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Equilibrium Mean Value of Random Fuzzy Variable and Its Convergence Properties

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Abstract

The equilibrium measure is a natural extension of both probability and credibility measures. The convergence modes of random fuzzy variables with respect to equilibrium measure is an important issue for research. In this paper, we first introduce several convergence concepts for sequences of random fuzzy variables, including convergence in equilibrium measure and convergence in equilibrium distribution. Then we deal with the properties of the convergence modes of random fuzzy variables. The equilibrium mean value of random fuzzy variable with respect to equilibrium measure is also defined by nonlinear integral. For sequence of integrable random fuzzy variables, we deal with the important monotone convergence theorems as well as dominated convergence theorems. The convergent results obtained in this paper have potential applications in the approximation scheme of equilibrium optimization models.

Keywords: random fuzzy variable, equilibrium measure, convergence mode, equilibrium expected value, convergence theorem

1 Introduction

Random fuzzy variable is to describe the phenomena in which fuzziness and randomness appear simultaneously in a decision process [8], it is a useful tool to model random fuzzy optimization problems, including random fuzzy dependent-chance programming [7], random fuzzy chance-constrained programming [15], and random fuzzy expected value model [16]. On the basis of the work mentioned above, random fuzzy variable was well-studied and applied to a number of fields. For instance, Zhu and Liu [29, 30, 31] discussed the properties of chance distribution of random fuzzy variable; Zhao et al. [28] studied random fuzzy renewal process; Zhang et al. [27] discussed the optimal run lengths in deteriorating production processes; Wen and Kang [25] proposed some optimal models for facility location-allocation problem with random fuzzy demands; Wang et al. [24] established random fuzzy EOQ model with imperfect quality items; Shen et al. [20, 21] studied random fuzzy alternating renewal processes; Sakalli et al. [19] gave an application of investment decision with random fuzzy outcomes; Liu et al. [17] studied random fuzzy shock models; Li et al. [6] dealt with random fuzzy delayed renewal processes; Huang [3, 4, 5] discussed optimal portfolio selection with random fuzzy returns; Guo and Guo [2] gave random fuzzy variable foundation for grey differential equation modeling, and Feng et al. [1] discussed transmission line maintenance scheduling considering both randomness and fuzziness.

Since random fuzzy optimization problems include uncertain parameters defined through probability and possibility distributions, the main difficulty of such models is due to optimal decisions that have to be taken before the observation of uncertain parameters. In this case, one can hardly find any decision which would definitely exclude later constraint violation caused by unexpected uncertain effects. Sometimes, such constraint violation can be balanced afterwards by some compensating decision taken in a second stage. As long as the costs of compensating decisions are known, these may be considered as a penalization for constraint violation. This idea leads to the important class of two-stage random fuzzy optimization problems [12]. When uncertain parameters have continuous distribution functions, algorithms to solve such optimization problems often rely on approximation schemes, which result in finite-dimensional optimization problems that can be tackled easily. This consideration motivates us to introduce several new convergence modes in equilibrium measure

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theory [15], which provide the theoretical foundation for the approximation schemes of fuzzy optimization models [13, 18].

The paper is organized as follows. Section 2 recalls some basic concepts, including credibility measure, random fuzzy variable, and the equilibrium measure of a random fuzzy event. In Section 3, we first introduce the concepts of convergence in equilibrium measure and convergence in equilibrium distribution, then discuss the properties of convergence modes of random fuzzy variables. Section 4 comprises the convergence modes of random fuzzy variables in equilibrium measure theory. In Section 5, we first define the equilibrium mean value of random fuzzy variable, then discuss the monotone convergence theorems and dominated convergence theorems for sequences of integrable random fuzzy variables. Section 6 gives the conclusions of this paper.

2 Fundamental Concepts

The equilibrium measure theory established in [15] is an extension of both credibility theory [9, 10] and probability theory. In this section, we review some concepts in this theory.

Given a universe Γ, an ample field $\mathcal{A}$ on Γ is a class of subsets of Γ that is closed under the formation of arbitrary unions, arbitrary intersections and complement, and Pos is a possibility measure defined on $\mathcal{A}$ (see, [23, 26]).

A self-dual set function Cr, called *credibility measure*, was defined as follows (see, [11])

$$\mathrm{Cr}(A) = \frac{1}{2}\left(1 + \mathrm{Pos}(A) - \mathrm{Pos}(A^c)\right), A \in \mathcal{A}, \tag{1}$$

where $A^c = \Gamma \backslash A$. The triplet $(\Gamma, \mathcal{A}, \mathrm{Cr})$ is called a credibility space.

