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Rough Modular Lattice

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Abstract

This paper presents the study of modular lattice under rough set environment and referred to herein a concept is called rough modular lattice. We study the properties of lattice in an approximation space and defined as rough lattice, rough sublattice and bounded rough lattice. We consider the approximation space by means of an equivalence relation and also we present the rough set as pair of set (lower and upper approximation sets). The aim of this paper is to study the properties of lattice in an approximation space based on Pawlak's notion of indiscernibility relation between the objects in a set. Some important results are established. Finally some examples are presented to usefulness and truthfulness the lemma and proposition.

Keywords: lattice, rough set, approximation space, rough lattice, rough modular lattice

1 Preliminaries

Lattice is a simple algebraic structure whose basic philosophy include only orders, least upper and greatest lower bounds. It is widely discussed and studied in classical algebraic theory. A poset $(L, \leq)$ is a lattice if $sup\{a, b\}$ and $inf\{a, b\}$ exist for all $a, b \in L$ [3], where $sup\{a, b\}$ and $inf\{a, b\}$ are the least upper and greatest lower bounds of a and b respectively. Here $sup\{a, b\}$ and $inf\{a, b\}$ are also denoted by $a \vee b$ and $a \wedge b$ respectively where $\vee$ and $\wedge$ are join and meet operators respectively. A meet-semi lattice $(L, \leq)$ is a poset such that for any pair (a, b) of its elements $a \wedge b \in L$. The definition of join-semi lattice is dual one. The modular inequality, $a \wedge (b \vee c) \geq b \vee (a \wedge c), \forall a, b, c \in L, a \geq b$ holds in all lattices. A modular lattice is a special type of lattice where the modular inequality becomes an equality.

2 Introduction

Rough set was first introduced by Pawlak [11] which is a framework for systematic study of incomplete knowledge. Rough set consists of two key notions: rough set approximations and information systems. Rough set approximations are defined by means of an equivalence relation namely indiscernibility relation. For every rough set, we associate two crisp sets, called lower and upper approximations and viewed as the sets of elements which certainly and possibly belong to the set.

Pawlak [12] introduced the theory of rough set as an extension of set theory for the study of incomplete information. In [5], Iwinski defined rough lattice and rough order and he described that rough set can be viewed as a pair of approximations. Again he defined the rough lattice without using any indiscernibility concept of rough set. Davey and Priestly [1] introduced the concept of lattice theory and order. Järvinen [6] proposed the lattice structure on rough sets which played an important role in rough set and Pawlak's information system. Xiao [17] introduced the notion of rough prime ideals and rough fuzzy prime ideals in semigroup. Rana and Roy [13] introduced an unique approach to form lattice by choice function in rough set. Rasouli and Davvaz [14] considered a relationship between rough sets and MV-algebra theory and introduced the notion of rough ideal with respect to an ideal of an MV-algebra, which is an extended notion of ideal in an MV-algebra. Estaji et al. [2] introduced the concept of upper and lower rough ideals in a lattice and they

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studied some properties on prime ideals. Samanta and Chakroborty [15] categorized the various generalized approaches for lower and upper approximations of a set in term of implication lattice. Yang [19] formulated a new lattice structure named as rough concept lattice. Kong et al. [7] introduced the method construction of rough lattice based on compressed matrix which can solve the redundancy of construction process and obtain corresponding rule. Wang et al. [16] studied object oriented concept lattice and property oriented concept lattice.

Several methodology have been proposed to form lattice by rough set and most of these are based on tolerance relation. We constructed the rough modular lattice based on equivalence relation which is the main motivation of our paper and established a connection between rough set and lattice structure. Also we have defined the modular lattice in the rough set environment using indiscernibility relation. Here we recall some basic properties and definitions related to lattice under the light of rough set environment.

Definition 2.1 *Let U be the set of universe and ρ be an equivalence relation on U. An equivalence class of $x(\in U)$ is denoted by $[x]_\rho$ and defined as follows: $[x]_\rho = \{y \in U : x\rho y\}$, where $x\rho y$ imply $(x, y) \in \rho$. The lower and upper approximations of $X \subseteq U$ are denoted by $A_\star(X)$ and $A^\star(X)$ respectively and defined as follows:*

$$A_\star(X) = \{x \in U : [x]_\rho \subseteq X\} \text{ and } A^\star(X) = \{x \in U : [x]_\rho \cap X \neq \phi\}.$$

The pair (U, ρ) is called an approximation space and is denoted by S. Then $A(X) = (A_\star(X), A^\star(X))$ is called the rough set of X in S.

