Journal of Uncertain Systems
Vol.7, No.2, pp.152-160, 2013
Online at: www.jus.org.uk

Generalized Fuzzy Filters (Ideals) of BE–Algebras

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Received 18 June 2012; Revised 31 December 2012

Abstract

In this paper, we introduce the concept of ε–generalized fuzzy filters (ideals) and δ–multiplication of fuzzy set μ, of BE–algebras and their basic properties are investigated.

Keywords: BE-algebra, fuzzy filter(ideal), ε–generalized fuzzy filter(ideal), fuzzy δ–multiplication

1 Introduction and Preliminaries

Imai and Iseki [5] introduced two classes of abstract algebras: BCK–algebras and BCI–algebras. It is known that the class of BCK–algebras is a proper subclass of the class BCI–algebras. There exist several generalization of BCK/BCI–algebras, as such BCH–algebras [3], d–algebras [8], etc. Especially, Kim and Kim [7] introduced the notion of BE–algebras which was deeply studied by Ahn and Kim in [1], Ahn and So [2], Walendziak [14], Rezaei and Borumand Saeid [9, 10, 11]. As it is well known Zadeh [15], introduced the notion of a fuzzy subset μ of a nonempty set X as a function from X to unit real interval $I = [0, 1]$. Then many authors have studied about it for engineering, medical science, social science, physics, statistics, graph theory, etc. Sun in [13] introduced the concepts of generalized fuzzy subalgebras and generalized fuzzy ideals of Boolean algebras and discuss some properties of it. Jun in [6] discussed fuzzy translation, fuzzy extensions and fuzzy multiplications of fuzzy ideals in BCK/BCI–algebras.

In the present paper we introduce the notions of ε–generalized fuzzy filters (ideals) in BE–algebra. Then we prove some theorems and review some basic properties on this algebra by using our definitions and show that:
(1) Every filter of BE–algebra X is a level filter(ideal) of an ε–generalized fuzzy filter(ideal) of X, for some $\varepsilon \in (0, 1]$.
(2) If μ and ν are ε–generalized fuzzy filter(ideal) of X, then ε–cartesian product μ and ν is so.
(3) If μ is an ε–generalized fuzzy filter(ideal) of X , then μ^+ is so.
(4) If μ is an ε–generalized fuzzy filter(ideal) of X and $\varepsilon, \delta \in (0, 1]$, then μ_δ is $\varepsilon.\delta$–generalized fuzzy filter(ideal).

Definition 1. [7] *An algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE–algebra if*

(BE1) $x * x = 1$,

(BE2) $x * 1 = 1$,

(BE3) $1 * x = x$,

(BE4) $x * (y * z) = y * (x * z)$, *for all* $x, y, z \in X$.

The binary relation "$\leq$" on X is defined by $x \leq y$ if and only if $x * y = 1$, for all $x, y \in X$.
From now on, in this paper X is a BE–algebra, unless otherwise is stated.

Definition 2. *A nonempty subset S of X is said to be a subalgebra of X if it satisfies: $x, y \in S$ implies $x * y \in S$, for all $x, y \in X$.*

Definition 3. [1] *A nonempty subset I of X is called an ideal if it satisfies:*

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(I1) $x * a \in I$,

(I2) $(a * (b * x)) * x \in I$, *for all* $x \in X$ *and* $a, b \in I$.

Lemma 1. [12] *A nonempty subset* I *of* X *is called an ideal if and only if it satisfies:*

(1) $1 \in I$,

(2) $x * (y * z) \in I$ *implies that* $x * z \in I$, *for all* $x, z \in X$ *and* $y \in I$.

Definition 4. [7] *A subset* F *of* X *is called a filter if it satisfies:*

(F1) $1 \in F$,

(F2) $x \in F$ *and* $x * y \in F$ *imply* $y \in F$.

Definition 5. [7] X *is said to be self distributive if* $x * (y * z) = (x * y) * (x * z)$, *for all* $x, y, z \in X$.

Definition 6. [7] X *is said to be transitive if* $y * z \leq (x * y) * (x * z)$, *for any* $x, y, z \in X$.

Definition 7. [7] *Let* $x, y \in X$. *Define*

$$A(x, y) = \{z \in X : x * (y * z) = 1\} \quad and \quad A(x) = \{z \in X : x * z = 1\}.$$

Applying $(BE3)$ *we conclude that* $A(x) = A(1, x)$.