Definition 1. *Let* $(\Gamma, \mathcal{A}, \mathrm{Cr})$ *be a credibility space. A map* X *from* Γ *to* $\Re$ *is called a fuzzy variable if for every* $t \in \Re$,

$$\{\gamma \in \Gamma \mid X(\gamma) \leq t\} \in \mathcal{A}. \tag{2}$$

The possibility distribution of the fuzzy variable X is defined by

$$\mu_X(t) = \mathrm{Pos}\{\gamma \in \Gamma \mid X(\gamma) = t\}, t \in \Re. \tag{3}$$

Definition 2 ([8])**.** *Let* $(\Gamma, \mathcal{A}, \mathrm{Cr})$ *be a credibility space. A random fuzzy variable is a map* $\xi : \Gamma \to \mathcal{R}_v$ *such that for any Borel subset* B *of* $\Re$, *the following probability function*

$$\xi^*(B)(\gamma) = \Pr\{\omega \in \Omega \mid \xi_\gamma(\omega) \in B\} \tag{4}$$

is measurable with respect to γ, *where* $\mathcal{R}_v$ *is the collection of random variables defined on a probability space.*

Definition 3 ([15])**.** *Let* ξ *be a random fuzzy variable, and* B *a Borel subset of* $\Re$. *Then the equilibrium measure* Ch *of an event* $\{\xi \in B\}$ *is defined as*

$$\mathrm{Ch}\{\xi \in B\} = \bigvee_{0<p\leq 1} \left[p \wedge \mathrm{Cr}\{\gamma \in \Gamma \mid \Pr\{\omega \in \Omega \mid \xi_\gamma(\omega) \in B\} \geq p\}\right]. \tag{5}$$

In equilibrium measure theory, we have the following convergence modes of random fuzzy variables.

Definition 4 ([14])**.** *A sequence* $\{\xi_n\}$ *of random fuzzy variables is said to converge almost surely to a random fuzzy variable* ξ, *denoted by* $\xi_n \xrightarrow{a.s} \xi$, *if there exist* $E \in \mathcal{A}, F \in \Sigma$ *with* $\mathrm{Cr}(E) = \Pr(F) = 0$ *such that for every* $(\gamma, \omega) \in \Gamma\backslash E \times \Omega\backslash F$, *we have* $\lim_{n\to\infty} \xi_{n,\gamma}(\omega) \to \xi_\gamma(\omega)$.

Definition 5 ([14])**.** *A sequence* $\{\xi_n\}$ *of random fuzzy variables is said to converge uniformly to a random fuzzy variable* ξ *on* $\Gamma \times \Omega$, *denoted by* $\xi_n \xrightarrow{u.} \xi$, *if*

$$\lim_{n\to\infty} \sup_{(\gamma,\omega)\in\Gamma\times\Omega} |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| = 0.$$

Definition 6 ([14])**.** *A sequence* $\{\xi_n\}$ *of random fuzzy variables is said to converge almost uniformly to a random fuzzy variable* ξ, *denoted by* $\xi_n \xrightarrow{a.u.} \xi$, *if there exist two nonincreasing sequences* $\{E_m\} \subset \mathcal{A}$, $\{F_m\} \subset \Sigma$ *with* $\lim_m \mathrm{Cr}(E_m) = \lim_m \Pr(F_m) = 0$ *such that for each* m, *we have* $\xi_n \xrightarrow{u.} \xi$ *on* $\Gamma\backslash F_m \times \Omega\backslash E_m$.

3 Convergence Modes in Equilibrium Theory

Let ξ be a random fuzzy variable defined on a credibility space $(\Gamma, \mathcal{A}, \mathrm{Cr})$. Then the equilibrium distribution of ξ is defined as $G_\xi(t) = \mathrm{Ch}\{\xi \geq t\}, t \in \Re$. It is evident that G_ξ is a nonincreasing $[0,1]$-valued function.

Let $\{F_n\}$ and F be nonincreasing real-valued functions. The sequence $\{F_n\}$ is said to converge weakly to F, denoted by $F_n \xrightarrow{w.} F$, if $F_n(t) \to F(t)$ for all continuity points t of F.

As for the convergence modes with respect to equilibrium measure, we have:

Definition 7. *Let $\{\xi_n\}$ be a sequence of random fuzzy variables. The sequence $\{\xi_n\}$ is said to converge in equilibrium measure* Ch *to a random fuzzy variable ξ, denoted by $\xi_n \xrightarrow{\mathrm{Ch}} \xi$, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathrm{Ch}\{|\xi_n - \xi| \geq \varepsilon\} = 0.$$

Definition 8. *Let G_{ξ_n} be the equilibrium distribution of random fuzzy variable ξ_n, and G_ξ the equilibrium distribution of random fuzzy variable ξ. The sequence $\{\xi_n\}$ is said to converge in equilibrium distribution to ξ, denoted by $\xi_n \xrightarrow{e.d.} \xi$, if $G_{\xi_n} \xrightarrow{w.} G_\xi$.*

The following proposition deals with the property of convergence almost sure.

