma 3 immediately.

This Lemma gives the upper bound for the treewidth of the product graph of 2-tree and partial k-tree.

Note that G is 2-tree with $|V(G)| = m \geq 4, H$ is a k-connected partial k-tree with $|V(H)| = n \geq 4$, there is $S \subseteq V(H)$ such that $|S| = k$, and both S and $V(H) \setminus S$ are vertex cuts of H, the following proposition is true.

Lemma 7 *Let (T, X) be a tree decomposition of $G \times H$ such that $TW(G \times H; T, X) = TW(G \times H)$. If there is a certain $t \in V(T)$ such that $X_t \cap V(G_{v_t}) \neq \phi$ for every $v \in V(H)$, $|X_t| = 2n \leq mk$, then there is a branch in $(G \times H) \setminus X_t$ such that every $x \in X_t$ is contained in a triangle.*

Proof: Let (T, X) be a tree decomposition of $G \times H$ such that

$$TW(G \times H; T, X) = TW(G \times H).$$

By Lemma 4, there exist $t, t' \in V(T)$ with

$$t \neq t', |X_t \cap X_{t'}| < min\{|X_t)|, |X_{t'}|\}.$$

This is possible.

Now suppose that there is a certain $t \in V(T)$ such that $X_t \cap V(G_{v_t}) \neq \phi$ for every $v \in V(H)$. Because it is 2-tree, then $|X_t| \geq 2n$.

We suppose $|X_t| = 2n \leq mk$, then for any $v \in V(H)$, $|X_t \cap V(G_{v_t}| \geq 2$.

If, for each $v \in V(T)$, there is a certain $v_t \in V(H)$, then $H_u \cong H$ for every $u \in V(G)$. The two cases exist as follows:

(i) There exists $u_0 \in V(G)$ such that $|X_t \cap V(H_{u_0})| \leq k-1$ or $|X_t \cap V(H_{u_0})| = k$, and $H_{u_0} \setminus X_t \cap V(H_{u_0})$ is connected.

In this case, because H is k-connected, $H_{u_0} \setminus X_t \cap V(H_{u_0})$ is connected.

Let C_0 be a component of $G \times H \setminus X_t$ such that $V(H_{u0}) \setminus X_t \in V(C_0)$, and for every $x \in X_t \cap V(H_{u_0})$, there must be a certain vertex $x' \in V(H_{u_0}) \setminus X_t$ such that $xx' \in E(H_{u_0}) \subseteq E(G \times H)$.

For two vertices $(u_1, v_1), (u_2, v_2) \in X_t \backslash V(H_{u_0})$ are only two points in $X_t \cap V(G_v)$, So $(u_1, v), (u_2, v), (u_0, v) \in V(H_{u_0}) \setminus X_t$, they form a triangle. In a word, every $x \in X_t$ is contained in a triangle xyz.

(ii) For every $u \in V(G)$, $|X_t \cap V(H_u)| \geq k$ and $H_u \setminus X_t \cap V(H_u)$ are not connected.

In this case, because $|X_t| = 2n \leq mk$, $|X_t \cap V(H_{u_0})| = k$ for $u \in V(G)$. This also implies that $X_t \cap V(H_u)$ is the minimum vertex cut of H_u for $u \in V(G)$.

By the condition $m \geq 4$, let $u \in V(G)$ is a vertex of degree 2 in G, $u_1, u_2 \in V(G) \setminus u$, and $uu_1, uu_2 \in E(G)$.

If $S_t = \{v \in V(H) : (u_i, v) \in X_t, i = 1, 2\}$, then $|S_1| = |S_2| = k$, and both S_1 and S_2 are minimum vertex cut for H. So there exist $v_1 \in S_1, v_2 \in V(H) \setminus S_1 \cup S_2$ such that $v_1, v_2 \in V(H)$.

If $v_1, v_2 \in V(H)$, then $Q = \{(u_1, v_1), (u_2, v_1), (v_1, v_2)\}$ is a triangle in $G \times H \setminus X_t$.

Lemma 7 has been proved.

The following Lemma will give the lower bound for the treewidth of the product graph of 2-tree and partial k-tree.