The set $A(x)$ *(resp.* $A(x, y)$*) is called an upper set of* x *(resp. of* x *and* y*). We say a subset* A *of* X *is an upper set of* X *if* $A = A(x, y)$, *for some* $x, y \in X$.

Definition 8. [11] *A fuzzy set* μ *is called a fuzzy filter of* X *if it satisfies:*

(FF1) $\mu(1) \geq \mu(x)$,

(FF2) $\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$, *for all* $x, y \in X$.

Proposition 1. [11] *Let* μ *be a fuzzy filter in* X. *Then*

(i) $x \leq y * z \Rightarrow \mu(y * z) \geq \mu(x)$,

(ii) $\mu(x * y) = \mu(1) \Rightarrow \mu(y) \leq \mu(x)$,*for all* $x, y, z \in X$.

Definition 9. [11] *A fuzzy set* μ *is called a fuzzy ideal of* X *if it satisfies:*

(FI1) $\mu(x * y) \geq \mu(y)$,

(FI2) $\mu((x * (y * z)) * z) \geq \min\{\mu(x), \mu(y)\}$, *for all* $x, y, z \in X$.

Definition 10. [9] *A mapping* $f : X \to Y$ *of* BE*–algebras is called a* BE*–homomorphism if* $f(x * y) = f(x) * f(y)$, *for all* $x, y \in X$.

Definition 11. [11] *A fuzzy filter* μ *of* X *is said to be normal if there exists* $x \in X$ *such that* $\mu(x) = 1$.

Theorem 1. [11] *Let* μ *be a fuzzy filter of* X. *Let* μ^+ *be a fuzzy set defined by* $\mu^+(x) = \mu(x) + 1 - \mu(1)$, *for all* $x \in X$. *Then* μ^+ *is a normal fuzzy filter of* X.

Corollary 1. [11] *Let* μ *and* μ^+ *be as in Theorem 1. If there exists* $x \in X$ *such that* $\mu^+(x) = 1$, *then* $\mu(x) = 1$.

Theorem 2. [11] *A fuzzy filter* μ *of* X *is normal if and only if* $\mu^+ = \mu$.

Proposition 2. [11] *If* μ *is a fuzzy filter of* X, *then* $(\mu^+)^+ = \mu^+$.

2 ε–Generalized Fuzzy Filters (Ideals) of BE–Algebras

Definition 12. *Let $\varepsilon \in \mathbb{R}^+$, μ be a fuzzy subset of X. μ is called an ε–generalized fuzzy ideal if it satisfies:*

(GFI1) $\mu(x * y) \geq \min\{\mu(y), \varepsilon\}$,

(GFI2) $\mu((x * (y * z)) * z) \geq \min\{\mu(x), \mu(y), \varepsilon\}$, *for all $x, y, z \in X$.*

Note. If $\varepsilon \geq 1$, then every ε–generalized fuzzy ideal is a fuzzy ideal of X. If $\varepsilon < 0$, then we can redefine Definition 12, with ε–generalized anti fuzzy ideal of X if it satisfies:
(GAFI1) $\mu(x * y) \leq \max\{\mu(y), \varepsilon\}$,
(GAFI2) $\mu((x * (y * z)) * z) \leq \max\{\mu(x), \mu(y), \varepsilon\}$,for all $x, y, z \in X$.

Example 1: Let $X = \{1, a, b\}$ be a set with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	a	1

Then $(X; *, 1)$ is a BE-algebra.

Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(1) = 0.6$, $\mu(a) = 0.4$ and $\mu(b) = 0.3$. Then μ is a 0.2–generalized fuzzy ideal of X.

In the following we show that any ε–generalized fuzzy ideal need not be a fuzzy ideal.