Proposition 1. *Let $\{\xi_n\}$ and ξ be random fuzzy variables, and* Ch *the equilibrium measure. Then $\xi_n \xrightarrow{a.s.} \xi$ if and only if for every $\varepsilon > 0$,*

$$\mathrm{Ch}\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{|\xi_n - \xi| \geq \varepsilon\}\right) = 0. \tag{6}$$

Proof. The llimit $\xi_n \xrightarrow{a.s.} \xi$ holds if and only if the limit $\xi_{n,\gamma} \xrightarrow{a.s.} \xi_\gamma$ holds with credibility 1 (w.c.1), i.e.,

$$\mathrm{Cr}\left\{\gamma \mid \xi_{n,\gamma} \xrightarrow{a.s.} \xi_\gamma\right\} = 1.$$

Then, there is $E \in \mathcal{A}$ with $\mathrm{Cr}(E) = 0$ such that for every $\gamma \in \Gamma \backslash E$, one has

$$\xi_{n,\gamma} \xrightarrow{a.s.} \xi_\gamma.$$

Hence, for every $\varepsilon > 0$, the following equality

$$\Pr\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\}\right) = 0$$

holds with credibility 1, which, by the property of equilibrium measure, is equivalent to

$$\mathrm{Ch}\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{|\xi_n - \xi| \geq \varepsilon\}\right) = 0.$$

The proof of proposition is complete. □

The property of convergence almost uniform is discussed by the following proposition.

Proposition 2. *Let $\{\xi_n\}$ and ξ be random fuzzy variables, and* Ch *the equilibrium measure. If $\xi_n \xrightarrow{a.u.} \xi$, then for every $\varepsilon > 0$, the following limit holds w.c.1*

$$\lim_{m\to\infty} \Pr\left(\bigcup_{n=m}^{\infty}\{\omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\}\right) = 0. \tag{7}$$

Conversely, if γ is a discrete fuzzy variable assuming a finite number of values, then Eq. (7) implies $\xi_n \xrightarrow{a.u.} \xi$.

Proof. If $\xi_n \xrightarrow{a.u.} \xi$, then there exist two nonincreasing sequences $\{E_m\} \subset \mathcal{A}$, $\{F_m\} \subset \Sigma$ with $\lim_m \mathrm{Cr}(E_m) = \lim_m \Pr(F_m) = 0$ such that for each m, $\xi_n \xrightarrow{u.} \xi$ on $\Gamma \backslash E_m \times \Omega \backslash F_m$.

Let$E = \bigcap_{m=1}^{\infty} E_m$. Then $\mathrm{Cr}(E) = 0$, and for every $\gamma \in \Gamma \backslash E$, there is a positive integer m_γ such that $\gamma \in \Gamma \backslash E_{m_\gamma}$. Since $\{E_m\}$ is nonincreasing, we have $\gamma \in \Gamma \backslash E_m$ whenever $m \geq m_\gamma$. Therefore, there is a subsequence $\{F_m, m \geq m_\gamma\}$ of $\{F_m\}$ such that for each $m_\gamma, m_\gamma + 1, \ldots$, the sequence $\{\xi_{n,\gamma}\}$ converges to ξ_γ uniformly on F_m. Thus, $\xi_{n,\gamma} \xrightarrow{a.u.} \xi_\gamma$ with credibility 1.

We now show that $\xi_{n,\gamma} \xrightarrow{a.u.} \xi_\gamma$ w.c.1 implies Eq. (7).

In fact, for any $\delta > 0$, there exists $F_\gamma \in \Sigma$ with $\Pr(F_\gamma) < \delta$ such that $\{\xi_{n,\gamma}\}$ converges to ξ_γ uniformly on $\Omega \backslash F_\gamma$. Thus, for every $\varepsilon > 0$, there exists a positive integer $m(\varepsilon, \gamma)$ such that for all $\omega \in \Omega \backslash F_\gamma$,

$$|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| < \varepsilon$$

whenever $n \geq m$. Therefore, one has

$$\Omega \backslash F_\gamma \subset \bigcap_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| < \varepsilon\},$$

or

$$\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \subset F_\gamma,$$

which implies

$$\Pr\left(\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\}\right) \leq \Pr(F_\gamma) < \delta.$$

Letting $\delta \to 0$, then we have

$$\lim_{m \to \infty} \Pr\left(\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\}\right) = 0,$$

which verifies Eq. (7).

Conversely, suppose Eq. (7) is valid, we prove $\xi_n \xrightarrow{a.u.} \xi$. By the supposition of proposition, we assume that γ has the following possibility distribution

$$\gamma \sim \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_N \\ p_1, & p_2, & \ldots, & p_N \end{pmatrix}$$

with $p_i > 0$ and $\max_{i=1}^N p_i = 1$. Since for each $i = 1, 2, \ldots, N$, we have

$$\lim_{m \to \infty} \Pr\left(\bigcup_{n=m}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq \varepsilon\}\right) = 0.$$

Then for every $\delta \in (0, 1)$, and each $k = 1, 2, \ldots$, there exists a positive integer m_k such that for $i = 1, 2, \ldots, N$, we have

$$\Pr\left(\bigcup_{n=m_k}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq 1/k\}\right) < \delta/2^{k+i}.$$

