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Existence and Uniqueness Theorems for Fuzzy Differential Equations

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Abstract

Fuzzy differential equation is an important tool to deal with dynamic systems in fuzzy environments. However, it is difficult to find the solutions to all fuzzy differential equations. In this paper, methods to solve linear fuzzy differential equations and reducible fuzzy differential equations are given. Moreover, existence and uniqueness theorem for homogeneous fuzzy differential equations are obtained.

Keywords: fuzzy variable, fuzzy process, fuzzy differential equation, Liu process

1 Introduction

Fuzziness is a basic type of uncertainty in real world. To describe a set without definite boundary, fuzzy set was initiated by Zadeh [25] in 1965, whose membership function indicates the degree of an element belonging to it. In order to measure a fuzzy event, a self-duality credibility measure was introduced by Liu and Liu [14] in 2002. Since an axiomatic foundation for credibility theory was constructed by Liu [10] and a sufficient and necessary condition was given by Li and Liu [9], credibility theory became a perfect mathematical system. A survey of credibility theory can be found in Liu [11], and interested reader may consult the book [12].

In order to describe dynamic fuzzy phenomena, a fuzzy process was proposed by Liu [13], and then a fuzzy integral and a chain rule were introduced by Liu [13]. For the importance and usefulness, they were renamed as Liu process, Liu integral and Liu formula, which are the counterparts of Brownian motion, Ito integral and Ito formula. Some researches concerning Liu process have be done. You [24] extended Liu process, Liu integral and Liu formula to the case of multi-dimensional and higher-order Liu formula was given. Complex Liu process was studied by Qin [19]. Moreover, Liu process is Lipschitz continuous and has finite variation, which was proved by Dai [2]. These results provide a theoretical foundation for the application of Liu process. As for the applications of Liu process, some researches were made. Under the assumption that stock price is modeled by geometric Liu process, a basic stock model was proposed by Liu [13], which is called Liu's stock model. By studying this model, Qin and Li [20] deduced the European option pricing formula. For different application background, two extended stock models were given by Gao [4] and Peng [18], respectively. Besides, Liu process was applied to fuzzy optimal control and an optimal equation was obtained by Zhu [26].

Differential equations with fuzzy parameters were studied in many literatures, such as [5, 6, 23, 3, 21, 8, 22, 15, 16, 7, 1, 17]. To solve such differential equations, we can use classical method without requiring fuzzy calculus. In 2008, fuzzy differential equation was defined by Liu [13] as

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{1.1}$$

where C_t is a standard Liu process, and f, g are some given functions. Such a fuzzy differential equation is a type of differential equations driven by Liu process. Its solution is a fuzzy process. The fuzziness of X_t results, on the one hand, from the initial condition, and on the other hand, from the noise generated by Liu process. It is just the fuzzy counterpart of stochastic differential equation. In this paper, we will study the fuzzy differential equation proposed by Liu [13].

In Section 2 of this paper, some concepts and results of Liu process will be given as a preliminaries. The solutions of some special fuzzy differential equations such as linear fuzzy differential equations and reducible fuzzy differential equations will be discussed in Section 3. In Section 4, three existence and uniqueness theorems are obtained. In the end, a brief summary is given in Section 4.

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2 Preliminaries

A fuzzy process $X_t(\theta)$ is defined as a function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers. In other words, $X_t*(\theta)$ is a fuzzy variable for each t^*, where t is time and θ is a point in credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. $X_t*(\theta)$ is a function of t for any given $\theta^* \in \Theta$, such a function is called a sample path of $X_t(\theta)$. For simplicity, we use the symbol X_t to replace $X_t(\theta)$ in the following section.

A fuzzy process X_t is called continuous if the sample paths of X_t are all continuous functions of t for almost all $\theta \in \Theta$.

Definition 2.1 (Liu [13]) *A fuzzy process C_t is said to be a Liu process if*
(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{t+s} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \ -\infty < x < +\infty.$$

The Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

Definition 2.2 (Liu [13]) *Let X_t be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\triangle = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the Liu integral of X_t with respect to C_t is defined as follows:

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\triangle \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limitation exists almost surely and is a fuzzy variable. In this case, X_t is called Liu integrable.

