Yager’s Combination of Probabilistic and Possibilistic Knowledge: Beyond t-Norms

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Abstract

Often, about the same real-life system, we have both measurement-related probabilistic information expressed by a probability measure \( P(S) \) and expert-related possibilistic information expressed by a possibility measure \( M(S) \). To get the most adequate idea about the system, we must combine these two pieces of information. For this combination, R. Yager – borrowing an idea from fuzzy logic – proposed to use a t-norm \( f(a,b) \) such as the product \( f(a,b) = a \cdot b \), i.e., to consider a set function \( f(S) = f_\&(P(S), M(S)) \).

A natural question is: can we uniquely reconstruct the two parts of knowledge from this function \( f(S) \)? In our previous paper, we showed that such a unique reconstruction is possible for strictly Archimedean t-norms; in this paper, we extend this result to a more general class of combination operations.

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1 Formulation of the Problem

Need to combine probabilistic and possibilistic knowledge. In many practical situations, we have both probabilistic information about some objects – e.g., information coming from measurements with known probability of measurement errors – and possibilistic information – describing expert knowledge. In the probabilistic case, for every set \( S \), we have a probability \( P(S) \in [0,1] \) that the actual (unknown) state \( s \) of the object belongs to the set \( S \). In the possibilistic case, for each set \( S \), we know the possibility \( M(S) \in [0,1] \) that \( s \) belongs to \( S \).

It is often desirable to combine these two numbers \( P(S) \) and \( M(S) \) into a single value \( f(S) \).

Yager’s approach: the use of t-norms [4]. We need to combine two degrees from the interval \([0,1]\). The desired combination must satisfy some reasonable properties; for example:

- if it is not possible for the state \( s \) to be in the set \( S \), i.e., if \( M(S) = 0 \), then the resulting degree \( f(S) \) must also reflect this impossibility, i.e., we should have \( f(S) = 0 \);
- if the probability \( P(S) \) of \( s \) being in the set \( S \) is equal to 0, i.e., if \( P(S) = 0 \), then we should also have \( f(S) = 0 \), etc.

Different procedures of combining such degrees have been actively analyzed in fuzzy logic; see, e.g., [2,3]. In particular, procedures that satisfy the above properties (and several other similar properties) are known as t-norms (or and-operations) \( f_\&(a,b) \). It is therefore reasonable to combine \( P(S) \) and \( M(S) \) by using a t-norm, i.e., to consider the set function \( f(S) = f_\&(P(S), M(S)) \).

One of the simplest (and most widely used) t-norms is the algebraic product \( f_\&(a,b) = a \cdot b \). In this case, we get a combination with a set function \( f(S) = P(S) \cdot M(S) \).

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Uniqueness: a natural question. A natural question is: once we have the combined measure \( f(S) = f_\mathcal{C}(P(S), M(S)) \), can we reconstruct both \( P(S) \) and \( M(S) \)?

Continuous case. We will consider a continuous case, in which the set \( \mathbb{R}^n \) or its open subset, and we restrict ourselves to open subsets \( S \subseteq X \). We assume that a probability measure \( P(S) \) is described by a continuous probability density function \( \rho(x) \geq 0 \) for which \( P(S) = \int_S \rho(x) \, dx \) and \( \int_X \rho(x) \, dx = 1 \). Similarly, we assume that a possibility measure is described by a continuous possibility function \( \mu(x) \geq 0 \) for which \( M(S) = \sup_{x \in S} \mu(x) \). We will also assume that a t-norm \( f_\mathcal{C}(a, b) \) is continuous.

What is known and what we do in this paper. In [1], we showed that reconstruction is unique for strictly Archimedean t-norms. In this paper, we extend this result to a more general class of combination operations.

2 First Result: Reconstructing \( P(S) \) from \( f(S) = g(P(S), M(S)) \)

Reminder. In this paper, we consider situations in which the universal set \( X \) is an open subset of an \( n \)-dimensional space \( \mathbb{R}^n \), a probability measure is defined by a continuous probability density function, and a possibility measure is defined by a continuous possibility function.

Definition 1. By a combination operation, we mean a continuous function \( g : [0, 1] \times [0, 1] \to [0, 1] \) which satisfies the following two properties:

- it is monotonic in each of the variables – i.e., \( a \leq a' \) and \( b \leq b' \) imply \( g(a, b) \leq g(a', b') \); and
- we have \( g(a, 1) = a \) for all \( a \in [0, 1] \).

Theorem 1. Let \( f(a, b) \) be a combination operation, let \( P(S) \) and \( P'(S) \) be probability measures on the same set \( X \), and let \( M(S) \) and \( M'(S) \) be possibility measures on \( X \). If for every open set \( S \subseteq X \), we have \( g(P(S), M(S)) = g(P'(S), M'(S)) \), then \( P(S) = P'(S) \) for all sets \( S \).