Letting

$$F = \bigcup_{i=1}^{N} \bigcup_{k=1}^{\infty} \bigcup_{n=m_k}^{\infty} \{\omega \in \Omega \mid |\xi_{n,\gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq 1/k\},$$

then

$$\begin{aligned} \Pr(F) &= \Pr\left(\bigcup_{i=1}^{N} \bigcup_{k=1}^{\infty} \bigcup_{n=m_k}^{\infty} \{\omega \mid |\xi_{n,\gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq 1/k\}\right) \\ &\leq \sum_{i=1}^{N} \sum_{k=1}^{\infty} \Pr\left(\bigcup_{n=m_k}^{\infty} \{\omega \mid |\xi_{n,\gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq 1/k\}\right) < \delta. \end{aligned}$$

In addition, for each k, one has

$$\sup_{1\le i\le N}\sup_{\omega\in\Omega\setminus F}|\xi_{n,\gamma_i}(\omega)-\xi_{\gamma_i}(\omega)|<1/k$$

whenever $n \ge m_k$, which implies $\xi_n \xrightarrow{a.u.} \xi$. The proof of proposition is complete. □

The next proposition deals with the property of convergence in equilibrium measure.

Proposition 3. *Suppose $\{\xi_n\}$ and ξ are random fuzzy variables, and* Ch *the equilibrium measure. Then $\xi_n \xrightarrow{\mathrm{Ch}} \xi$ if and only if for every $\varepsilon > 0$, $\Pr\{\omega \mid |\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)| \ge \varepsilon\} \xrightarrow{\mathrm{Cr}} 0$.*

Proof. Let $\xi_n \xrightarrow{\mathrm{Ch}} \xi$. Then for every $\eta > 0$, we have

$$\mathrm{Ch}\{|\xi_n-\xi|\ge\varepsilon\}\ge\eta\wedge\mathrm{Cr}\{\gamma\mid\Pr\{|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}\ge\eta\}.$$

Therefore, the limit $\Pr\{\omega \mid |\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)| \ge \varepsilon\} \xrightarrow{\mathrm{Cr}} 0$ holds, which is equivalent to

$$\Pr\{\omega\mid|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}\xrightarrow{\mathrm{a.u.}}0.$$

For any $\delta > 0$, there exists $E_\delta \in \mathcal{A}$ with $\mathrm{Cr}(E_\delta) < \delta$ such that on $\Gamma\setminus E_\delta$, we have

$$\Pr\{\omega\mid|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}\xrightarrow{\mathrm{u.}}0.$$

Therefore, there exists some positive integer N_δ such that for any $\gamma \in \Gamma\setminus E_\delta$, we have

$$\Pr\{\omega\mid|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}<\delta$$

whenever $n \ge N_\delta$. Thus, $\mathrm{Cr}\{\gamma \in \Gamma\setminus E_\delta \mid \Pr\{\omega \mid |\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)| \ge \varepsilon\} \ge \alpha\} = 0$ whenever $n \ge N_\delta$ and $\alpha \ge \delta$. As a consequence of the subadditivity of credibility measure, we have

$$\sup_{\alpha\in(0,\delta]}[\alpha\wedge\mathrm{Cr}\{\gamma\mid\Pr\{\omega\mid|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}\ge\alpha\}]\le\delta,$$

and

$$\begin{aligned}&\sup\nolimits_{\alpha\in(\delta,1]}[\alpha\wedge\mathrm{Cr}\{\gamma\mid\Pr\{\omega\mid|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}\ge\alpha\}]\\&\le\sup\nolimits_{\alpha\in(\delta,1]}[\alpha\wedge(\delta+\mathrm{Cr}\{\gamma\in\Gamma\setminus E_\delta\mid\Pr\{\omega\mid|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}\ge\alpha\})]\\&=\sup\nolimits_{\alpha\in(\delta,1]}[\alpha\wedge(\delta+0)]=\delta.\end{aligned}$$

It follows from the definition of equilibrium measure that $\mathrm{Ch}\{|\xi_n-\xi| \ge \varepsilon\} \le \delta$. By the arbitrary of δ, the proof of proposition is complete. □

4 Comparison of Convergence Modes

In this section, we compare the convergence modes of random fuzzy variables in equilibrium theory. First, the following theorem compares convergence almost uniform and convergence almost sure.

Theorem 1. *Suppose $\{\xi_n\}$ and ξ are random fuzzy variables. If $\xi_n \xrightarrow{a.u.} \xi$, then we have $\xi_n \xrightarrow{a.s.} \xi$.*

Proof. Let $\xi_n \xrightarrow{a.u.} \xi$. Then, by Proposition 2, the following limit

$$\lim_{m\to\infty}\Pr\left(\bigcup_{n=m}^{\infty}\{\omega\in\Omega\mid|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}\right)=0$$

holds with credibility 1. By the upper semicontinuity of probability, the equality

$$\Pr\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{\omega\in\Omega\mid|\xi_{n,\gamma}(\omega)-\xi_\gamma(\omega)|\ge\varepsilon\}\right)=0$$

holds with credibility 1, which implies

$$\mathrm{Ch}\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{|\xi_n-\xi|\ge\varepsilon\}\right)=0.$$

It follows from Proposition 1 that $\xi_n \xrightarrow{a.s.} \xi$. The proof of proposition is complete. □

The following theorem compares convergence almost uniform and convergence in equilibrium measure.