In order to simplify the calculation of Liu integral, Liu formula was introduced by Liu [13], which corresponds to Ito formula.

Theorem 2.1 (Liu Formula, Liu [13]) *Let C_t be a standard Liu process, and let $h(t, x)$ be a continuously differentiable function. If fuzzy process X_t is given by $\mathrm{d}X_t = u_t \mathrm{d}t + v_t \mathrm{d}C_t$, where u_t, v_t are absolutely integrable fuzzy process and Liu integrable fuzzy process, respectively. Define $Y_t = h(t, X_t)$. Then*

$$\mathrm{d}Y_t = \frac{\partial h}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial h}{\partial x}(t, X_t)\mathrm{d}X_t.$$

When there exist multiple fuzzy processes in a system, multi-dimensional Liu formula was given.

Theorem 2.2 (Multi-dimensional Liu Formula, You [24]) *Let $C_t = (C_{1t}, C_{2t}, \cdots, C_{mt})^T$ be an m-dimensional standard Liu process, and let*

$$\boldsymbol{h}(t, x_1, x_2, \cdots, x_n) = (h_1(t, x_1, x_2, \cdots, x_n), h_2(t, x_1, x_2, \cdots, x_n), \cdots, h_p(t, x_1, x_2, \cdots, x_n))^T,$$

where $h_i(t, x_1, x_2, \cdots, x_n)$, $i = 1, 2, \cdots, p$ are multivariate continuously differentiable functions. If fuzzy process $(X_{1t}, X_{2t}, \cdots, X_{nt})^T$ is given by

$$\begin{cases} \mathrm{d}X_{1t} = u_{1t}\mathrm{d}t + v_{11t}\mathrm{d}C_{1t} + \cdots + v_{1mt}\mathrm{d}C_{mt} \\ \quad \vdots \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ \mathrm{d}X_{nt} = u_{nt}\mathrm{d}t + v_{n1t}\mathrm{d}C_{1t} + \cdots + v_{nmt}\mathrm{d}C_{mt} \end{cases}$$

where u_{it}, v_{ijt} are absolutely integrable fuzzy processes and Liu integrable fuzzy processes, respectively. Define $\boldsymbol{Y}_t = \boldsymbol{h}(t, X_{1t}, X_{2t}, \cdots, X_{nt})$. Then

$$\mathrm{d}\boldsymbol{Y}_t = \begin{pmatrix} \mathrm{d}Y_{1t} \\ \mathrm{d}Y_{2t} \\ \vdots \\ \mathrm{d}Y_{pt} \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial t}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial h_1}{\partial x_i}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}X_{it} \\ \frac{\partial h_2}{\partial t}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial h_2}{\partial x_i}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}X_{it} \\ \vdots \\ \frac{\partial h_p}{\partial t}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial h_p}{\partial x_i}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}X_{it} \end{pmatrix}.$$

Differential equations are used to describe the evolution of a dynamic systems. Next, we will discuss the fuzzy differential equations proposed by Liu [13] when a fuzzy noise is introduced.

Definition 2.3 (Liu [13]) *Suppose C_t is a standard Liu process, and f and g are some given functions. Let X_t be an unknown fuzzy process. Then the equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{2.1}$$

is called a fuzzy differential equation. A solution is a fuzzy process X_t that satisfies (2.1) *identically in t.*

3 Solutions to Some Special Fuzzy Differential Equations

It follows from the definition of fuzzy differential equation and solution to fuzzy differential equation that a solution is some function of the given Liu process C_t. When $g(t, X_t) = 0$, fuzzy differential equation (1.1) becomes an ordinary differential equation.

3.1 Linear Fuzzy Differential Equations

Linear fuzzy differential equations form a class of fuzzy differential equations that can be solved explicitly.

Definition 3.1 *Let C_t be a standard Liu process, and let X_t be an unknown fuzzy process. Suppose that a, b, c, d are constants. Then the equation*

$$\mathrm{d}X_t = (aX_t + b)\mathrm{d}t + (cX_t + d)\mathrm{d}C_t \tag{1}$$

is called a linear fuzzy differential equation.