Example 2: In Example 1, define fuzzy set $\mu : X \to [0, 1]$ by $\mu(1) = 0.4$ and $\mu(a) = \mu(b) = 0.5$ and let $\varepsilon = 0.2$. Then μ is an ε–generalized fuzzy ideal but it is not fuzzy ideal of X because

$$\mu((a * (b * 1)) * 1) = \mu((a * 1) * 1) = \mu(1 * 1) = 0.4 \ngeq \min\{\mu(a), \mu(b)\} = 0.5.$$

Lemma 2. *Let $\varepsilon \in (0, 1]$. Then every ε–generalized fuzzy ideal μ of X satisfies the following inequality:*

(i) $\mu(1) \geq \min\{\mu(x), \varepsilon\}$,

(ii) $\mu((x * y) * y) \geq \min\{\mu(x), \varepsilon\}$, *for all $x, y \in X$.*

Proof. (i) Using $(BE1)$ and $(GFI1)$, we have

$$\mu(1) = \mu(x * x) \geq \min\{\mu(x), \varepsilon\},$$

for all $x \in X$. (ii) Taking $y := 1$ and $z := y$ in $(GFI2)$ and using $(CI2)$ and (i), we have

$$\mu((x * y) * y) = \mu((x * (1 * y) * y) \geq \{\mu(x), \mu(1), c\} = \{\mu(x), \varepsilon\},$$

for all $x, y \in X$.

Proposition 3. *Let $\varepsilon \in (0, 1]$ and μ be a fuzzy set in X which satisfies:*

(i) $\mu(1) \geq \min\{\mu(x), \varepsilon\}$,

(ii) $\mu(x * y) \geq \min\{\mu(x * (z * y)), \mu(z), \varepsilon\}$, *for all $x, y, z \in X$.*
Then $x \leq y$ implies that $\mu(y) \geq \min\{\mu(x), \varepsilon\}$.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so by hypothesis and $(BE1)$, $(BE3)$ we have

$$\mu(y) = \mu(1 * y) \geq \min\{\mu(1 * (x * y)), \mu(x), \varepsilon\} = \min\{\mu(1 * 1), \mu(x), \varepsilon\} = \min\{\mu(x), c\}.$$

Proposition 4. *Let μ be an ε–generalized fuzzy ideal of X and $\varepsilon \in (0, 1]$. Then if $x \leq y$, then $\min\{\mu(x), \varepsilon\} \leq \mu(y)$, for all $x, y \in X$.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so by $(BE3)$ and Lemma 2 we have $\mu(y) = \mu(1 * y) = \mu((x * y) * y) \geq \min\{\mu(x), \varepsilon\}$.

Proposition 5. *If X is transitive, then every ε–generalized fuzzy ideal of X satisfies the following assertion: $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y), \varepsilon\}$, for all $x, y, z \in X$.*

Proof. Since X is transitive, then $((y*z)*z)*((x*(y*z))*(x*z)) = 1$, for all $x, y, z \in X$. Using $(BE3)$ and Lemma 2, we have

$$\begin{aligned} \mu(x*z) = \mu((1*(x*z)) &= \mu((((y*z)*z)*((x*(y*z))*(x*z)))*(x*z)) \\ &\geq \min(\mu(y*z)*z), \mu(x*(y*z)), \varepsilon\} \\ &= \min(\mu(y), \mu(x*(y*z), \varepsilon\}. \end{aligned}$$

Definition 13. *Let $\varepsilon \in \mathbb{R}^+$ and μ be a fuzzy subset of X. μ is called an ε–generalized fuzzy filter if it satisfies:*

(GFF1) $\mu(1) \geq \min\{\mu(x), \varepsilon\}$,

(GFF2) $\mu(y) \geq \min\{\mu(x), \mu(x*y), \varepsilon\}$, *for all $x, y \in X$.*

Note. *If $\varepsilon \geq 1$, then every ε–generalized fuzzy filter is a fuzzy filter of X. If $\varepsilon < 0$, then we can redefine Definition 13, with ε–generalized anti fuzzy filter of X if it satisfies:*

(GAFF1) $\mu(1) \leq \max\{\mu(x), \varepsilon\}$,

(GAFF2) $\mu(y) \leq \max\{\mu(x), \mu(x*y), \varepsilon\}$, *for all $x, y, z \in X$.*

If $\varepsilon \geq 1$, then every ε–generalized fuzzy filter is a fuzzy filter of X.

Example 3: Let $\varepsilon := 0.5$ and $X = \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	c	c	1

Then $(X; *, 1)$ is a BE-algebra. Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(1) = 0.7$, $\mu(a) = \mu(b) = 0.2$ and $\mu(c) = 0.1$. Then μ is a 0.5-generalized fuzzy filter of X.