Theorem 2. *Suppose $\{\xi_n\}$ and ξ are random fuzzy variables. If $\xi_n \xrightarrow{a.u.} \xi$, then we have $\xi_n \xrightarrow{\mathrm{Ch}} \xi$.*

Proof. If $\xi_n \xrightarrow{a.u.} \xi$, then for any given $\delta > 0$, there exist $E \in \mathcal{A}$ and $F \in \Sigma$ with $\mathrm{Cr}(E) < \delta$, $\Pr(F) < \delta$ such that $\{\xi_n\}$ converges to ξ uniformly on $\Gamma \backslash E \times \Omega \backslash F$. For any given $\varepsilon > 0$, there exists some positive integer N such that for every $(\gamma, \omega) \in \Gamma \backslash E \times \Omega \backslash F$, we have $|\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| < \varepsilon$ whenever $n \geq N$. As a consequence, when $n \geq N$, we have

$$\sup_{\alpha \in (0,\delta]} [\alpha \wedge \mathrm{Cr}\{\gamma \mid \Pr\{\omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq \alpha\}] \leq \delta,$$

and

$$\begin{aligned}
&\sup_{\alpha \in (\delta,1]} [\alpha \wedge \mathrm{Cr}\{\gamma \mid \Pr\{\omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq \alpha\}] \\
&\leq \sup_{\alpha \in (\delta,1]} [\alpha \wedge (\delta + \mathrm{Cr}\{\gamma \in \Gamma \backslash E \mid \Pr\{\omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq \alpha\})] \\
&= \sup_{\alpha \in (\delta,1]} [\alpha \wedge (\delta + \mathrm{Cr}\{\gamma \in \Gamma \backslash E \mid \Pr\{\omega \in F \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \geq \alpha\})] \\
&= \sup_{\alpha \in (\delta,1]} [\alpha \wedge (\delta + 0)] = \delta.
\end{aligned}$$

Combining the inequalities above, we conclude that $\mathrm{Ch}\{|\xi_n - \xi| \geq \varepsilon\} \leq \delta$ whenever $n \geq N$. It follows from the arbitrary of δ that $\xi_n \xrightarrow{\mathrm{Ch}} \xi$. The proof of theorem is complete. □

The following theorem compares convergence in equilibrium measure and convergence almost sure.

Theorem 3. *Let γ be a discrete fuzzy variable assuming finite number of values, and* Ch *the equilibrium measure. If $\xi_n \xrightarrow{\mathrm{Ch}} \xi$, then there exists some subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\xi_{n_k} \xrightarrow{a.s.} \xi$.*

Proof. Suppose that γ has the following possibility distribution

$$\gamma \sim \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_N \\ p_1, & p_2, & \ldots, & p_N \end{pmatrix}$$

with $p_i > 0$ and $\max_{i=1}^N p_i = 1$. Let $\xi_n \xrightarrow{\mathrm{Ch}} \xi$. Then, by Proposition 3, for every $\varepsilon > 0$, we have

$$\Pr\{\omega \mid |\xi_{n,\gamma}(\omega) - \xi_\gamma(\omega)| \geq \varepsilon\} \xrightarrow{\mathrm{Cr}} 0.$$

Since convergence in credibility implies convergence almost sure [9], one has

$$\Pr\{\omega \mid |\xi_{n,\gamma_i}(\omega) - \xi_{\gamma_i}(\omega)| \geq \varepsilon\} \to 0, \quad i = 1, 2, \ldots, N.$$

That is, the limit $\xi_{n,\gamma_i} \xrightarrow{\Pr} \xi_{\gamma_i}$ holds for $i = 1, 2, \ldots, N$. By Riesz's theorem [22], there exists some subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\xi_{n_k,\gamma_i}(\omega) \to \xi_{\gamma_i}(\omega)$ for every $\omega \in \Omega$ and $i = 1, 2, \ldots, N$. The proof of theorem is complete. □

Before ending this section, we next compare the convergence in equilibrium measure and convergence in equilibrium distribution.

Theorem 4. *Let $\{\xi_n\}$ and ξ be random fuzzy variables, and* Ch *the equilibrium measure. If γ is a discrete fuzzy variable assuming finite number of values, then $\xi_n \xrightarrow{\mathrm{Ch}} \xi$ implies $\xi_n \xrightarrow{e.d.} \xi$.*

Proof. For every $\gamma \in \Gamma$, let $G_{n,\gamma}$ and G_γ be the probability distribution functions of $\xi_{n,\gamma}$ and ξ_γ, respectively. Then for every $t \in \Re, \varepsilon > 0$ and integer n, one has

$$\begin{aligned}
\Pr\{\xi_{n,\gamma} \geq t\} &\leq \Pr\{\xi_{n,\gamma} \geq t, |\xi_{n,\gamma} - \xi_\gamma| < \varepsilon\} + \Pr\{\xi_{n,\gamma} \geq t, |\xi_{n,\gamma} - \xi_\gamma| \geq \varepsilon\} \\
&\leq \Pr\{\xi_\gamma \geq t - \varepsilon\} + \Pr\{|\xi_{n,\gamma} - \xi_\gamma| \geq \varepsilon\}.
\end{aligned}$$

