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A Note on the Evolution of Solutions of a System of Ordinary Differential Equations with Fuzzy Initial Conditions and Fuzzy-Inputs

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Abstract

In this paper, we investigate the solutions of n-dimensional nonlinear ordinary differential equations with fuzzy initial conditions and fuzzy inputs. In our analysis we employ the tools of levelwise approach and real analysis. We have shown with a complete proof that the solutions of such systems are described by a system of $2n$-ordinary differential equations with crisp initial conditions and crisp inputs corresponding to the end points of the alpha cuts of fuzzy states. A numerical example is given to illustrate the results.

Keywords: fuzzy differential equations, fuzzy number, fuzzy initial condition, extension principle

1 Introduction

In the theory of fuzzy differential equations many approaches have been suggested to define a solution for a fuzzy differential equation. Among them Hukuhara approach or H-differentiability approach [4, 8, 9, 13, 14, 17, 18], differential inclusion [2, 7, 15], and the levelwise or α-cut approach [3, 5, 16, 20] and the references there in, are the most referred in the literature. Unlike other approaches levelwise approach allows to translate a system of fuzzy differential equations in to a system of ordinary differential equations corresponding to the end points of α-cuts of the state. Due to this reason levelwise approach has attracted the attention of many researchers (for example, [1],[5],[11],[19],[20]) to study and analyze the various aspects of solutions of fuzzy differential equations. Recently in the literature other alternative formulations of levelwise approach are also suggested in order to overcome certain disadvantages like unboundedness of the diameter of the solution $x(t)$ of a fuzzy differential equation with time (see [1]).

The early work in the direction of levelwise approach is initiated by Siekkala [16]. In [16], it is shown that by using the Zadeh's extension principle, the evolution of fuzzy initial value problem $\dot{x}(t) = f(t, x(t)), x(t_0) = x_0$, where $f : R^+ \times \mathbb{R} \to \mathbb{R}$ is measurable in t and continuous in x, and x_0 is a fuzzy number on $\mathbb{R}$, is expressed by a system of 2-ordinary differential equations corresponding to the end points of the α-cuts of x. However, the proof is missing. Following the idea of Siekkala [16], Xu et el. [19] has considered the linear fuzzy initial value problem of the type $\dot{x}(t) = Ax(t), x(t_0) = X_0$, where A is $n \times n$ real matrix and X_0 is an n-vector of fuzzy numbers on $\mathbb{R}$. In [19] it is stated (see [19, Lemma 3.2]) that the evolution of aforesaid system can be described by a system of $2n$-ordinary differential equations corresponding to the end points of the α-cuts of x. Similar results are stated in [3] for the systems of the type $\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = X_0$, where A and B are $n \times n$ and $n \times m$ real matrices, respectively and X_0, $u(t)$ are the vectors of fuzzy numbers on $\mathbb{R}$.

These results motivate us to investigate the solutions of a general nonlinear system with fuzzy initial conditions and fuzzy inputs by using the levelwise approach. To the best of our knowledge we feel that a detailed proof of the results describing the evolution of solutions of such systems in terms of α-cuts is missing in the literature. The novelty of our results lies in their applicability to nonlinear systems with uncertain but fuzzily modelled initial conditions and control variables. Our results can also be regarded as the generalizations of some of the results in the literature (for example, [3],[16],[19]).

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In this paper, we consider an n-dimensional nonlinear fuzzy initial value problem

$$\dot{x}(t) = f(t, x(t), u(t)), x(t_0) = x_0, \tag{1}$$

where $f : R^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function which is measurable in t and is continuous in x and u. The fuzzy initial condition x_0 is an n-vector of fuzzy numbers on $\mathbb{R}$, and the fuzzy input $u(t)$ is an m-vector of fuzzy numbers on $\mathbb{R}$. By using the extension principle, we prove that the evolution of system (1) can be described by a system of $2n$-ordinary differential equations corresponding to the end points of the α-cuts of x.

The organization of the paper is as follows. In Section 2, we state some of the preliminary results from fuzzy set theory. In Section 3, we establish that the evolution of system (1) is described by a system of $2n$-crisp differential equations corresponding to the end points of the α-cuts of x. In Section 4, an example is given to demonstrate the results obtained. We conclude the paper in Section 5.