Lemma 8 *Suppose that G is 2-tree with $|V(G)| = m \geq 4, H$ is a k-connected partial k-tree with $|V(H)| = n \geq 4$. If there is $S \subseteq V(H)$ such that $|S| = k$, and both S and $V(H) \setminus S$ are vertex cuts of H, then*

$$TW(G \times H) \geq min\{2n, mk\}.$$

Proof: Let (T, X) be a tree decomposition of $G \times H$ such that

$$TW(G \times H; T, X) = TW(G \times H).$$

By Lemma 4, there exist $t, t' \in V(T)$ with

$$t \neq t', |X_t \cap X_{t'}| < min\{|X_t)|, |X_{t'}|\}.$$

This is possible.

For $v \in V(H)$, let G_v be the induced subgraph of $G \times H$ with $V(G_v) = \{(u, v)|u \in V(G)\}$.

If, for each $v \in V(T)$, there is a certain $v_t \in V(H)$ such that $X_t \cap V(G_{v_t}) = \phi$, by Lemma 1, then there exist $t, t' \in V(T)$ and $V_t, V_{t'} \in V(H)$, such that $X_t \cap X_{t'}$ separates $V(G_{v_t})$ and $V(G_{v'_t})$ in $G \times H$. Note that

$$V(G_{v_t}) \cong V(G_{v'_t}) \cong G$$

and H is k-connected, there are at least mk internally vertex-disjoint paths of $G \times H$ between $V(G_{vt})$ and $V(G_{vt'})$. By Menger's theorem, also note that

$$(X_t \cup X_{t'}) \cap (V(G_{v_t}) \cup V(G_{v'_t})) = \phi.$$

We must have $|X_t \cap X_{t'}| \geq mk$. Hence we have

$$TW(G \times H; T, X) \geq |X_t| - 1 \geq |X_t \cap X_{t'}| \geq mk.$$

The result holds.

Now suppose that there is a certain $t \in V(T)$ such that $|X_t \cap V(G_{vt}) \neq \phi$ for every $v \in V(H)$.

Because it is 2-tree, then $|X_t| \geq 2n$. If $|X_t| \geq 2n+1$ or $mk+1$, the result clearly holds.

Without loss of generality, we suppose $|X_t| = 2n \leq mk$, then $|X_t \cap V(G_{vt}| \geq 2$, for any $v \in V(H)$.

For $u \in V(G)$, let H_u be the induced subgraph of $G \times H$, and $V(H_u) = \{(u, v)|v \in V(H)\}$.

If, for each $v \in V(T)$, there is a certain $v_t \in V(H)$, then $H_u \cong H$ for every $u \in V(G)$. By Lemma 7, there is a branch in $(G \times H) \setminus X_t$ to make every $x \in X_t$ are contained in a triangle.

Denote $E(X_t) = \{xy \notin E(G \times H) : x, y \in X_t, x \neq y\}$, let $F = G \times H + E(X_t)$, then X_t is a clique of F, and (T, X) is a tree decomposition for F, so $TW(G \times H) = TW(F)$.

K_{2n+1} is isomorphism for F, by Lemma 2, so $TW(F) = TW(G \times H) \geq 2n$.

Thus we complete the proof.

To summarize Lemma 6 and Lemma 8, the following result is clear immediately.

Theorem *Suppose that G is 2-tree with $|V(G)| = m \geq 4, H$ is a k-connected partial k-tree with $|V(H)| = n \geq 4$, and $S \subseteq V(H), |S| = k$, both S and $V(H) \setminus S$ are vertex cuts of H, then*

$$TW(G \times H) = min\{2n, mk\}.$$

The same method can be applied to more families of special graphs such as the product graph of circle and a patrial k-tree. Because the treewidth problem is NP-complete, our study has to be concentrated on some typically special cases, the lower and upper bounds, etc. Furthermore, in combinatorial optimization problem, the objective functions usually have two types, one is the minimum sum, another is minimum in the maximal value. The treewidth problem belongs to the latter type. Exchanging a form can generate new problem, and this new problem is waiting to be studied and discussed. Most results are expected.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No.10971053, 11226047), and the Natural Science Foundation of Education Department of Henan Province (No. 2009A110005), the Henan Youth Teacher Foundation(2008).

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