In the following we show that any ε–generalized fuzzy filter need not be a fuzzy filter.

Example 4: In the Example 2 define fuzzy set $\mu : X \to [0, 1]$ by $\mu(1) = 0.7$, $\mu(a) = 0.3$ and $\mu(b) = \mu(c) = 0.4$ and let $\varepsilon = 0.2$. Then μ is an ε–generalized fuzzy filter but it is not fuzzy filter of X because

$$\mu(a) = 0.3 \not\geq \min\{\mu(c), \mu(c*a)\} = \min\{\mu(c), \mu(c)\} = \{0.4, 0.4\} = 0.4.$$

Proposition 6. *Let μ be an ε–generalized fuzzy filter(ideal) of X and $\varepsilon \in (0, 1]$. Then μ is a β–generalized fuzzy filter(ideal) of X, where $\beta \in (0, 1]$ and $\beta \leq \varepsilon$.*

Proof. Since μ is an ε–generalized fuzzy filter of X and $\beta \leq \varepsilon$, then

$$\mu(1) \geq \min\{\mu(x), \varepsilon\} \geq \min\{\mu(x), \beta\}, \text{ for any } x \in X;$$

$$\mu(y) \geq \min\{\mu(x), \mu(x*y), \varepsilon\} \geq \min\{\mu(x), \mu(x*y), \beta\}, \text{ for all } x, y \in X.$$

Hence μ is a β–generalized fuzzy filter of X.

Proposition 7. *Let $\varepsilon \in (0, 1]$ and μ be an ε–generalized fuzzy filter of X. Then*

(i) $x \leq y*z \Rightarrow \mu(y*z) \geq \min\{\mu(x), c\}$,

(ii) $\mu(x*y) = \mu(1) \Rightarrow \mu(y) \geq \min\{\mu(x), \varepsilon\}$, *for all $x, y, z \in X$.*

Proof. (i) Since $x \leq y*z$, then $x*(y*z) = 1$ and so we have

$$\mu(y*z) \geq \min\{\mu(x), \mu(x*(y*z)), \varepsilon\} = \min\{\mu(x), \mu(1), \varepsilon\} = \min\{\mu(x), \varepsilon\}.$$

(ii) By hypothesis and Definition 13, we have

$$\mu(y) \geq \min\{\mu(x), \mu(x*y), \varepsilon\} = \min\{\mu(x), \mu(1), \varepsilon\} = \min\{\mu(x), \varepsilon\}.$$

Theorem 3. *Let $\varepsilon \in (0, 1]$ and μ be an ε–generalized fuzzy filter(ideal) of X. If there exists a sequence $\{x_n\}_{n \in N}$ in X, such that $\lim_{n\to\infty} \mu(x_n) = \varepsilon$, then $\mu(1) \geq \varepsilon$.*

Proof. By Lemma 2, we have $\mu(1) \geq \min\{\mu(x), \varepsilon\}$, for all $x \in X$, thus $\mu(1) \geq \min\{\mu(x_n), \varepsilon\}$, for every positive integer n. Consider

$$\mu(1) \geq \min\{\lim_{n\to\infty} \mu(x_n), \varepsilon\} = \min\{\varepsilon, \varepsilon\} = \varepsilon.$$

Hence $\mu(1) \geq \varepsilon$.

Theorem 4. *Let $\{\mu_{\varepsilon_\alpha}\}_{\alpha\in\Lambda}$ are family of ε_α–generalized fuzzy filter(ideal) of X, and $\varepsilon_\alpha \in (0, 1]$, for all $\alpha \in \Lambda$. Then μ is an ε–generalized fuzzy filter(ideal) of X, where $\varepsilon = \inf\{\varepsilon_\alpha : \alpha \in \Lambda\}$. Hence forms a complete distributive lattice under the ordering of fuzzy set inclusion $\subset$.*