2 Preliminaries

Let $\mathbb{R}$ be the set of all real numbers. $\mathbb{R}^+$ denotes the set of all nonnegative real numbers. $\mathbb{N}$ is the set of all natural numbers. For $n \in \mathbb{N}$, $\mathbb{E}^n$ is the set of all n-dimensional vectors of fuzzy numbers on $\mathbb{R}$. $\mathbb{F}(X)$ denotes the set of all fuzzy sets defined in a set X.

Definition 2.1 ([23]). *If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x)) | x \in X\}$, where $\mu_A(x)$ is called the membership function or grade of membership of x in A. The range of membership function is a subset of nonnegative real numbers whose supremum is finite.*

Definition 2.2. *By a fuzzy number on $\mathbb{R}$, we mean a mapping $\mu : \mathbb{R} \longrightarrow [0,1]$ with the following properties:*

1. *μ is upper semi-continuous.*
2. *μ is fuzzy convex, that is, $\mu(\alpha x + (1-\alpha)y) \geq min(\mu(x), \mu(y))$ for all $x, y \in \mathbb{R}$.*
3. *μ is normal, that is, there exists $x_0 \in X$ such that $\mu(x_0) = 1$.*
4. *Closure of the support of μ is compact, that is, $cl(x \in \mathbb{R} : \mu(x) > 0)$ is compact in $\mathbb{R}$.*

Definition 2.3. *Let A be a fuzzy number or a fuzzy set defined on the universe X which could, in general, be a subset of $\mathbb{R}^n$. The α-cut or α-level set of A is denoted by A^α or $[A]_\alpha$ and is defined as $A^\alpha = \{x \in X : A(x) \geq \alpha\}$ for $\alpha \in (0,1]$. For $\alpha = 0$, the 0-cut of A is defined as the closure of union of all nonzero α-cuts of A. That is*

$$A^0 = \overline{\bigcup_{\alpha \in (0,1]} A^\alpha}.$$

It is well known that for every $\mathcal{A} \in \mathbb{E}^1$, the α-level sets of $\mathcal{A}$ are closed and bounded intervals defined by $[\mathcal{A}]_\alpha := [\underline{\mathcal{A}^\alpha}, \overline{\mathcal{A}^\alpha}]$, where $\underline{\mathcal{A}^\alpha}$, $\overline{\mathcal{A}^\alpha}$ are called the lower α-cut and the upper α-cut of $\mathcal{A}$, respectively. Every fuzzy set can be uniquely represented in terms of its α-cuts. The following decomposition theorem of fuzzy sets depicts this fact.

Theorem 2.4 ([10]). *Let X be an arbitrary set. Then for every $A \in \mathbb{F}(X)$ we have $A = \bigcup_{\alpha \in [0,1]} {}_\alpha A$, in which $\bigcup$ denotes the standard fuzzy union and ${}_\alpha A$ are defined by*

$${}_\alpha A(x) = \begin{cases} \alpha & if\, x \in A^\alpha \\ 0 & if\, x \in X \setminus A^\alpha. \end{cases} \tag{2}$$

It can be easily shown that a fuzzy number $\mu \in \mathbb{E}^1$ is characterized by the endpoints of the intervals μ^α. Thus a fuzzy number $\mu \in \mathbb{E}^1$ can be identified by a parameterized triple

$$\{(\underline{\mu^\alpha}, \overline{\mu^\alpha}, \alpha)|, \alpha \in [0,1]\}.$$

The following lemma provides a characterization of fuzzy numbers on $\mathbb{R}$.

Lemma 2.5 ([6])**.** *Assume that $I = [0, 1]$, and $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ satisfy the conditions:*

(a) $a : I \to \mathbb{R}$ *is a bounded increasing function.*

(b) $b : I \to \mathbb{R}$ *is a bounded decreasing function.*

(c) $a(1) \leq b(1)$.

(d) For $0 < k \leq 1, \lim_{\alpha \to k^-} a(\alpha) = a(k)$ *and* $\lim_{\alpha \to k^-} b(\alpha) = b(k)$.