Definition 3.2 *Let C_t be a standard Liu process, and let X_t be an unknown fuzzy process. Suppose that a, c are constants. Then the equation*

$$\mathrm{d}X_t = aX_t\mathrm{d}t + cX_t\mathrm{d}C_t \tag{2}$$

is called a homogeneous linear fuzzy differential equation.

Lemma 3.1 *The solution of the homogeneous linear fuzzy differential equation* (2) *is*

$$X_t = \exp(at + cC_t). \tag{3}$$

Proof: Since equation (2) is equivalent to

$$\mathrm{d}\ln X_t = a\mathrm{d}t + c\mathrm{d}C_t,$$

$$\ln \frac{X_t}{X_0} = \int_0^t a\mathrm{d}s + \int_0^t c\mathrm{d}C_s,$$

i.e.

$$X_t = X_0 \exp(at + cC_t).$$

It is obvious that $X_0 = 1$ and

$$X_t = \exp(at + cC_t).$$

The lemma is proved.

Lemma 3.1 states that the solution of a homogeneous linear fuzzy differential equation is a geometric Liu process.

Theorem 3.1 *The solution of the linear fuzzy differential equation* (1) *is*

$$X_t = \exp(at + cC_t)\left(X_0 + \int_0^t \frac{b}{\exp(as + cC_s)}\mathrm{d}s + \int_0^t \frac{d}{\exp(as + cC_s)}\mathrm{d}C_s\right). \tag{4}$$

Proof: Let

$$X_t = U_t V_t, \tag{5}$$

where

$$\mathrm{d}U_t = aU_t\mathrm{d}t + cU_t\mathrm{d}C_t, \quad \mathrm{d}V_t = a_t\mathrm{d}t + b_t\mathrm{d}C_t.$$

It follows from Lemma 3.1 that

$$U_t = \exp(at + cC_t).$$

Therefore, $U_0 = 1$ and $V_0 = X_0$. Taking the differentials of both sides of (5), we have

$$\mathrm{d}X_t = U_t\mathrm{d}V_t + V_t\mathrm{d}U_t$$

by using multi-dimensional Liu formula, i.e.,

$$\mathrm{d}X_t = (U_t a_t + aV_t U_t)\mathrm{d}t + (U_t b_t + cV_t U_t)\mathrm{d}C_t. \tag{6}$$

Comparing (6) with (1), we can choose coefficients a_t and b_t such that $X_t = U_t V_t$. The desired coefficients satisfy equations

$$U_t a_t = b, \text{ and } U_t b_t = d.$$

Thus, the solution of the fuzzy differential equation (1) is

$$X_t = U_t\left(X_0 + \int_0^t \frac{b}{\exp(as + cC_s)}\mathrm{d}s + \int_0^t \frac{d}{\exp(as + cC_s)}\mathrm{d}C_s\right).$$

Example 3.1: Let C_t be a standard Liu process. Then the fuzzy differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t \tag{7}$$

has a solution $X_t = at + bC_t$, where a, b are two constants. It is just a Liu process.

Example 3.2: Let C_t be a standard Liu process. Then the fuzzy differential equation

$$\mathrm{d}X_t = aX_t\mathrm{d}t + b\mathrm{d}C_t \tag{8}$$

has a solution $X_t = \exp(at + bC_t)$, where a, b are two constants. It is just a geometric Liu process.

Multiplying both sides of (8) by the integrating factor $\exp(-at)$, we have

$$\exp(-at)\mathrm{d}X_t - a\exp(-at)X_t\mathrm{d}t = b\exp(-at)\mathrm{d}C_t,$$

i.e.,

$$\mathrm{d}\left(\exp(-at)X_t\right) = b\exp(-at)\mathrm{d}C_t.$$

Thus

$$X_t = \exp(-at)\left(X_0 + b\int_0^t \exp(-as)\mathrm{d}C_s\right).$$

Example 3.3: It follows from Theorem 3.1 that the solution of the fuzzy differential equation

$$\mathrm{d}X_t = (aX_t + b)\mathrm{d}t + c\mathrm{d}C_t \tag{9}$$

is

$$X_t = \exp(at)\left(X_0 + \frac{b}{a}(1 - \exp(-at)) + c\int_0^t \exp(-as)\mathrm{d}C_s\right).$$

3.2 General Linear Fuzzy Differential Equations

Just as general linear stochastic differential equations, the solution of a general linear fuzzy differential equation can be solved. Thus, the community take more interested in this type of equations.