Comment. In other words, if we know the combined measure \( f(S) = g(P(S), M(S)) \), then we can uniquely reconstruct the probability measure.

Proof. This proof is similar to the one from [1].

1°. For every point \( x_0 \in X \) and for every positive real number \( \delta \), let \( B_\delta(x_0) \) denote an open ball with a center in \( x \) and radius \( \delta \). In this proof, we will consider sets of the type \( S \cup B_\delta(x_0) \) in the limit \( \delta \to 0 \).

We want to know the limit of \( f(S \cup B_\delta(x_0)) = g(P(S \cup B_\delta(x_0)), M(S \cup B_\delta(x_0))) \) when \( \delta \to 0 \). Since the combination operation \( g(a, b) \) is continuous, it is sufficient to find the limits of \( P(S \cup B_\delta(x_0)) \) and \( M(S \cup B_\delta(x_0)) \); then, the limit of \( f(S \cup B_\delta(x_0)) \) is simply equal to the result of applying the combination operation \( g(a, b) \) to the limits of \( P(S \cup B_\delta(x_0)) \) and \( M(S \cup B_\delta(x_0)) \).

2°. Let us start with computing the limit of \( P(S \cup B_\delta(x_0)) \). A probability measure is monotonic and additive, so we have \( P(S) \leq P(S \cup B_\delta(x_0)) \leq P(S) + P(B_\delta(x_0)) \). Let us show that \( P(B_\delta(x_0)) \to 0 \) as \( \delta \to 0 \); this will imply that \( P(S \cup B_\delta(x_0)) \to P(S) \).

Indeed, since the probability density function \( \rho(x) \) is continuous, for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( d(x, x_0) \leq \delta \) implies that \( |\rho(x) - \rho(x_0)| \leq \varepsilon \). Let us pick any \( \varepsilon_0 > 0 \) (e.g., \( \varepsilon_0 = 1 \)). Then, there exists a \( \delta_0 > 0 \) for which \( d(x, x_0) \leq \delta_0 \) implies that \( |\rho(x) - \rho(x_0)| \leq \varepsilon_0 \).

In this case, for every \( \delta \leq \delta_0 \), if \( x \in B_\delta(x_0) \), then \( d(x, x_0) < \delta \leq \delta_0 \) hence \( \rho(x) \leq \rho(x_0) + \varepsilon_0 \). Thus,

\[
0 \leq P(B_\delta(x_0)) = \int_{B_\delta(x_0)} \rho(x) \, dx \leq (\rho(x_0) + \varepsilon_0) \cdot V(B_\delta(x_0)).
\]
When \( \delta \rightarrow 0 \), the sum \( \rho(x_0) + \varepsilon_0 \) is a constant and the volume \( V(B_\delta(x_0)) \sim \delta^n \) tends to 0, so indeed \( P(B_\delta(x_0)) \rightarrow 0 \) and \( P(S \cup B_\delta(x_0)) \rightarrow P(S) \).

3°. Let us now compute the limit of \( M(S \cup B_\delta(x_0)) \) when \( \delta \rightarrow 0 \). From the definition of a possibility measure, it follows that \( M(A \cup B) = \max(M(A), M(B)) \) for all \( A \) and \( B \); in particular, \( M(S \cup B_\delta(x_0)) = \max(M(S), M(B_\delta(x_0))) \). Since \( \max(a, b) \) is a continuous function, it is sufficient to find the limit of \( M(B_\delta(x_0)) \).

The possibility function \( \mu(x) \) is also assumed to be continuous, so for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( d(x, x_0) \leq \delta \) implies that \( |\mu(x) - \mu(x_0)| \leq \varepsilon \), i.e., for all \( x \in B_\delta(x_0) \), we have

\[
\mu(x_0) - \varepsilon \leq \mu(x) \leq \mu(x_0) + \varepsilon.
\]

Since all the values \( \mu(x) \) are between \( \mu(x_0) - \varepsilon \) and \( \mu(x_0) + \varepsilon \), the largest of these values \( M(B_\delta(x_0)) = \sup_{B_\delta(x_0)} \mu(x) \) also lies within the same interval:

\[
\mu(x_0) - \varepsilon \leq M(B_\delta(x_0)) \leq \mu(x_0) + \varepsilon.
\]

Thus, for every \( \varepsilon > 0 \) there exists a \( \delta \) for which \( |M(B_\delta(x_0)) - \mu(x)| \leq \varepsilon \). By definition of the limit, this means that \( M(B_\delta(x_0)) \rightarrow \mu(x) \). So, due to the continuity of the maximum function,

\[
M(S \cup B_\delta(x_0)) = \max(M(S), M(B_\delta(x_0))) \rightarrow \max(M(S), \mu(x)).
\]