(e) $\lim_{\alpha \to 0^+} a(\alpha) = a(0)$ *and* $\lim_{\alpha \to 0^+} b(\alpha) = b(0)$.

Then $\mu : \mathbb{R} \to I$ *defined by*

$$\mu(x) = \sup\{\alpha \mid a(\alpha) \leq x \leq b(\alpha)\}$$

is a fuzzy number with parametrization given by $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$. *Moreover, if* $\mu : \mathbb{R} \to I$ *is a fuzzy number with parametrization given by* $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$, *then functions* $a(\alpha)$ *and* $b(\alpha)$ *satisfy conditions* $(a) - (e)$.

Let $F : T = [a, b] \to \mathbb{E}^1$ be a fuzzy process and suppose that the parametric form of $F(t)$ is represented by

$$F(t) = \{(F_1(t, \alpha), F_2(t, \alpha), \alpha) : \alpha \in [0, 1], t \in T\}.$$

The Seikkala [16] derivative $\dot{F}(t)$ of a fuzzy function $F(t)$ is defined by

$$\dot{F}(t) = \{(\dot{F}_1(t, \alpha), \dot{F}_2(t, \alpha), \alpha) : \alpha \in [0, 1], t \in T\}$$

provided that the above equation defines a fuzzy number.

3 Evolution of the Solutions of Fuzzified Nonlinear Ordinary Differential Equations

Let $f : X \longrightarrow Y$ be a given function. Then the Zadeh's extension principle leads to the extension $f^* : \mathbb{F}(X) \longrightarrow \mathbb{F}(Y)$ of f defined as follows:

$$f^*(A)(y) = \sup_{x \mid f(x) = y} A(x) \qquad \forall A \in \mathbb{F}(X).$$

The function f^* is called fuzzy extension of f. Further suppose, $f : X_1 \times X_2 \times \cdots \times X_n \longrightarrow Y$. Then the extension principle leads to the following definition of $f^* : \mathbb{F}(X_1) \times \mathbb{F}(X_2) \times \cdots \times \mathbb{F}(X_n) \longrightarrow \mathbb{F}(Y)$. For $A_i \in \mathbb{F}(X_i)$, $1 \leq i \leq n$, we have

$$f^*(A_1, A_2, \ldots, A_n)(y) = \sup_{x = (x_1, x_2, \ldots, x_n) \mid f(x) = y} (A_1(x_1) \wedge A_2(x_2) \wedge \cdots \wedge A_n(x_n)), \tag{3}$$

where $\wedge$ denotes the standard fuzzy intersection operator. Further details on the extension principle could be found in [22, 12, 21, 10, 23].

We now consider a fuzzy initial value problem in which a fuzzy valued input parameter $u(\cdot)$ is introduced. Let $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a given function such that $f(t, x, u)$ is measurable with respect to t and continuous with respect to x and u. Then the fuzzy initial value problem is formulated as

$$\dot{x}(t) = f(t, x(t), u(t)), \, x(t_0) = x_0 \in \mathbb{E}^n, \, u(t) \in \mathbb{E}^m, \, t_0 \in \mathbb{R}^+. \tag{4}$$

In the following theorem, we will show that by using the extension principle the evolution of the solutions of system (4) is described by a $2n$-differential equations corresponding to the end points of the α-cuts of x.

Theorem 3.1. *Let* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *and* $[x_k(t)]_\alpha = [\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ *be the* α*-cut of* $x_k(t)$ *for* $1 \leq k \leq n$. $u(t) = (u_1(t), u_2(t), \ldots, u_m(t))$ *and* $[u_k(t)]_\alpha = [\underline{u_k^\alpha}(t), \overline{u_k^\alpha}(t)]$ *be the* α*-cut of* $u_k(t)$ *for* $1 \leq k \leq m$.

The evolution of system (4) is described by the following set of $2n$-levelwise differential equations corresponding to the end points of the α-cuts of $x(t)$. That is, for each $\alpha \in [0,1]$ and $1 \le k \le n$, we have

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = min(f_k(t,z,w) : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = max(f_k(t,z,w) : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases} \tag{5}$$

where $z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$, $w = (w_1, w_2, \dots, w_m) \in \mathbb{R}^m$ and f_k is the k^{th} component of f.