Definition 3.3 *Let C_t be a standard Liu process, and let X_t be an unknown fuzzy process. Suppose that $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ are some given continuous fuzzy processes. Then*

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t \tag{10}$$

is called a general linear fuzzy differential equation.

Definition 3.4 *Let C_t be a standard Liu process, and let X_t be an unknown fuzzy process. Suppose that u_{1t}, u_{2t}, v_{2t} are some given continuous fuzzy processes. Then*

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + v_{2t}\mathrm{d}C_t \tag{11}$$

is called a general linear fuzzy differential equation with additive noise.

Definition 3.5 *Let C_t be a standard Liu process, and let X_t be an unknown fuzzy process. Suppose that u_t, v_t are some given continuous fuzzy processes. Then*

$$\mathrm{d}X_t = u_tX_t\mathrm{d}t + v_tX_t\mathrm{d}C_t \tag{12}$$

is called a general homogeneous linear fuzzy differential equation.

Next, we will discuss the solution of general linear fuzzy differential equations (11) and (12).

Lemma 3.2 *The solution of the general linear fuzzy differential equation with additive noise* (11) *is*

$$X_t = z_t^{-1}\left(X_0 + \int_0^t u_{2s}z_s\mathrm{d}s + \int_0^t v_{2s}z_s\mathrm{d}C_s\right),$$

where

$$z_t = \exp\left(-\int_0^t u_{1s}\mathrm{d}s\right).$$

Proof: Following the method of Example 3.2, let

$$X_t = z_t^{-1}Y_t,$$

where

$$z_t = \exp\left(-\int_0^t u_{1s}\mathrm{d}s\right).$$

Then

$$Y_t = z_tX_t, \quad Y_0 = X_0.$$

It follows from Liu formula that

$$\mathrm{d}Y_t = u_{2t}z_t\mathrm{d}t + v_{2t}z_t\mathrm{d}C_t.$$

Thus,

$$Y_t = X_0 + \int_0^t u_{2s}z_s\mathrm{d}s + \int_0^t v_{2s}z_s\mathrm{d}C_s.$$

Therefore, the solution of (11) is

$$X_t = z_t^{-1}\left(X_0 + \int_0^t u_{2s}z_s\mathrm{d}s + \int_0^t v_{2s}z_s\mathrm{d}C_s\right).$$

Lemma 3.3 *The solution of* (12) *is*

$$X_t = X_0 \exp\left(\int_0^t u_{1s} \mathrm{d}s + \int_0^t v_{1s} \mathrm{d}C_s\right).$$

Proof: In a similar proof of Lemma 3.1, the lemma is obtained easily.

Example 3.4: Let C_t be a standard Liu process. Then $X_t = C_t/(1+t)$ is a solution of

$$\mathrm{d}X_t = -\frac{X_t}{1+t}\mathrm{d}t + \frac{1}{1+t}\mathrm{d}C_t \tag{13}$$

In fact, it follows from Liu formula that

$$\mathrm{d}X_t = -\frac{C_t}{(1+t)^2}\mathrm{d}t + \frac{1}{1+t}\mathrm{d}C_t,$$

which is equivalent to the fuzzy differential equation (13).

Example 3.5: Let C_t be a standard Liu process. Suppose that u_t is a Liu integrable fuzzy process, and $f(x)$ is an absolutely integrable and continuous function. Then the solution of the fuzzy differential equation

$$\mathrm{d}X_t = u_t \mathrm{d}t + f(C_t)\mathrm{d}C_t \tag{14}$$

is

$$X_t = X_0 + \int_0^t u \mathrm{d}t + \int_0^t f(C_s)\mathrm{d}C_s.$$

Example 3.6: Let C_t be a standard Liu process, and a a constant. Suppose that u_t is a Liu integrable fuzzy process, and $f(x)$ is an absolutely integrable and continuous function. Considering the solution of the fuzzy differential equation

$$\mathrm{d}X_t = \frac{a - X_t}{1-t}\mathrm{d}t + \mathrm{d}C_t, \quad 0 \le t < 1. \tag{15}$$