Proof. Let $\{\mu_{\varepsilon_\alpha}\}_{\alpha\in\Lambda}$ be a family of ε_α–generalized fuzzy filter of X and $\varepsilon_\alpha \in (0, 1]$. Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$, it is sufficient to show that $\bigcap_{\alpha\in\Lambda} \mu_{\varepsilon_\alpha}$ is an ε–generalized fuzzy filter of X. Set $\varepsilon = \inf\{\varepsilon_\alpha : \alpha \in \lambda\}$ and let $x \in X$. Then

$$\begin{aligned}
(\bigcap_{\alpha\in\Lambda} \mu_{\varepsilon_\alpha})(1) &= \inf\{\mu_{\varepsilon_\alpha}(1) : \alpha \in \Lambda\} \\
&\geq \inf\{\min\{\mu_{\varepsilon_\alpha}(x), \varepsilon_\alpha\} : \alpha \in \Lambda\} \\
&= \min(\inf\{\mu_{\varepsilon_\alpha}(x) : \alpha \in \Lambda\}, \inf\{\varepsilon_\alpha : \alpha \in \Lambda\}) \\
&= \min(\bigcap_{\alpha\in\Lambda} \mu_{\varepsilon_\alpha}(x), \varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
(\bigcap_{\alpha\in\Lambda} \mu_{\varepsilon_\alpha})(y) &= \inf\{\mu_{\varepsilon_\alpha}(y) : \alpha \in \Lambda\} \\
&\geq \inf\{\min\{\mu_{\varepsilon_\alpha}(x), \mu_{\varepsilon_\alpha}(x * y), \varepsilon_\alpha : \alpha \in \Lambda\} \\
&= \min(\inf\{\mu_{\varepsilon_\alpha}(x) : \alpha \in \Lambda\}, \inf\{\mu_{\varepsilon_\alpha}(x * y) : \alpha \in \Lambda\}, \inf\{\varepsilon_\alpha : \alpha \in \Lambda\}) \\
&= \min(\bigcap_{\alpha\in\Lambda} \mu_{\varepsilon_\alpha}(x), \bigcap_{\alpha\in\Lambda} \mu_{\varepsilon_\alpha}(x * y), \varepsilon).
\end{aligned}$$

Hence $\bigcap_{\alpha\in\Lambda} \mu_{\varepsilon_\alpha}$ is an ε–generalized fuzzy filter of X.

Definition 14. *Let $\varepsilon \in (0, 1]$ and μ be an ε–generalized fuzzy filter(ideal) in X and $\lambda \in [0, 1]$. Then the level filter(ideal) $U(\mu; \lambda)$ of μ and strong level filter(ideal) $U(\mu; >, \lambda)$ of X are defined as following:*

$$U(\mu; \lambda) := \{x \in X \mid \mu(x) \geq \lambda\},$$

$$U(\mu; >, \lambda) := \{x \in X \mid \mu(x) > \lambda\}.$$

Note. If $\lambda \geq \varepsilon$, then $U(\mu; \lambda) \subseteq U(\mu; \varepsilon)$.

Example 5: In Example 2, $U(\mu; 0.5) = \{a, b\}$, $U(\mu; >, 0.3) = \emptyset$, $U(\mu; 0.3) = \{b\}$.

Theorem 5. *Let μ be a fuzzy subset ideal(filter) of X. Then μ is ε–generalized fuzzy ideal(filter) if and only if $U(\mu; \lambda)$ is an ideal(filter), for $\lambda \in (0, 1]$ and $\lambda \leq \epsilon$.*

Proof. Let μ be an ε–generalized fuzzy ideal of X, $\lambda \in (0, 1]$, $\lambda \leq \varepsilon$ and $x \in X$, $y \in U(\mu; \lambda)$. Then $\mu(x * y) \geq \min\{\mu(y), \varepsilon\} \geq \lambda$, so $x * y \in U(\mu; \lambda)$. Now, if $x \in X$ and $a, b \in U(\mu; \lambda)$, then

$$\mu((a * (b * x)) * x) \geq \min\{\mu(a), \mu(b), \varepsilon\} \geq \lambda.$$

So $(a * (b * x)) * x \in U(\mu; \lambda)$. Thus $U(\mu; \lambda)$ is an ideal of X.

Conversely, let $U(\mu; \lambda)$ be an ideal of X, for all $\lambda \in (0, 1]$ and $\lambda \leq \varepsilon$. If there exists $a, b \in X$, such that

$$\mu((a * (b * x)) * x) < \min\{\mu(a), \mu(b), \varepsilon\} = \kappa,$$

then $a, b \in U(\mu;\kappa)$, for $\kappa \in (0,1]$ and $\mu((a*(b*x))*x) < \kappa$. Since $U(\mu;\kappa)$ is an ideal of X, hence $((a*(b*x))*x \in U(\mu;\kappa)$ and $\mu((a*(b*x))*x \geq \kappa$, which is a contradiction. Therefore $\mu((a*(b*x))*x) \geq \min\{\mu(a), \mu(b), \varepsilon\}$.