Proof. Since $x(t) \in \mathbb{E}^n$, $u(t) \in \mathbb{E}^m$ for all $t \ge 0$, therefore, $f_k(\cdot,\cdot,\cdot)$ is extended by using the extension principle (3). The fuzzy extension $f_k^*(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{E}^n \times \mathbb{E}^m \longrightarrow \mathbb{E}$ of f_k is defined as follows:

$$\begin{aligned} & f_k^*(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t))(y) \\ & = \sup_{(\tau,\nu)=(\tau_1,\dots,\tau_n,\nu_1,\dots,\nu_m) \mid f_k(t,\tau,\nu)=y} \{min(x_1(t)(\tau_1), \dots, x_n(t)(\tau_n), u_1(t)(\nu_1), \dots, u_m(t)(\nu_m))\}. \end{aligned} \tag{6}$$

Fuzzification of system (4) yields for each $1 \le k \le n$,

$$\dot{x_k}(t) = f_k^*(t, x(t), u(t)),\ x_k(t_0) = x_{0k}, \tag{7}$$

in which f_k^* is the fuzzy extension of f_k. By taking the α-cuts from both sides of (7), we have

$$[\dot{x_k}(t)]_\alpha = [f_k^*(t, x(t), u(t))]_\alpha, [x_k(t_0)]_\alpha = [x_{0k}]_\alpha. \tag{8}$$

From [8, Th. 5.2] we have that

$$[\dot{x_k}(t)]_\alpha = [\dot{\underline{x_k^\alpha}}(t), \dot{\overline{x_k^\alpha}}(t)]. \tag{9}$$

We shall now show the following

$$\begin{aligned} [f_k^*(t, x(t), u(t))]_\alpha = [& min(f_k(t,z,w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha), \\ & max(f_k(t,z,w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha)]. \end{aligned} \tag{10}$$

Let $s \in [min(f_k(t,z,w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha), max(f_k(t,z,w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha)]$. Since f_k is continuous with respect to x and u, therefore there exist $\tau_i \in [x_i(t)]_\alpha$ for $1 \le i \le n$ and $\nu_j \in [u_j(t)]_\alpha$ for $1 \le j \le m$, such that $f_k(t, \tau_1, \dots, \tau_n, \nu_1, \dots, \nu_m) = s$. From which it follows that

$$\begin{aligned} & f_k^*(t, x(t), u(t))(s) \\ & = \sup_{(\tau,\nu)=(\tau_1,\dots,\tau_n,\nu_1,\dots,\nu_m) \mid f_k(t,\tau,\nu)=s} \{min(x_1(t)(\tau_1), \dots, x_n(t)(\tau_n), u_1(t)(\nu_1), \dots, u_m(t)(\nu_m))\} \ge \alpha. \end{aligned} \tag{11}$$

Hence $s \in [f_k^*(t, x(t), u(t))]_\alpha$.

Conversely, assume that $s \in [f_k^*(t, x(t), u(t))]_\alpha$, which implies that

$$f_k^*(t, x(t), u(t))(s) \ge \alpha. \tag{12}$$

Define

$$\rho := \sup_{(\tau,\nu)=(\tau_1,\dots,\tau_n,\nu_1,\dots,\nu_m) \mid f_k(t,\tau,\nu)=s} \{min(x_1(t)(\tau_1), \dots, x_n(t)(\tau_n), u_1(t)(\nu_1), \dots, u_m(t)(\nu_m))\}. \tag{13}$$

Clearly $\rho \ge \alpha$. From Eq.(13) it follows that, there exists a sequence $\{\varsigma^p\}$, where

$$\varsigma^p := (\tau_1^p, \dots, \tau_n^p, \nu_1^p, \dots, \nu_m^p)$$

with $f_k(t, \tau_1^p, \dots, \tau_n^p, \nu_1^p, \dots, \nu_m^p) = s$, such that

$$\rho = \sup_{\varsigma=(\tau_1,\dots,\tau_n,\nu_1,\dots,\nu_m) \in \{\varsigma^p\}} \{min(x_1(t)(\tau_1), \dots, x_n(t)(\tau_n), u_1(t)(\nu_1), \dots, u_m(t)(\nu_m))\}. \tag{14}$$