Theorem 2.1 *Let $A(X) = (A_\star(X), A^\star(X))$ and $A(Y) = (A_\star(Y), A^\star(Y))$ be two rough sets under the approximation space $S[= (U, \rho)]$, then*
(i) $A(X) \cup A(Y) = (A_\star(X) \cup A_\star(Y), A^\star(X) \cup A^\star(Y))$;
(ii) $A(X) \cap A(Y) = (A_\star(X) \cap A_\star(Y), A^\star(X) \cap A^\star(Y))$;
(iii) $A(X) \subseteq A(Y)$ if and only if $A_\star(X) \subseteq A_\star(Y)$ and $A^\star(X) \subseteq A^\star(Y)$.

Definition 2.2 *The cartesian product of two rough sets $A(X) = (A_\star(X), A^\star(X))$ and $A(Y) = (A_\star(Y), A^\star(Y))$ is defined as follows:*

$$A(X) \times A(Y) = \{(x, y) : x \in A^\star(X) \text{ and } y \in A^\star(Y)\}.$$

Definition 2.3 *A rough set $A(Y)$ is said to be rough subset of a rough set $A(X)$ if $A_\star(Y) \subseteq A_\star(X)$ and $A^\star(Y) \subseteq A^\star(X)$ and it is denoted by $A(Y) \subseteq A(X)$.*

Definition 2.4 ([3]) *A poset $(L, \leq)$ is a non-empty set L together with a binary relation '$\leq$' that satisfy the following conditions:*
(i) $a \leq a$ for all $a \in L$ (reflexivity);
(ii) If $a \leq b$ and $b \leq a$ then $a = b$, $\quad a, b \in L$ (anti-symmetricity);
(iii) If $a \leq b$ and $b \leq c$ then $a \leq c$, $\quad a, b, c \in L$ (transitivity).

Definition 2.5 ([3]) *A poset $(L, \leq)$ is called a meet-semi lattice if for all $a, b \in L$, $inf\{a, b\}$ exists. The definition of join-semi lattice is dual one.*

Definition 2.6 ([3]) *A poset $(L, \leq)$ is a lattice if $inf\{a, b\}$ and $sup\{a, b\}$ exist, for all $a, b \in L$.*

Definition 2.7 ([3]) *A non-empty subset K of a lattice L is called a sublattice of L if K itself forms a lattice with respect to the same operation in L.*

Definition 2.8 *A lattice L is said to be modular lattice if $\forall x, y, z \in L$ with $x \geq y$ such that $x \wedge (y \vee z) = y \vee (x \wedge z)$ where $\vee$, $\wedge$ are join and meet operators considered in L.*

Lemma 2.1 *A sublattice of a modular lattice is modular.*
Proof: Let K be a sublattice of a modular lattice L. Then $K \subseteq L$, therefore $x, y, z \in K$ with $x \geq y$ which imply, $x, y, z \in L$ with $x \geq y$. Since L is modular, therefore, $x \wedge (y \vee z) = y \vee (x \wedge z)$. Hence the lemma.

3 Rough Lattice

In this section, we have introduced the rough lattice and rough modular lattice and some properties of them. Here we have described the rough lattice based on Pawlak's notion of roughness. Let $(L, \vee, \wedge)$ be a lattice and also let $S[= (L, \rho)]$ be an approximation space. Let $X \subseteq U$ and $A(X) = (A_\star(X), A^\star(X))$ be the rough set of X in S.

Definition 3.1 *$A(X)$ is said to be rough join-semi lattice if $x \vee y \in A^\star(X)$, $\forall x, y \in X$, and $A(X)$ is said to be rough meet-semi lattice if $x \wedge y \in A^\star(X)$, $\forall x, y \in X$.*