Taking a substitution

$$X_t = (y_t)^{-1} Y_t,$$

where $y_t = 1/(1-t)$, it follows from Liu formula that

$$\mathrm{d}Y_t = \mathrm{d}(y_t X_t) = \frac{a}{(1-t)^2}\mathrm{d}t + \frac{\mathrm{d}C_t}{1-t}.$$

Then

$$Y_t = X_0 + \frac{a}{1-t} - a + \int_0^t \frac{\mathrm{d}C_s}{1-s}.$$

Furthermore,

$$X_t = X_0(1-t) + at + (1-t)\int_0^t \frac{\mathrm{d}C_s}{1-s}.$$

Example 3.7: Let C_t be a standard Liu process, and u_t a continuous fuzzy process. Considering the solution of fuzzy differential equation

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + f(C_t) X_t \mathrm{d}C_t, \tag{16}$$

where $f(x)$ is an absolutely integrable and continuous function.

Since

$$\ln\frac{X_t}{X_0} = \int_0^t u_s \mathrm{d}s + \int_0^t f(C_s)\mathrm{d}C_s,$$

the solution of the fuzzy differential equation (16) is

$$X_t = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t f(C_s)\mathrm{d}C_s\right).$$

Example 3.8: Let $(C_{1t}, C_{2t}, \cdots, C_{nt})$ be an n-dimensional standard Liu process, and X_t a fuzzy process defined by

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + X_t \sum_{k=1}^{n} \alpha_k \mathrm{d}C_{kt}, \tag{17}$$

where u_t is a Liu integrable fuzzy process, α_k are constants for all k. We immediately have

$$\ln\left(\frac{X_t}{X_0}\right) = \int_0^t u_s \mathrm{d}s + \int_0^t \sum_{k=1}^{n} \alpha_k \mathrm{d}C_{ks},$$

i.e., the solution of the fuzzy differential equation (17) is

$$X_t = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t \sum_{k=1}^{n} \alpha_k \mathrm{d}C_{kt}\right) = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \sum_{k=1}^{n} \alpha_k C_{kt}\right).$$

Integrating the solving method of the general linear fuzzy differential equation with that of general homogeneous linear fuzzy differential equation, we can solve the general linear fuzzy differential equation.

Theorem 3.2 *The solution of the general linear fuzzy differential equation* (10) *is*

$$X_t = U_t\left(X_0 + \int_0^t \frac{u_{2s}}{U_s} \mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s} \mathrm{d}C_s\right),$$

where

$$U_t = \exp\left(\int_0^t u_{1s} \mathrm{d}s + \int_0^t v_{1s} \mathrm{d}C_s\right).$$

Proof: It is easily obtained in a similar proof of Theorem 3.1.

Example 3.9: Let C_t be a standard Liu process, and u_t, v_t are two continuous fuzzy processes. Consider the solution of the fuzzy differential equation

$$\mathrm{d}X_t = u_t \mathrm{d}t + v_t X_t \mathrm{d}C_t. \tag{18}$$

It follows from Theorem 3.2 that

$$X_t = \exp\left(\int_0^t v_s \mathrm{d}C_s\right)\left(X_0 + \int_0^t u_s \exp\left(-\int_0^s v_r \mathrm{d}C_r\right) \mathrm{d}s\right).$$

3.3 Reducible Fuzzy Differential Equations

For certain fuzzy differential equation, the solution can be found by performing a substitution which reduces the given fuzzy differential equation to a general linear fuzzy differential equation.

Definition 3.6 *A fuzzy differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is called reducible if there exists a substitution $Y_t = U(t, X_t)$ such that it deduces to a general linear fuzzy differential equation

$$\mathrm{d}Y_t = (u_{1t} Y_t + u_{2t})\mathrm{d}t + (v_{1t} Y_t + v_{2t})\mathrm{d}C_t$$

where $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ are chosen as fuzzy processes satisfying the conditions

$$\begin{cases} u_{2t} + u_{1t} U = \dfrac{\partial U}{\partial t} + \dfrac{\partial U}{\partial x} f \\ v_{2t} + v_{1t} U = \dfrac{\partial U}{\partial x} g. \end{cases}$$

Example 3.10: Let C_t be a standard Liu process. Suppose a fuzzy process X_t is defined by

$$\mathrm{d}X_t = rX_t(a - X_t)\mathrm{d}t + bX_t\mathrm{d}C_t,\ X_0 = x > 0 \tag{19}$$

where a, b, r are constants and $a > 0$.