From Eq.(14) it follows that

$$\limsup_{p\to\infty} x_1(t)(\tau_1^p) \geq \limsup_{p\to\infty}(min(x_1(t)(\tau_1^p),\ldots,x_n(t)(\tau_n^p),u_1(t)(\nu_1^p),\ldots,u_m(t)(\nu_m^p))) \geq \alpha. \tag{15}$$

Without loss of generality it can be assumed that $\{\tau_1^p\} \in [x_1(t)]_0$ for otherwise the entries in the sequence $\{\tau_1^p\}$ which do not belong to $[x_1(t)]_0$ can be simply ignored. Since $[x_1(t)]_0$ is compact, therefore there exists a subsequence $\{\tau_1^{p_r^{(1)}}\}$ of $\{\tau_1^p\}$(indeed a subsequence of $\{\tau_1^p\}$) such that $\tau_1^{p_r^{(1)}} \longrightarrow \tau_1^*$ as $r \to \infty$ for some $\tau_1^* \in \mathbb{R}$ and

$$\limsup_{r\to\infty} x_1(t)(\tau_1^{p_r^{(1)}}) = \limsup_{p\to\infty} x_1(t)(\tau_1^p) \geq \alpha. \tag{16}$$

Now by using upper semi-continuity of $x_1(t)$ we have

$$x_1(t)(\tau_1^*) \geq \limsup_{r\to\infty} x_1(t)(\tau_1^{p_r^{(1)}}) \geq \alpha. \tag{17}$$

Therefore, we have

$$\tau_1^* \in [x_1(t)]_\alpha. \tag{18}$$

Since $\{\varsigma^{p_r^{(1)}}\}$ is a subsequence of $\{\varsigma^p\}$, therefore we have

$$\rho = \sup_{\varsigma=(\tau_1,\ldots,\tau_n,\nu_1,\ldots,\nu_m)\in\{\varsigma^{p_r^{(1)}}\}} \{min(x_1(t)(\tau_1),\ldots,x_n(t)(\tau_n),u_1(t)(\nu_1),\ldots,u_m(t)(\nu_m))\}. \tag{19}$$

Using the same arguments as for $x_1(t)$, we will have a subsequence $\{\varsigma^{p_r^{(2)}}\}$ of $\{\varsigma^{p_r^{(1)}}\}$ such that $\tau_2^{p_r^{(2)}} \longrightarrow \tau_2^*$ as $r \longrightarrow \infty$ and $\tau_2^* \in [x_2(t)]_\alpha$.

Continuing in the same fashion, we will have nested subsequences of $\{\varsigma^{p_r^{(1)}}\}$, and points $\tau_i^* \in \mathbb{R}$ for $1 \leq i \leq n$, and $\nu_j^* \in \mathbb{R}$ for $1 \leq j \leq m$. That is,

$$\{\varsigma^{p_r^{(n+m)}}\} \subset \ldots \{\varsigma^{p_r^{(n+1)}}\} \subset \{\varsigma^{p_r^{(n)}}\} \subset \{\varsigma^{p_r^{(n-1)}}\} \ldots \subset \{\varsigma^{p_r^{(2)}}\} \subset \{\varsigma^{p_r^{(1)}}\}, \tag{20}$$

with the property that for $1 \leq i \leq n$,

$$\tau_i^{p_r^{(i)}} \longrightarrow \tau_i^* \text{ as } r \longrightarrow \infty, \text{ and } \tau_i^* \in [x_i(t)]_\alpha, \tag{21}$$

and for $1 \leq j \leq m$,

$$\nu_j^{p_r^{(n+j)}} \longrightarrow \nu_j^* \text{ as } r \longrightarrow \infty, \text{ and } \nu_j^* \in [u_j(t)]_\alpha. \tag{22}$$