That is, $G_{n,\gamma}(t) \leq G_\gamma(t - \varepsilon) + \Pr\{|\xi_{n,\gamma} - \xi_\gamma| \geq \varepsilon\}$. From the proof of Theorem 3, letting $n \to \infty$, and then $\varepsilon \to 0$, we obtain

$$\limsup_{n \to \infty} G_{n,\gamma}(t) \leq G_\gamma(t - 0).$$

On the other hand, according to the following inequality

$$\begin{aligned}\Pr\{\xi_\gamma \ge t+\varepsilon\} &\le \Pr\{\xi_\gamma \ge t+\varepsilon, |\xi_{n,\gamma}-\xi_\gamma| < \varepsilon\} + \Pr\{\xi_\gamma \ge t+\varepsilon, |\xi_{n,\gamma}-\xi_\gamma| \ge \varepsilon\} \\ &\le \Pr\{\xi_{n,\gamma} \ge t\} + \Pr\{|\xi_{n,\gamma}-\xi_\gamma| \ge \varepsilon\},\end{aligned}$$

we have $G_\gamma(t+\varepsilon) \le G_{n,\gamma}(t) + \Pr\{|\xi_{n,\gamma}-\xi_\gamma| \ge \varepsilon\}$, and

$$\liminf_{n\to\infty} G_{n,\gamma}(t) \ge G_\gamma(t+0).$$

Therefore, $G_{n,\gamma} \xrightarrow{w.} G_\gamma$ uniform with respect to γ. By the property of equilibrium measure, we have $G_n \xrightarrow{w.} G$, i.e., $\xi_n \xrightarrow{e.d.} \xi$. □

5 Equilibrium Mean Value of Random Fuzzy Variable

In the following, we first define the equilibrium mean value of a random fuzzy variable.

Definition 9. *Let ξ be a random fuzzy variable, and* Ch *the equilibrium measure. Then the equilibrium mean value of ξ is defined as*

$$\mathcal{E}[\xi] = \int_0^\infty \mathrm{Ch}\{\,\xi \ge r\,\}\,\mathrm{d}r - \int_{-\infty}^0 \mathrm{Ch}\{\,\xi \le r\,\}\,\mathrm{d}r \tag{8}$$

provided one of the two integrals is finite.

We next define the equilibrium fractile function of a random fuzzy variable.

Definition 10. *Let ξ be a random fuzzy variable with equilibrium distribution G_ξ. Then the equilibrium fractile function of ξ is defined by*

$$\mathrm{V@R}_\alpha(\xi) = \sup\{t \mid G_\xi(t) \ge \alpha\}, \alpha \in (0,1]. \tag{9}$$

The equilibrium distribution $G_\xi(t)$ is nonincreasing with respect to t. As a result, the equilibrium fractile function $\mathrm{V@R}_\alpha(\xi)$ is also nonincreasing with respect to α, and it is a pseudo-inverse function of $G_\xi(t)$.

In the case that γ has a continuous possibility distribution, the equilibrium fractile function of ξ can be represented equivalently by

$$\mathrm{V@R}_\alpha(\xi) = \sup\{t \mid \mathrm{Cr}\{\gamma \in \Gamma \mid \Pr\{\omega \in \Omega \mid \xi_\gamma(\omega) \ge t\} \ge \alpha\} \ge \alpha\}, \alpha \in (0,1]. \tag{10}$$

Definition 11. *Let $\{\xi_n\}$ and ξ be random fuzzy variables, and their equilibrium fractile functions are $\{\mathrm{V@R}_\alpha(\xi_n)\}$ and $\mathrm{V@R}_\alpha(\xi)$, respectively. If $\mathrm{V@R}_\alpha(\xi_n) \xrightarrow{w.} \mathrm{V@R}_\alpha(\xi)$, then the sequence $\{\xi_n\}$ is said to converge in equilibrium fractile to ξ, and denoted by $\xi_n \xrightarrow{e.f.} \xi$.*

For sequence of monotone increasing random fuzzy variables, we have the following convergence result:

Theorem 5. (**Monotone Convergence Theorem**) *Let $\{\xi_n\}$ be a sequence of random fuzzy variables such that for any $n \ge 1$, $\xi_n \le \xi_{n+1}$ almost sure and $\mathcal{E}[\xi_1] > -\infty$. If $\xi_n \xrightarrow{e.d.} \xi$ with $\xi = \lim_{n\to\infty} \xi_n$, then we have $\lim_{n\to\infty} \mathcal{E}[\xi_n] = \mathcal{E}[\lim_{n\to\infty} \xi_n]$.*