Example 3.1: Consider the lattice $(L, \vee, \wedge)$ of all positive integers where $x \vee y$= l.c.m of $\{x, y\}$ and $x \wedge y$= g.c.d. of $\{x, y\}$, $\forall x, y \in L$, where l.c.m. and g.c.d stand for least common multiple and greatest common divisor respectively. Let ρ be an equivalence relation on L defined by $x\rho y$ iff $x = y$, $\forall x, y \in L$. Let $X = \{2, 3, 4, 6, 12\}$ then $A_\star(X) = X$ and $A^\star(X) = X$ under the approximation space (L, ρ). Clearly $x \vee y \in A^\star(X)$, $\forall x, y \in X$. But $4 \wedge 3 = 1 \notin A^\star(X)$. Therefore $A(X)$ is a rough join semi-lattice. In the other hand if we take $Z = \{1, 2, 3, 4, 5, 6\}$. Then $x \wedge y \in A^\star(Z)$, $\forall x, y \in Z$. But $4 \vee 6 = 12 \notin A^\star(Z)$. Therefore $A(Z)$ is a rough meet-semi lattice.

Definition 3.2 *$A(X)$ is said to be rough lattice in S if it is rough join-semi lattice as well as rough meet-semi lattice, i.e., $\forall\ x, y \in X$*
(i) $x \vee y \in A^\star(X)$;
(ii) $x \wedge y \in A^\star(X)$.
Let $x, y, z \in X$, then the rough lattice$< A(X), \vee, \wedge >$ satisfies the following properties:
(i) $x \vee x = x$ and $x \wedge x = x$ (Idempotency);
(ii) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (Commutativity);
(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (Associativity);
(iv) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ (Absorption);
(v) $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$ (Consistency).

Definition 3.3 *A rough lattice $A(X)$ is said to be bounded rough lattice if X have zero and unit in $A^\star(X)$.*

Definition 3.4 *A rough subset $A(Y)$ of a rough lattice $A(X)$ in an approximation space $S[= (L, \rho)]$ is said to be rough sublattice if $A(Y)$ itself forms a rough lattice with respect to the same operation.*

Proposition 3.1 *If $A(X) = (A_\star(X), A^\star(X))$ is a rough lattice in an approximation space $S[= (L, \rho)]$ such that $A^\star(X) = X$, then $A^\star(X)$ is a sublattice of L.*
Proof: Since $A(X) = (A_\star(X), A^\star(X))$ is a rough lattice in the approximation space $S[= (L, \rho)]$, then clearly, $x \vee y \in A^\star(X)$, and $x \wedge y \in A^\star(X)$, $\forall x, y \in A^\star(X)$. Hence the proposition.

Proposition 3.2 *If L is a modular lattice and $A(X)$ is a rough lattice in $S[= (L, \rho)]$ such that $A^\star(X) = X$ then $A^\star(X)$ is a modular lattice.*
Proof: Since $A^\star(X) = X$ and $A(X)$ is a rough lattice, therefore by **Proposition 3.1** and **Lemma 2.1**, $A^\star(X)$ is a sublattice of L and hence $A^\star(X)$ is a modular lattice.

Definition 3.5 *Let $< A(X), \vee, \wedge >$ is a rough lattice under an approximation space $S[= (L, \rho)]$, then $< A(X), \vee, \wedge >$ is said to be rough modular lattice (RML) if $\forall\ x, y, z \in A^\star(X)$ with $x \geq y$, $x \wedge (y \vee z) = y \vee (x \wedge z)$.*

Example 3.2: Let $U = \{a, b, c\}$. The power set of U, $P(U) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, U\}$ forms a lattice where the operators $\vee$ and $\wedge$ are defined as $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ and the order relation is set inclusion. Consider an equivalence relation ρ on $P(U)$ by $A\rho B$ iff $O(A) = O(B)\ \forall\ A, B \in P(U)$, where $O(A)$ and $O(B)$ denote the number of elements in the set A and B respectively. Let $X = \{\phi, \{a\}, \{a, b, c\}\}$. Then $A_\star(X) = \{\phi, \{a, b, c\}\}$ and $A^\star(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$. Clearly $A(X)$ is a rough lattice. Also $\forall A, B, C \in A^\star(X)$ with $A \supseteq B$, $A \wedge (B \vee C) = B \vee (A \wedge C)$ i.e, $A(X)$ is a RML.

Proposition 3.3 *Rough sublattice of a RML is RML.*
Proof: Let $A(M)$ is a RML and $A(N)$ is a rough sublattice of $A(M)$. Therefore, $A^\star(N) \subseteq A^\star(M)$ and hence if $x, y, z \in A^\star(N)$ with $x \geq y$ imply $x, y, z \in A^\star(M)$ with $x \geq y$. Since $A(M)$ is modular rough lattice, therefore, $x \wedge (y \vee z) = y \vee (x \wedge z)$. Therefore, if $x, y, z \in A^\star(N)$ with $x \geq y$, $x \wedge (y \vee z) = y \vee (x \wedge z)$. Hence the proposition.