Clearly $f_k(t,\tau_1^{p_r^{(n+m)}},\ldots,\tau_n^{p_r^{(n+m)}},\nu_1^{p_r^{(n+m)}},\ldots,\nu_m^{p_r^{(n+m)}}) = s$ for all $r \in \mathbb{N}$. Also, for $1 \leq i \leq n$, $\tau_i^{p_r^{(n+m)}} \longrightarrow \tau_i^*$ as $r \longrightarrow \infty$ and for $1 \leq j \leq m$, $\nu_j^{p_r^{(n+m)}} \longrightarrow \nu_j^*$ as $r \longrightarrow \infty$. Now by continuity of f_k we have

$$f_k(t,\tau_1^{p_r^{(m+n)}},\ldots,\tau_n^{p_r^{(m+n)}},\nu_1^{p_r^{(m+n)}},\ldots,\nu_m^{p_r^{(m+n)}}) \longrightarrow f_k(t,\tau_1^*,\ldots,\tau_n^*,\nu_1^*,\ldots,\nu_m^*). \tag{23}$$

Hence it follows that $f_k(t,\tau_1^*,\ldots,\tau_n^*,\nu_1^*,\ldots,\nu_m^*) = s$ with $\tau_i^* \in [x_i(t)]_\alpha$ for $1 \leq i \leq n$, and $\nu_j^* \in [u_j(t)]_\alpha$ for $1 \leq j \leq m$. Thus we must have

$$s \in [min(f_k(t,z,w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha), max(f_k(t,z,w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha)]. \tag{24}$$

Hence we have established (10). Now the proof of the theorem follows from Eqs.(8), (9) and (10). □

In [3, Lemma 1], a similar result is employed in order to compute the controllable initial fuzzy states for the linear systems of the form $\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = x_0 \in \mathbb{E}^n, u(t) \in \mathbb{E}^m$, $t_0 \in \mathbb{R}^+$, where A and B are real matrices of size $n \times n$, $n \times m$, respectively.

Corollary 3.2 ([3])**.** *Let $f(t, x(t), u(t)) = Ax(t) + Bu(t)$, where A, B are real matrices of size $n \times n$ and $n \times m$, respectively. Then, the evolution of system* (4) *is described by $2n$-differential equations corresponding to the end points of the α-cuts of $x(t)$. That is, for $1 \leq k \leq n$,*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = min((Az + Bw)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = max((Az + Bw)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases} \tag{25}$$

in which $(Az + Bw)_k$ denotes the k^{th} row of $(Az + Bw)$.

If $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be such that $f(t, x)$ is measurable with respect to t, and continuous with respect to x. Then the evolution of fuzzy initial value problem

$$\dot{x}(t) = f(t, x(t)), x(t_0) = x_0 \in \mathbb{E}^n, t_0 \in \mathbb{R}^+ \tag{26}$$

will be given by a $2n$-differential equations corresponding to the end points of the α-cuts of x. The following theorem follows immediately from Theorem 3.1.

Theorem 3.3. *Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, and $[x_k(t)]_\alpha = [\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ be the α-cut of $x_k(t)$ for $1 \leq k \leq n$. The evolution of system (26) is described by the following set of $2n$-levelwise equations corresponding to the end points of the α-cuts of $x(t)$. That is, for each $\alpha \in [0, 1]$ and $1 \leq k \leq n$, we have*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = min(f_k(t, z) : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = max(f_k(t, z) : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases} \tag{27}$$

in which $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ and f_k is the k^{th} component of f.

Proof. The proof follows exactly on the same lines as for Theorem 3.1. □

As a special case, a similar result for the linear time-invariant systems $\dot{x}(t) = Ax(t), x(t_0) = x_0 \in \mathbb{E}^n$, where A is $n \times n$ real matrix, $t_0 \in \mathbb{R}^+$, is used by Xu et al. in [19].

Corollary 3.4 ([19])**.** *Let $f(t, x(t)) = Ax(t)$, where A is a real matrix of size $n \times n$. Then, the evolution of the system* (26) *is described by $2n$-differential equations corresponding to the end points of α-cuts of $x(t)$. That is, for $1 \leq k \leq n$,*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = min((Az)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = max((Az)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases} \tag{28}$$

where $(Az)_k$ denotes the k^{th} row of (Az).

