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Yager's Combination of Probabilistic and Possibilistic Knowledge: Beyond t-Norms

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Abstract

Often, about the same real-life system, we have both measurement-related probabilistic information expressed by a probability measure $P(S)$ and expert-related possibilistic information expressed by a possibility measure $M(S)$. To get the most adequate idea about the system, we must combine these two pieces of information. For this combination, R. Yager – borrowing an idea from fuzzy logic – proposed to use a t-norm $f_\&(a,b)$ such as the product $f_\&(a,b) = a \cdot b$, i.e., to consider a set function $f(S) = f_\&(P(S), M(S))$. A natural question is: can we uniquely reconstruct the two parts of knowledge from this function $f(S)$? In our previous paper, we showed that such a unique reconstruction is possible for strictly Archimedean t-norms; in this paper, we extend this result to a more general class of combination operations.

Keywords: Yager's combination, possibility measure, probability measure, t-norms, combination operations

1 Formulation of the Problem

Need to combine probabilistic and possibilistic knowledge. In many practical situations, we have both *probabilistic* information about some objects – e.g., information coming from measurements with known probability of measurement errors – and *possibilistic* information – describing expert knowledge. In the probabilistic case, for every set S, we have a probability $P(S) \in [0,1]$ that the actual (unknown) state s of the object belongs to the set S. In the possibilistic case, for each set S, we know the possibility $M(S) \in [0,1]$ that s belongs to S.

It is often desirable to combine these two numbers $P(S)$ and $M(S)$ into a single value $f(S)$.

Yager's approach: the use of t-norms [4]**.** We need to combine two degrees from the interval $[0,1]$. The desired combination must satisfy some reasonable properties; for example:

- if it is not possible for the state s to be in the set S, i.e., if $M(S) = 0$, then the resulting degree $f(S)$ must also reflect this impossibility, i.e., we should have $f(S) = 0$;
- if the probability $P(S)$ of s being in the set S is equal to 0, i.e., if $P(S) = 0$, then we should also have $f(S) = 0$, etc.

Different procedures of combining such degrees have been actively analyzed in fuzzy logic; see, e.g., [2, 3]. In particular, procedures that satisfy the above properties (and several other similar properties) are known as *t-norms* (or and-operations) $f_\&(a,b)$. It is therefore reasonable to combine $P(S)$ and $M(S)$ by using a t-norm, i.e., to consider the set function $f(S) = f_\&(P(S), M(S))$.

One of the simplest (and most widely used) t-norms is the algebraic product $f_\&(a,b) = a \cdot b$. In this case, we get a combination with a set function $f(S) = P(S) \cdot M(S)$.

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Uniqueness: a natural question. A natural question is: once we have the combined measure $f(S) = f_\&(P(S), M(S))$, can we reconstruct both $P(S)$ and $M(S)$?

Continuous case. We will consider a continuous case, in which the set X of all possible states is either an n-dimensional space $\mathbb{R}^n$ or its open subset, and we restrict ourselves to open subsets $S \subseteq X$. We assume that a probability measure $P(S)$ is described by a continuous probability density function $\rho(x) \geq 0$ for which $P(S) = \int_S \rho(x)\,dx$ and $\int_X \rho(x)\,dx = 1$. Similarly, we assume that a possibility measure is described by a continuous possibility function $\mu(x) \geq 0$ for which $M(S) = \sup\limits_{x \in S} \mu(x)$ and $\sup\limits_{x \in X} \mu(x) = 1$. We will also assume that a t-norm $f_\&(a, b)$ is continuous.

What is known and what we do in this paper. In [1], we showed that reconstruction is unique for strictly Archimedean t-norms. In this paper, we extend this result to a more general class of combination operations.

2 First Result: Reconstructing $P(S)$ from $f(S) = g(P(S), M(S))$

Reminder. In this paper, we consider situations in which the universal set X is an open subset of an n-dimensional space $\mathbb{R}^n$, a probability measure is defined by a continuous probability density function, and a possibility measure is defined by a continuous possibility function.

Definition 1. *By a* combination operation, *we mean a continuous function* $g : [0, 1] \times [0, 1] \to [0, 1]$ *which satisfies the following two properties:*

- *it is monotonic in each of the variables – i.e.,* $a \leq a'$ *and* $b \leq b'$ *imply* $g(a, b) \leq g(a', b')$*; and*
- *we have* $g(a, 1) = a$ *for all* $a \in [0, 1]$.