Proof. First, by the integral relationship of equilibrium distribution and its fractile function, we have

$$\mathcal{E}[\xi] = \int_0^1 \mathrm{V@R}_\alpha(\xi)\mathrm{d}\alpha, \quad \mathcal{E}[\xi_n] = \int_0^1 \mathrm{V@R}_\alpha(\xi_n)\mathrm{d}\alpha$$

for $n = 1, 2, \ldots$. By the supposition of theorem, we have $G_{\xi_n} \overset{e.c.}{\le} G_{\xi_{n+1}}$, which implies

$$\mathrm{V@R}_\alpha(\xi_n) \overset{e.c.}{\le} \mathrm{V@R}_\alpha(\xi_{n+1}), n = 1, 2, \ldots.$$

We next prove $\xi_n \xrightarrow{e.d.} \xi$ implies $\xi_n \xrightarrow{e.f.} \xi$. Suppose $\alpha \in (0,1)$ is such that there is at most one value t having $G_\xi(t) = \alpha$. Denote $z = \mathrm{V@R}_\alpha(\xi)$.

On the one hand, we have $G_\xi(t) > \alpha$ for $t < z$. Thus $G_{\xi_n}(t) > \alpha$ for $n \geq N_t$ (some positive integer), provided that $t < z$ is a continuity point of G_ξ. Hence $\mathrm{V@R}_\alpha(\xi_n) \geq t$, provided that $t < z$ is a continuity point of G_ξ. Therefore, $\liminf_{n\to\infty} \mathrm{V@R}_\alpha(\xi_n) \geq t$. Since there is an increasing sequence $\{t_n\}$ of continuity points of G_ξ converging to z, we have

$$\liminf_{n\to\infty} \mathrm{V@R}_\alpha(\xi_n) \geq z.$$

On the other hand, as $t > z$, we have $G_\xi(t) < \alpha$. Thus $G_{\xi_n}(t) < \alpha$ for $n \geq N_t'$ (some positive integer), provided that $t > z$ is a continuity point of G_ξ. Hence $\mathrm{V@R}_\alpha(\xi_n) \leq t$, provided that $t > z$ is a continuity point of G_ξ. Therefore, $\limsup_{n\to\infty} \mathrm{V@R}_\alpha(\xi_n) \leq t$. Since there is a decreasing sequence $\{t_n\}$ of continuity points converging to z, we have

$$\limsup_{n\to\infty} \mathrm{V@R}_\alpha(\xi_n) \leq z.$$

Therefore, $\mathrm{V@R}_\alpha(\xi_n) \to \mathrm{V@R}_\alpha(\xi)$ for all except at most countably infinite number of α's, i.e., for all except those α's that have many values of t such that $G_\xi(t) = \alpha$, which correspond to the heights of flat spots of G_ξ, and these flat spot height α's are exactly the discontinuity points of $\mathrm{V@R}_\alpha(\xi_n)$. That is, $\xi_n \xrightarrow{e.f.} \xi$. Hence, the limit $\mathrm{V@R}_\alpha(\xi_n) \xrightarrow{w.} \mathrm{V@R}_\alpha(\xi)$ holds. It follows from the monotone convergence theorem that

$$\lim_{n\to\infty} \int_0^1 \mathrm{V@R}_\alpha(\xi_n)\mathrm{d}\alpha = \int_0^1 \mathrm{V@R}_\alpha(\xi)\mathrm{d}\alpha.$$

That is, $\lim_{n\to\infty} \mathcal{E}[\xi_n] = \mathcal{E}[\xi]$. The proof of theorem is complete. □

Similarly, for sequence of monotone decreasing random fuzzy variables, we have:

Theorem 6. (**Monotone Convergence Theorem**) *Let $\{\xi_n\}$ be a sequence of random fuzzy variables such that for any $n \geq 1$, $\xi_n \geq \xi_{n+1}$ almost sure and $\mathcal{E}[\xi_1] < \infty$. If $\xi_n \xrightarrow{e.d.} \xi$ with $\xi = \lim_{n\to\infty} \xi_n$, then we have $\lim_{n\to\infty} \mathcal{E}[\xi_n] = \mathcal{E}[\lim_{n\to\infty} \xi_n]$.*

Proof. The proof of theorem is similar to that of Theorem 5. □

For sequence of integrable random fuzzy variables, we have the following general dominated convergence theorem:

Theorem 7. *Let $\{\xi_n\}$ be a sequence of random fuzzy variables, and η and ζ integrable random fuzzy variables such that $\eta \leq \xi_n \leq \zeta$ almost sure. If $\xi_n \xrightarrow{e.d.} \xi$, then we have $\lim_{n\to\infty} \mathcal{E}[\xi_n] = \mathcal{E}[\xi]$.*

Proof. The proof of theorem is similar to that of Theorem 5. □

Theorem 8. *Let $\{\xi_n\}$ be a sequence of random fuzzy variables, and η and ζ integrable random fuzzy variables such that $\eta \leq \xi_n \leq \zeta$ almost sure. If γ is a discrete fuzzy variable assuming finite number of values and $\xi_n \xrightarrow{\mathrm{Ch}} \xi$, then we have $\lim_{n\to\infty} \mathcal{E}[\xi_n] = \mathcal{E}[\xi]$.*