4 Numerical Example

In this section, we provide an example to explain the fuzzification procedure and the evolution of fuzzy solutions as prescribed in Theorem 3.1.

Example 1: Consider the following nonlinear fuzzy differential equation

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_2^2(t) + cos(t) + u_1(t) \\ -x_1^2(t) + sin(t) + u_2(t) \end{pmatrix}. \tag{29}$$

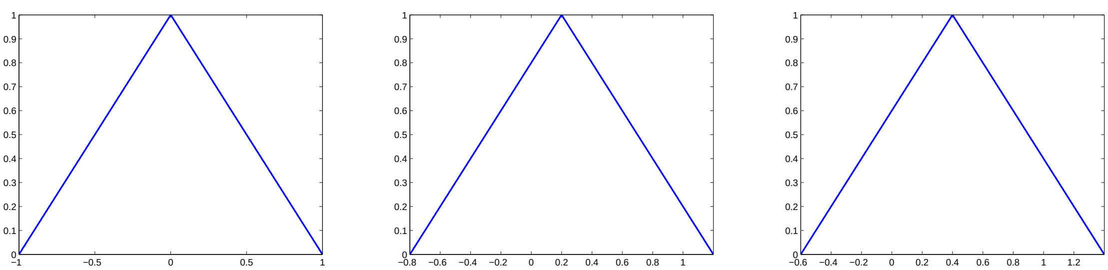
Let the initial membership functions $x_1(0) = \mu_1(s)$ and $x_2(0) = \mu_2(s)$ be defined as below:

$$\mu_1(s) = \begin{cases} s & 0 \leq s \leq 1 \\ 2 - s & 1 \leq s \leq 2, \end{cases} \qquad \mu_2(s) = \begin{cases} 2s & 0 \leq s \leq 1/2 \\ 2 - 2s & 1/2 \leq s \leq 1. \end{cases}$$

Taking the fuzzy inputs $u_1(t), u_2(t) \in \mathbb{E}$ as defined by the following fuzzy numbers:

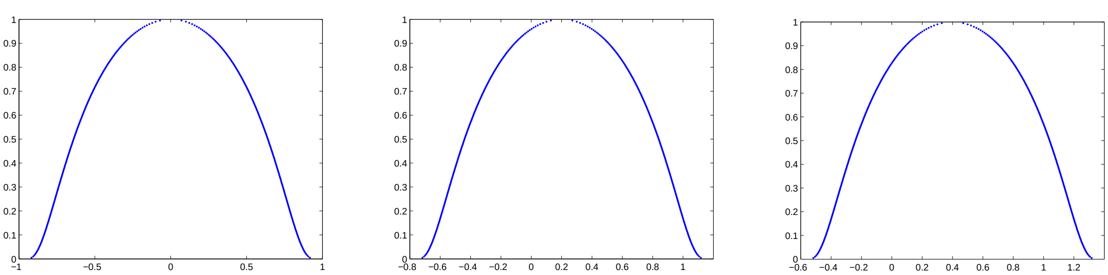
$$u_1(t)(s) = \begin{cases} s - t + 1, & t - 1 \leq s \leq t \\ -s + t + 1, & t \leq s \leq t + 1, \end{cases} \quad u_2(t)(s) = \begin{cases} ee^{-\frac{1}{1 - |s-t|^2}}, & \mid s - t \mid \leq 1 \\ 0, & \mid s - t \mid \geq 1. \end{cases}$$

The input functions $u_1(t)$ and $u_2(t)$ at various time instants are shown in Fig.1 and Fig.2, respectively. The



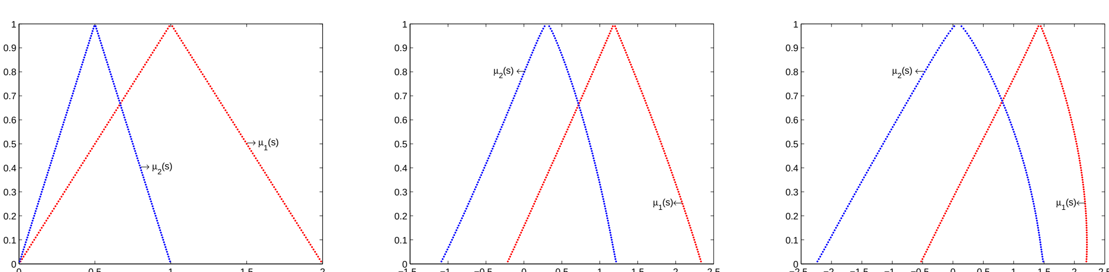
(a) u_1 at time t=0 (b) u_1 at time t=·2 (c) u_1 at time t=·4