Theorem 1. *Let* $f(a, b)$ *be a combination operation, let* $P(S)$ *and* $P'(S)$ *be probability measures on the same set* X*, and let* $M(S)$ *and* $M'(S)$ *be possibility measures on* X*. If for every open set* $S \subseteq X$*, we have* $g(P(S), M(S)) = g(P'(S), M'(S))$*, then* $P(S) = P'(S)$ *for all sets* S.

Comment. In other words, if we know the combined measure $f(S) = g(P(S), M(S))$, then we can uniquely reconstruct the probability measure.

Proof. This proof is similar to the one from [1].

1°. For every point $x_0 \in X$ and for every positive real number δ, let $B_\delta(x_0) \stackrel{\text{def}}{=} \{x : d(x, x_0) < \delta\}$ denote an open ball with a center in x and radius δ. In this proof, we will consider sets of the type $S \cup B_\delta(x_0)$ in the limit $\delta \to 0$.

We want to know the limit of $f(S \cup B_\delta(x_0)) = g(P(S \cup B_\delta(x_0)), M(S \cup B_\delta(x_0)))$ when $\delta \to 0$. Since the combination operation $g(a, b)$ is continuous, it is sufficient to find the limits of $P(S \cup B_\delta(x_0))$ and $M(S \cup B_\delta(x_0))$; then, the limit of $f(S \cup B_\delta(x_0))$ is simply equal to the result of applying the combination operation $g(a, b)$ to the limits of $P(S \cup B_\delta(x_0))$ and $M(S \cup B_\delta(x_0))$.

2°. Let us start with computing the limit of $P(S \cup B_\delta(x_0))$. A probability measure is monotonic and additive, so we have $P(S) \leq P(S \cup B_\delta(x_0)) \leq P(S) + P(B_\delta(x_0))$. Let us show that $P(B_\delta(x_0)) \to 0$ as $\delta \to 0$; this will imply that $P(S \cup B_\delta(x_0)) \to P(S)$.

Indeed, since the probability density function $\rho(x)$ is continuous, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x, x_0) \leq \delta$ implies that $|\rho(x) - \rho(x_0)| \leq \varepsilon$. Let us pick any $\varepsilon_0 > 0$ (e.g., $\varepsilon_0 = 1$). Then, there exists a $\delta_0 > 0$ for which $d(x, x_0) \leq \delta_0$ implies that $|\rho(x) - \rho(x_0)| \leq \varepsilon_0$.

In this case, for every $\delta \leq \delta_0$, if $x \in B_\delta(x_0)$, then $d(x, x_0) < \delta \leq \delta_0$ hence $\rho(x) \leq \rho(x_0) + \varepsilon_0$. Thus,

$$0 \leq P(B_\delta(x_0)) = \int_{B_\delta(x_0)} \rho(x)\,dx \leq (\rho(x_0) + \varepsilon_0) \cdot V(B_\delta(x_0)).$$

When $\delta \to 0$, the sum $\rho(x_0) + \varepsilon_0$ is a constant and the volume $V(B_\delta(x_0)) \sim \delta^n$ tends to 0, so indeed $P(B_\delta(x_0)) \to 0$ and $P(S \cup B_\delta(x_0)) \to P(S)$.

3°. Let us now compute the limit of $M(S \cup B_\delta(x_0))$ when $\delta \to 0$. From the definition of a possibility measure, it follows that $M(A \cup B) = \max(M(A), M(B))$ for all A and B; in particular, $M(S \cup B_\delta(x_0)) = \max(M(S), M(B_\delta(x_0)))$. Since $\max(a, b)$ is a continuous function, it is sufficient to find the limit of $M(B_\delta(x_0))$.

The possibility function $\mu(x)$ is also assumed to be continuous, so for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x, x_0) \le \delta$ implies that $|\mu(x) - \mu(x_0)| \le \varepsilon$, i.e., for all $x \in B_\delta(x_0)$, we have

$$\mu(x_0) - \varepsilon \le \mu(x) \le \mu(x_0) + \varepsilon.$$

Since all the values $\mu(x)$ are between $\mu(x_0) - \varepsilon$ and $\mu(x_0) + \varepsilon$, the largest of these values $M(B_\delta(x_0)) = \sup_{B_\delta(x_0)} \mu(x)$ also lies within the same interval:

$$\mu(x_0) - \varepsilon \le M(B_\delta(x_0)) \le \mu(x_0) + \varepsilon.$$

Thus, for every $\varepsilon > 0$ there exists a δ for which $|M(B_\delta(x_0)) - \mu(x)| \le \varepsilon$. By definition of the limit, this means that $M(B_\delta(x_0)) \to \mu(x)$. So, due to the continuity of the maximum function,

$$M(S \cup B_\delta(x_0)) = \max(M(S), M(B_\delta(x_0))) \to \max(M(S), \mu(x)).$$