Letting $Y_t = (t+1)/X_t$, we have $Y_0 = x^{-1}$ and the fuzzy differential equation (19) reduces to

$$\mathrm{d}Y_t = \left(r(t+1) + \left(\frac{1}{t+1} - ra\right)Y_t\right)\mathrm{d}t - bY_t\mathrm{d}C_t.$$

Thus,

$$Y_t = (t+1)\exp(-rat - bC_t)\left(Y_0 + \int_0^t \frac{r}{\exp(-ras - bC_s)}\mathrm{d}s\right),$$

i.e.,

$$X_t = \frac{\exp(rat + bC_t)}{x^{-1} + r\int_0^t \exp(ras + bC_s)\mathrm{d}s}.$$

Example 3.11: Let C_t be a standard Liu process. Suppose a fuzzy process X_t is defined by

$$\mathrm{d}X_t = k(a - \ln X_t)X_t\mathrm{d}t + bX_t\mathrm{d}C_t,\ X_0 = x > 0 \tag{20}$$

where k, a, b are positive constants.

Letting $Y_t = \ln X_t$, the fuzzy differential equation (20) is transformed as

$$\mathrm{d}Y_t = k(a - Y_t)\mathrm{d}t + b\mathrm{d}C_t.$$

Thus

$$Y_t = \exp(-kt)\ln x + a + b\exp(-kt)\int_0^t \exp(ks)\mathrm{d}C_s,$$

i.e.,

$$X_t = \exp\left(\exp(-kt)\ln x + a + b\exp(-kt)\int_0^t \exp(ks)\mathrm{d}C_s\right). \tag{21}$$

Remark 3.1: The process (21) is called a geometric mean reversion fuzzy process.

Example 3.12: Let C_t be a standard Liu process. Suppose a fuzzy process X_t is defined by

$$\mathrm{d}X_t = (aX_t^n + bX_t)\mathrm{d}t + cX_t\mathrm{d}C_t,\ X_0 = x > 0, \tag{22}$$

where a, b, c are positive constants.

Letting $Y_t = X_t^{1-n}$, the fuzzy differential equation (22) reduces to

$$\mathrm{d}Y_t = (1-n)\left((a + bY_t)\mathrm{d}t + cY_t\mathrm{d}C_t\right).$$

It follows from Theorem 3.1 that

$$Y_t = (1-n)\exp(bt + cC_t)\left(x^{1-n} + a\int_0^t \exp(-bs - cC_s)\mathrm{d}s\right),$$

i.e.,

$$X_t = \exp\left(\frac{bt + cC_t}{1-n}\right)\left((1-n)x^{1-n} + a(1-n)\int_0^t \exp(-bs - cC_s)\mathrm{d}s\right)^{\frac{1}{1-n}}.$$

Example 3.13: Let C_t be a standard Liu process. Suppose a fuzzy process X_t is defined by

$$\mathrm{d}X_t = (a\exp(cX_t) + b)\mathrm{d}t + c\mathrm{d}C_t, \tag{23}$$

where a, b, c are positive constants.

Letting $Y_t = \exp(-cX_t)$, the fuzzy differential equation (23) reduces to

$$\mathrm{d}Y_t = (-ac - bcY_t)\mathrm{d}t - cY_t\mathrm{d}C_t.$$

It follows from Theorem 3.1 that

$$Y_t = \exp(-bct - cC_t)\left(\exp(-cX_0) - ac\int_0^t \exp(bcs + cC_s)\mathrm{d}s\right),$$

i.e.,

$$X_t = (bt + C_t) - \frac{1}{c}\ln\left(\exp(-cX_0) - ac\int_0^t \exp(bcs + cC_s)\mathrm{d}s\right).$$

Besides above fuzzy differential equations, there is a type of special equation in nonlinear fuzzy differential equations, it can be reduced to ordinary differential equations with fuzzy coefficients by some substitution.

For example, the fuzzy differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + v_tX_t\mathrm{d}C_t, \tag{24}$$

where $f : R \times R \to R$ is a given continuous function and $v_t = v_t(\theta)$ is a continuous fuzzy process.