Figure 1: Input function u_1 at various time instants



(a) u_2 at time t=0 (b) u_2 at time t=·2 (c) u_2 at time t=·4

Figure 2: Input function u_2 at various time instants



(a) at time t=0 (b) at time t=·2 (c) at time t=·4

Figure 3: Propagation of membership functions for x_1 and x_2

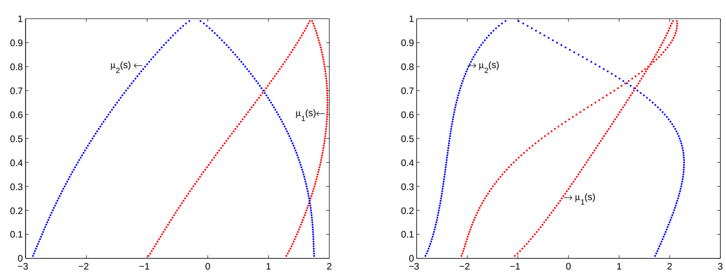
evolution of the solutions of system (29) is given by levelwise decomposed differential equations (30). For each $\alpha \in [0, 1]$,

$$\begin{pmatrix} \dot{\underline{x_1^\alpha}}(t) = -\overline{x_2^\alpha}^2(t) + cos(t) + (t + \alpha - 1) \\ \dot{\underline{x_2^\alpha}}(t) = -\overline{x_1^\alpha}^2(t) + sin(t) + (t - \mid (1 - \frac{1}{log\left(\frac{e}{\alpha}\right)})^{\frac{1}{2}} \mid) \\ \dot{\overline{x_1^\alpha}}(t) = -\underline{x_2^\alpha}^2(t) + cos(t) + (t + 1 - \alpha) \\ \dot{\overline{x_2^\alpha}}(t) = -\underline{x_1^\alpha}^2(t) + sin(t) + (t + \mid (1 - \frac{1}{log\left(\frac{e}{\alpha}\right)})^{\frac{1}{2}} \mid) \end{pmatrix}, \tag{30}$$

with the initial condition

$$[\underline{x_1^\alpha}(0), \underline{x_2^\alpha}(0), \overline{x_1^\alpha}(0), \overline{x_2^\alpha}(0)]^T = [\alpha, \alpha/2, 2 - \alpha, 1 - (\alpha/2)]^T.$$

The propagated fuzzy states at time $t = \cdot 2$ and $t = \cdot 4$ starting from the initial fuzzy state at time $t = 0$, are shown in Fig.3. It must be noted that the existence of the solutions of Eq.(29) is guaranteed only in the interval $[0, T]$ for some $T > 0$ (see [8, 17]). In this example the solutions of system (29) exist in the interval [0,T], where $\cdot 4 < T < \cdot 6$. For $t > T$ the solutions cease to exist as shown in Fig.4. It is clear from the Fig.4a



(a) at time t=·6 (b) at time t= 1

Figure 4: x_1 and x_2 at $t = \cdot 6$ and $t = 1$

that $x_1(\cdot 6) \notin \mathbb{E}$. Similarly at $t = 1$, $x_1(1) \notin \mathbb{E}$ and $x_2(1) \notin \mathbb{E}$ as indicated in Fig.4b.

5 Conclusion

We feel that the results obtained in this paper are of strong theoretical importance in the theory of fuzzy differential equations. The results of the paper can be treated as the generalizations of many existing results in the literature, for example, [3],[16],[19]. Moreover, the steps involved in the fuzzificaion of nonlinear crisp systems are illustrated in detail. Obviously, the present investigation gives more insight on the evolution of the solutions of the systems considered in this paper.

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