4°. Since the combination operation $g(a, b)$ is continuous and we know the limits for $P(S \cup B_\delta(x_0))$ and $M(S \cup B_\delta(x_0))$, we conclude that

$$f(S \cup B_\delta(x_0)) = g(P(S \cup B_\delta(x_0)), M(S \cup B_\delta(x_0))) \to g(P(S), \max(M(S), \mu(x))),$$

i.e., that

$$\lim_{\delta \to 0} f(S \cup B_\delta(x_0)) = g(P(S), \max(M(S), \mu(x_0))).$$

5°. We now want to find the largest value of $g(P(S), \max(M(S), \mu(x)))$, i.e.,

$$\sup_{x_0 \in X} g(P(S), \max(M(S), \mu(x_0))).$$

Since the combination operation is monotonic, it is sufficient to find the largest possible value of $\max(M(S), \mu(x_0))$:

$$\sup_{x_0 \in X} g(P(S), \max(M(S), \mu(x_0))) = g\left(P(S), \sup_{x_0 \in X} \max(S, \mu(x_0))\right).$$

By definition of a possibility measure,

$$M(X) = \sup_{x_0 \in X} \mu(x_0) = 1.$$

Since $\mu(x_0) \le \max(S, \mu(x_0)) \le 1$, we can thus conclude that $\sup\limits_{x_0 \in X} \max(S, \mu(x_0)) = 1$ and thus,

$$\sup_{x_0 \in X} g(P(S), \max(M(S), \mu(x_0))) = g(P(S), 1).$$

By definition of a combination operation, $g(a, 1) = a$, hence $g(P(S), 1) = P(S)$ and thus, for every set S,

$$\sup_{x_0 \in X} g(P(S), \max(M(S), \mu(x_0))) = P(S).$$

We already know how to describe $g(P(S), \max(M(S), \mu(x_0)))$ in terms of the combined function $f(S)$: $g(P(S), \max(M(S), \mu(x_0))) = \lim\limits_{\delta \to 0} f(S \cup B_\delta(x_0))$. Thus,

$$P(S) = \sup_{x_0 \in X} \lim_{\delta \to 0} f(S \cup B_\delta(x_0)).$$

This formula describes the probability measure in terms of the combined measure. So, the probability measure can indeed be uniquely reconstructed form the combined measure. The theorem is proven.

3 Second Result: For Strictly Monotonic Combination Operations, We Can Also Reconstruct $M(S)$ from $f(S) = g(P(S), M(S))$

Definition 1. *We say that a combination operation g is* strictly monotonic *if $0 < a$ and $0 < b < b'$ imply $g(a, b) < g(a, b')$.*

Discussion. In the previous section, we showed that we can uniquely reconstruct the probability measure $P(S)$ from the combined measure $f(S) = g(P(S), M(S))$. Let us show that for strictly monotonic combination operations, we can also reconstruct the possibility measure $M(S)$.

When $\rho(x) = 0$ for all points x from some region S, this means that the probability $P(S) = 0$ of this region is 0, so points x from this region are not possible. We can therefore exclude these points from our universal set X, and assume that $\rho(x) > 0$ for all $x \in X$. Such probability measures will be called *strictly positive.*

Theorem 2. *Let $g(a, b)$ be a strictly monotonic combination operation, let $P(S)$ and $P'(S)$ be strictly positive probability measures on the same set X, and let $M(S)$ and $M'(S)$ be possibility measures on X. If for every open set $S \subseteq X$, we have $g(P(S), M(S)) = g(P'(S), M'(S))$, then $P(S) = P'(S)$ and $M(S) = M'(S)$ for all sets S.*

Proof. According to Theorem 1, the fact that $g(P(S), M(S)) = g(P'(S), M'(S))$ for all open sets S implies that $P(S) = P'(S)$ for all such sets. Thus, for every open set S, we have $g(P(S), M(S)) = g(P(S), M'(S))$. For strictly positive probability measures, with continuous positive density function $\rho(x) > 0$, the probability $P(S) = \int_S \rho(x)\,dx$ is always positive $P(S) > 0$.

Thus, we cannot have $M(S) < M'(S)$, because then, due to the the definition of strict monotonicity, we would have $g(P(S), M(S)) < g(P(S), M'(S))$. Similarly, we cannot have $M'(S) < M(S)$, because then we would have $g(P(S), M'(S)) < g(P(S), M(S))$. Since we cannot have $M(S) < M'(S)$ and we cannot have $M'(S) < M(S)$, the only remaining possibility is $M(S) = M'(S)$. The theorem is proven.

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