In order to solve the fuzzy differential equation (24), we define

$$U_t = U_t(\theta) = \exp\left(-\int_0^t v_s\mathrm{d}C_s\right).$$

Then the fuzzy differential equation (24) is equivalent to

$$\mathrm{d}(U_tX_t) = U_tf(t, X_t)\mathrm{d}t$$

by using multi-dimensional Liu formula.

Letting

$$Y_t(\theta) = U_t(\theta)X_t(\theta),$$

the fuzzy differential equation (24) reduces to

$$\mathrm{d}Y_t(\theta) = U_t(\theta)f(t, U_t^{-1}(\theta)Y_t(\theta))\mathrm{d}t, \quad Y_0 = X_0. \tag{25}$$

In fact, the equation (25) is an ordinary differential equation for each fixed θ. Thus, the fuzzy differential equation (24) can be solved.

Example 3.14: Let C_t be a standard Liu process. The fuzzy process X_t is given by

$$\mathrm{d}X_t = \frac{1}{X_t}\mathrm{d}t + aX_t\mathrm{d}C_t, \quad X_0 = x > 0, \tag{26}$$

where a is a constant.

Letting $U_t = \exp(-aC_t)$ and $Y_t = U_tX_t$, we have

$$X_t = \exp(aC_t)Y_t,\ Y_0 = x$$

and

$$\mathrm{d}Y_t = \frac{\exp(-2aC_t)}{Y_t}\mathrm{d}t.$$

Then

$$X_t = \exp(aC_t)\left(x^2 + 2\int_0^t \exp(-2aC_s)\mathrm{d}s\right)^{\frac{1}{2}}.$$

Example 3.15: Let C_t be a standard Liu process. The fuzzy process X_t is given by

$$\mathrm{d}X_t = X_t^b \mathrm{d}t + aX_t \mathrm{d}C_t, \quad X_0 = x > 0, \tag{27}$$

where a and b are constants, and $b \leq 1$.

Letting $U_t = \exp(-aC_t)$ and $Y_t = U_t X_t$, we have

$$X_t = \exp(aC_t)Y_t, \quad Y_0 = x,$$

and

$$\mathrm{d}Y_t = \frac{\exp(-2aC_t)}{Y_t}\mathrm{d}t.$$

Next, we discuss the solution of the fuzzy differential equation (27) in two cases.

Case 1: If $b = 1$, the solution of the fuzzy differential equation (27) is

$$X_t = x\exp(t + aC_t).$$

Case 2: Otherwise, the solution of fuzzy differential equation (27) is

$$X_t = \exp(aC_t)\left(x^{1-b} + (1-b)\int_0^t \exp(a(b-1)C_s)\mathrm{d}s\right)^{\frac{1}{1-b}}.$$

4 Existence and Uniqueness Theorems

In this section, existence and uniqueness theorems for the homogeneous fuzzy differential equations will be given.

Definition 4.1 *Let C_t be a standard Liu process, and f, g are two given functions. Then*

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}C_t \tag{28}$$

is called a homogeneous fuzzy differential equation.

Theorem 4.1 *Suppose that $f : \Re \to \Re$ is a Lipschitz continuous function. Then the solution of the fuzzy differential equation*

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \mathrm{d}C_t \tag{29}$$

exists and is unique.

Proof: The integral equation

$$X_t = X_0 + \int_0^t f(X_s)\mathrm{d}s + C_t \tag{30}$$

is well defined by the continuity of f. Thus the equation (29) is equivalent to the equation (30).

Next, we will try a successive approximation method to construct a solution of the integral equation (30).

Define $X_0^{(0)} = X_0$, and then

$$X_t^{(n+1)} = x + \int_0^t f(X_s^{(n)})\mathrm{d}s + C_t \quad (t \geq 0) \tag{31}$$

for $n = 0, 1, 2, \ldots$. Next write

$$D_t^{(n)} = \max_{0\leq s\leq t}\left|X_s^{(n+1)} - X_s^{(n)}\right|, \ n = 0, 1, 2, \ldots,$$

and notice that C_t is a continuous fuzzy process, we have

$$D_t^{(0)} = \max_{0\leq s\leq t}\left|\int_0^s f(x)\mathrm{d}r + C_s\right| \leq M$$

for all times $0 \le t \le T$, where M depends on θ.