Proposition 3.4 *If L is a modular lattice and $A(X)$ is a rough lattice then $A(X)$ is a RML.*
Proof: Since $A(X)$ is a rough lattice, therefore $\forall x, y, z \in A^{\star}(X)$ with $x \geq y$ imply $x, y, z \in L$ and since L is modular, then $x \wedge (y \vee z) = y \vee (x \wedge z)$. So $A(X)$ is a RML.

Proposition 3.5 *If $A(X)$ is a RML in $S[= (L, \rho)]$ and if $A^{\star}(X) = X$ then X is modular sublattice of L and vice-versa.*
Proof: Since $A(X)$ is RML and $A^{\star}(X) = X$, therefore modular equality holds in X. Also by **Proposition 3.1**, $A^{\star}(X)$ is sublattice of L and therefore $A^{\star}(X) = X$ is modular sublattice of L. Conversely, let $A^{\star}(X) = X$ is modular sublattice of L. Therefore $A^{\star}(X)$ is rough lattice and also modular equality holds in $A^{\star}(X)$. So $A(X)$ is RML.

Proposition 3.6 *Two rough lattices $A(X)$ and $A(Y)$ are RML iff $A(X) \times A(Y)$ is RML.*
Proof: Let us consider that $A(X)$ and $A(Y)$ be RML and $A(X) = (A_{\star}(X), A^{\star}(X))$ and $A(Y) = (A_{\star}(Y), A^{\star}(Y))$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^{\star}(X) \times A^{\star}(Y)$ with $(x_1, y_1) \geq (x_2, y_2)$. Therefore

$$(x_1, y_1) \wedge \{(x_2, y_2) \vee (x_3, y_3)\} = (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3))$$
$$= (x_2 \vee (x_1 \wedge x_3), y_2 \vee (y_1 \wedge y_3)) = (x_2, y_2) \vee \{(x_1, y_1) \wedge (x_3, y_3)\}.$$

Therefore $A^{\star}(X) \times A^{\star}(Y)$ is RML. Conversely, let $A^{\star}(X) \times A^{\star}(Y)$ be RML. Let $x_1, x_2, x_3 \in A^{\star}(X)$ with $x_1 \geq x_2$ and $y_1, y_2, y_3 \in A^{\star}(Y)$ with $y_1 \geq y_2$ then $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A^{\star}(X) \times A^{\star}(Y)$ and $(x_1, y_1) \geq (x_2, y_2)$. Since $A^{\star}(X) \times A^{\star}(Y)$ is RML, we find

$$(x_1, y_1) \wedge \{(x_2, y_2) \vee (x_3, y_3)\} = (x_2, y_2) \vee \{(x_1, y_1) \wedge (x_3, y_3)\},$$

or,

$$(x_1, y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3) = (x_2, y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3),$$

or,

$$(x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) = (x_2 \vee (x_1 \wedge x_3), y_2 \vee (y_1 \wedge y_3)),$$

which indicates $x_1 \wedge (x_2 \vee x_3) = x_2 \vee (x_1 \wedge x_3)$ and $y_1 \wedge (y_2 \vee y_3) = y_2 \vee (y_1 \wedge y_3)$. Hence $A(X)$ and $A(Y)$ are RML.

4 Conclusion

Lattice and ordered set play an important role in the area of computer science. Lattice and ordered set can be applied in various fields such as area of knowledge representation, text categorization and data mining order in a fundamental ways. In this paper, the rough modular lattice has introduced based on indiscernibility relation. At first, the rough lattice has constructed and interpreted based on the equivalence relation. Then the different types of lattice under the rough set environment have established by incorporating a pair of sets in an approximation space. We also have showed that modularity property of ordinary lattice in crisp set is extended to area of uncertainty for rough set which is the generalization of lattice theory. This concept may be useful on lattice structure when the elements are imprecise. We addressed a connection between rough set and lattice theory both of which have wide field of application in the area of computer science. This paper has made special interest for lattice structure, when the elements of the set are imprecise. In our future study, we extent this concept in knowledge representation problems.

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