4°. Since the combination operation \( g(a, b) \) is continuous and we know the limits for \( P(S \cup B_\delta(x_0)) \) and \( M(S \cup B_\delta(x_0)) \), we conclude that

\[
f(S \cup B_\delta(x_0)) = g(P(S \cup B_\delta(x_0)), M(S \cup B_\delta(x_0))) \rightarrow g(P(S), \max(M(S), \mu(x))),
\]

i.e., that

\[
\lim_{\delta \rightarrow 0} f(S \cup B_\delta(x_0)) = g(P(S), \max(M(S), \mu(x))).
\]

5°. We now want to find the largest value of \( g(P(S), \max(M(S), \mu(x))) \), i.e.,

\[
\sup_{x_0 \in X} g(P(S), \max(M(S), \mu(x)))).
\]

Since the combination operation is monotonic, it is sufficient to find the largest possible value of \( \max(M(S), \mu(x)) \):

\[
\sup_{x_0 \in X} g(P(S), \max(M(S), \mu(x))) = g\left(P(S), \sup_{x_0 \in X} \max(M(S), \mu(x))\right).
\]

By definition of a possibility measure,

\[
M(X) = \sup_{x_0 \in X} \mu(x_0) = 1.
\]

Since \( \mu(x_0) \leq \max(S, \mu(x_0)) \leq 1 \), we can thus conclude that \( \sup_{x_0 \in X} \max(S, \mu(x_0)) = 1 \) and thus,

\[
\sup_{x_0 \in X} g(P(S), \max(M(S), \mu(x))) = g(P(S), 1).
\]

By definition of a combination operation, \( g(a, 1) = a \), hence \( g(P(S), 1) = P(S) \) and thus, for every set \( S \),

\[
\sup_{x_0 \in X} g(P(S), \max(M(S), \mu(x))) = P(S).
\]

We already know how to describe \( g(P(S), \max(M(S), \mu(x))) \) in terms of the combined function \( f(S) \):

\[
g(P(S), \max(M(S), \mu(x))) = \lim_{\delta \rightarrow 0} f(S \cup B_\delta(x_0)).
\]

Thus,

\[
P(S) = \sup_{x_0 \in X} \lim_{\delta \rightarrow 0} f(S \cup B_\delta(x_0)).
\]

This formula describes the probability measure in terms of the combined measure. So, the probability measure can indeed be uniquely reconstructed from the combined measure. The theorem is proven.
3 Second Result: For Strictly Monotonic Combination Operations, We Can Also Reconstruct $M(S)$ from $f(S) = g(P(S), M(S))$

**Definition 1.** We say that a combination operation $g$ is strictly monotonic if $0 < a$ and $0 < b < b'$ imply $g(a, b) < g(a, b')$.

**Discussion.** In the previous section, we showed that we can uniquely reconstruct the probability measure $P(S)$ from the combined measure $f(S) = g(P(S), M(S))$. Let us show that for strictly monotonic combination operations, we can also reconstruct the possibility measure $M(S)$.

When $\rho(x) = 0$ for all points $x$ from some region $S$, this means that the probability $P(S) = 0$ of this region is 0, so points $x$ from this region are not possible. We can therefore exclude these points from our universal set $X$, and assume that $\rho(x) > 0$ for all $x \in X$. Such probability measures will be called strictly positive.

**Theorem 2.** Let $g(a, b)$ be a strictly monotonic combination operation, let $P(S)$ and $P'(S)$ be strictly positive probability measures on the same set $X$, and let $M(S)$ and $M'(S)$ be possibility measures on $X$. If for every open set $S \subseteq X$, we have $g(P(S), M(S)) = g(P'(S), M'(S))$, then $P(S) = P'(S)$ and $M(S) = M'(S)$ for all sets $S$.

**Proof.** According to Theorem 1, the fact that $g(P(S), M(S)) = g(P'(S), M'(S))$ for all open sets $S$ implies that $P(S) = P'(S)$ for all such sets. Thus, for every open set $S$, we have $g(P(S), M(S)) = g(P(S), M'(S))$. For strictly positive probability measures, with continuous positive density function $\rho(x) > 0$, the probability $P(S) = \int_S \rho(x) \, dx$ is always positive $P(S) > 0$.

Thus, we cannot have $M(S) < M'(S)$, because then, due to the the definition of strict monotonicity, we would have $g(P(S), M(S)) < g(P'(S), M'(S))$. Similarly, we cannot have $M'(S) < M(S)$, because then we would have $g(P(S), M'(S)) < g(P'(S), M(S))$. Since we cannot have $M(S) < M'(S)$ and we cannot have $M'(S) < M(S)$, the only remaining possibility is $M(S) = M'(S)$. The theorem is proven.

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**References**