Since f is a Lipschitz continuous function, there exists a constant L such that

$$|f(x) - f(y)| \le L|x - y|$$

for all $x, y \in \Re$. We now claim that

$$D_t^{(n)} \le M \frac{L^n}{n!} t^n$$

for all n and $0 \le t \le T$. To see this note that

$$\begin{aligned} D_t^{(n)} &= \max_{0 \le s \le t} \left| \int_0^s f(X_r^{(n)}) - f(X_r^{(n-1)}) \mathrm{d}r \right| \\ &\le L \int_0^t \left| X_r^{(n)} - X_r^{(n-1)} \right| \mathrm{d}r \\ &\le L \int_0^t D_s^{(n-1)} \mathrm{d}s \\ &\le L \int_0^t M \frac{L^{n-1}}{(n-1)!} s^{n-1} \quad \text{by the induction assumption} \\ &= M \frac{L^n}{n!} t^n. \end{aligned}$$

In view of the claim, for $m \ge n$, we have

$$\max_{0 \le t \le T} |X_t^{(m)} - X_t^{(n)}| \le M \sum_{k=n}^{\infty} \frac{L^k T^k}{k!} \to 0, \text{as } n \to \infty.$$

Thus $X_t^{(n)}$ converges uniformly to a limit process X_t for $0 \le t \le T$ and almost every θ. It is easy to check that X_t is the solution of (30). The existence is proved.

Next we will prove that the solution of the integral equation (30) is unique. Assume that X_{1t} and X_{2t} are all solutions of (30). Then

$$|X_{1t} - X_{2t}| = \left| \int_0^t f(X_{1s}) - f(X_{2s}) \right| \le \int_0^t L|X_{1s} - X_{2s}| \mathrm{d}s.$$

It follows from Gronwall inequality that $|X_{1t} - X_{2t}| = 0$ for almost all θ, i.e., $X_{1t} = X_{2t}$. The uniqueness is proved.

Corollary 4.1 *Suppose that $f : \Re \to \Re$ is a continuously differentiable function and $|f'| \le L$. Then the solution of the fuzzy differential equation*

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \mathrm{d}C_t$$

exists and is unique.

Proof: It follows from the continuity of f that there exists a point x_0 such that

$$|f(x) - f(y)| = |f'(x_0)||x - y| \le L|x - y|$$

for any $x, y \in \Re$. It implies that f is a Lipschitz continuous function, the result can be obtained from Theorem 4.1.

Theorem 4.2 *(Existence and Uniqueness Theorem) Suppose that $f, g : \Re \to \Re$, and $g, f/g$ are all Lipschitz continuous functions. Then the solution of the homogeneous fuzzy differential equation* (28)

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}C_t, \quad X_0 = x$$

exists and is unique.

Proof: First, let us solve the ordinary differential equation

$$u'(z) = g(u(z)), \quad u(y) = x. \tag{32}$$

Since g is a Lipschitz continuous function, the solution of the equation (32) exists and is unique. Without loss of generality, denote the solution of (32) by $u(x)$.

Letting $X_t = u(Y_t)$, it follows from Liu formula that

$$\mathrm{d}X_t = u'(Y_t)\mathrm{d}Y_t = g(u(Y_t))\mathrm{d}Y_t = g(X_t)\mathrm{d}Y_t. \tag{33}$$

Comparing (28) with (33), we have

$$\mathrm{d}Y_t = \frac{f(X_t)}{g(X_t)}\mathrm{d}t + \mathrm{d}C_t, \quad Y_0 = y. \tag{34}$$

Since f/g is a Lipschitz continuous function, according to Theorem 4.1, the solution of the fuzzy differential equation (34) exists and is unique. Then the solution of the homogeneous fuzzy differential equation (28) exists and is unique.

5 Conclusions

This paper was mainly to discuss the solutions of some special fuzzy differential equations based on Liu process. The methods were provided to solve linear fuzzy differential equation, general linear fuzzy differential equation and reducible fuzzy differential equation. Then existence and uniqueness theorems for homogeneous fuzzy differential equation were given.

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