Proof. Since γ is a discrete fuzzy variable assuming finite number of values and $\xi_n \xrightarrow{\mathrm{Ch}} \xi$, it follows from Theorem 4 that $\xi_n \xrightarrow{e.d.} \xi$. By Theorem 5, we have the desired result. The proof of theorem is complete. □

Definition 12. *A random fuzzy variable ξ is said to be essentially bounded with respect to equilibrium measure* Ch *if there is a positive number a such that $G_\xi(-a) = 1$ and $G_\xi(a) = 0$.*

A sequence $\{\xi_k\}$ of random fuzzy variables is said to be uniformly essentially bounded with respect to equilibrium measure Ch *if there is a positive number a such that for each k, we have $G_{\xi_k}(-a) = 1$ and $G_{\xi_k}(a) = 0$.*

For essentially bounded random fuzzy variables, we have the following result:

Theorem 9. *Let $\{\xi_n\}$ and ξ be uniformly essentially bounded random fuzzy variables. If $\xi_n \xrightarrow{e.d.} \xi$, then we have $\lim_{n\to\infty} \mathcal{E}[\xi_n] = \mathcal{E}[\xi]$.*

Proof. By the supposition of theorem, $\{\xi_n\}$ is uniformly essentially bounded random fuzzy variables, there exist a positive number a such that for any n, $\mathrm{Ch}\{\xi_n \geq -a\} = 1$ and $\mathrm{Ch}\{\xi_n > a\} = 0$.

By the self-duality of equilibrium measure Ch, for any n, we have

$$\mathrm{Ch}\{\xi_n < -a\} = 0, \text{ and } \mathrm{Ch}\{\xi_n \leq a\} = 1.$$

By the definition of equilibrium measure, for any n, the following equalities

$$\mathrm{Cr}\{\xi_{n,\gamma} > a\} = \mathrm{Cr}\{\xi_{n,\gamma} < -a\} = 0$$

hold almost sure with respect to γ. If we denote $\eta = -a$, and $\zeta = a$, then $\eta \leq \xi_n \leq \zeta$ almost sure.

By the subadditivity of Pr, for every $t \in \Re$, the inequality

$$\Pr\{\xi_{n,\gamma} \geq t\} \leq \Pr\{\xi_{n,\gamma} \geq t, \xi_{n,\gamma} \geq \zeta_\gamma\} + \Pr\{\xi_{n,\gamma} \geq t, \xi_{n,\gamma} < \zeta_\gamma\} \leq \Pr\{\zeta_\gamma \geq t\}$$

holds almost sure with respect to γ. By the monotonicity of equilibrium measure, we have

$$\mathrm{Ch}\{\xi_n \geq t\} \leq \mathrm{Ch}\{\zeta \geq t\},$$

i.e., $G_{\xi_n} \leq G_\zeta$. Similarly, by

$$\Pr\{\eta_\gamma \geq t\} \leq \Pr\{\eta_\gamma \geq t, \xi_{n,\gamma} \geq \eta_\gamma\} + \Pr\{\eta_\gamma \geq t, \xi_{n,\gamma} < \eta_\gamma\} \leq \Pr\{\xi_{n,\gamma} \geq t\},$$

we have $G_\eta \leq G_{\xi_n}$. It follows from Theorem 7 that $\lim_{n\to\infty} E^e[\xi_n] = E^e[\xi]$. The proof of theorem is complete. □

Theorem 10. *Let $\{\xi_n\}$ and ξ be uniformly essentially bounded random fuzzy variables. If γ is a discrete fuzzy variable assuming finite number of values and $\xi_n \xrightarrow{\mathrm{Ch}} \xi$, then we have $\lim_{n\to\infty} \mathcal{E}[\xi_n] = \mathcal{E}[\xi]$.*

Proof. The assertion of theorem can be proved by combining Theorems 4 and 7. □

6 Conclusions

In present paper, we developed the convergence modes of random fuzzy variables in equilibrium measure theory, and obtained the following major results.

(*i*) Based on equilibrium measure, we introduced some new convergence modes of random fuzzy variables such as convergence in equilibrium measure and convergence in equilibrium distribution.

(*ii*) After discussing the properties of convergence modes of random fuzzy variables, we established the interconnections between convergence almost uniform and convergence almost sure, convergence almost uniform and convergence in equilibrium measure, convergence in equilibrium measure and convergence almost sure, and convergence in equilibrium measure and convergence in equilibrium distribution.

(*iii*) We introduced the equilibrium mean value of random fuzzy variable, and established the important monotone convergence theorems and dominated convergence theorems for sequences of integrable random fuzzy variables.

Equilibrium measure theory is a fertile area of research. The present paper has resolved some convergent results of integrable random fuzzy variables, some new issues have been exposed. In our future research, we will consider the potential applications of the convergence modes in the approximation scheme of equilibrium optimization models.

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