If there exists $a, b \in X$, such that

$$\mu(a*b) < \min\{\mu(b), \varepsilon\} = \kappa.$$

Then $b \in U(\mu;\kappa)$, and $\kappa \in (0,1]$, $\kappa \leq \lambda$. Hence $\mu(b) < \kappa$. Since $U(\mu;\kappa)$ is an ideal of X, hence $a*b \in U(\mu;\kappa)$ and $\mu(a*b) \geq \kappa$, which is a contradiction.

Theorem 6. *Let μ be a fuzzy subset in X and $\varepsilon \in (0,1]$. Then μ is ε–generalized fuzzy ideal if and only if μ satisfies the following assertion: $(\forall a, b \in X) a, b \in U(\mu;\lambda) \Rightarrow A(a,b) \subseteq U(\mu;\min\{\lambda,\varepsilon\})$, for all $\lambda \in [0,1]$.*

Proof. Assume that μ is an ε–generalized fuzzy ideal of $X, a, b \in U(\mu;\lambda), \lambda \in [0,1]$ and $\varepsilon \in (0,1]$. Let $x \in A(a,b)$. Then $a*(b*x) = 1$. Using $(BE3)$ and $(GFI2)$ we have $\mu(x) = \mu(1*x) = \mu((a*(b*x))*x) \geq \min\{\mu(a), \mu(b), \varepsilon\} \geq \min\{\lambda, \varepsilon\}$. Hence $A(a,b) \subseteq U(\mu;\min\{\lambda,\varepsilon\})$.

Conversely, suppose that μ satisfies the above assertion. Note that since $1 \in A(a,b) \subseteq U(\mu;\min\{\lambda,\varepsilon\}$, for all $a, b \in X$. Let $x, y, z \in X$ be such that $x*(y*z) \in U(\mu;\min\{\lambda,\varepsilon\})$ and $y \in U(\mu;\min\{\lambda,\varepsilon\})$. By $(BE3)$ and $(BE1)$ we have $(x*(y*z))*(y*(x*z)) = (x*(y*z))*(x*(y*z)) = 1$. Therefore we have $x*z \in A(x*(y*z), y) \subseteq \mu\{\lambda,\varepsilon\}$. It follows form Lemma 1, that $U(\mu;\min\{\lambda,\varepsilon\})$ is an ideal of X for all $\lambda \in (0,1]$.

Theorem 7. *Let $\varepsilon \in (0,1]$. If μ is an ε–generalized fuzzy filter(ideal), then the set $X_\mu := \{x \in X \mid \mu(x) \geq \min\{\mu(1), \varepsilon\}\}$ is a filter(ideal).*

Proof. Let $x, x*y \in X_\mu$ and $\varepsilon \in (0,1]$. Then $\mu(x) = \mu(x*y) \geq \min\{\mu(1), \varepsilon\}$, and so

$$\mu(y) \geq \min\{\mu(x), \mu(x*y), \varepsilon\} \geq \min\{\mu(1), \mu(1), \varepsilon\} = \min\{\mu(1), \varepsilon\}.$$

Hence $\mu(y) \geq \min\{\mu(1), \varepsilon\}$, which means that $y \in X_\mu$.

Theorem 8. *Let F be a crisp subset of X and $\varepsilon \in (0,1]$. Suppose that μ is a fuzzy set defined by:*

$$\mu(x) = \begin{cases} \min\{\alpha, \varepsilon\} & \text{if } x \in F \\ \beta & \text{otherwise} \end{cases}$$

for all $\alpha, \beta \in [0,1]$ with $\min\{\alpha,\varepsilon\} \geq \beta$. Then μ is a ε–generalized fuzzy filter(ideal) if and only if F is a filter(ideal). Moreover, in this case $X_\mu = F$.

Proof. Let μ be ε–generalized fuzzy filter and $x, y \in X$ be such that $x, x*y \in F$. Then $\mu(y) \geq \min\{\mu(x), \mu(x*y), \varepsilon\} = \min\{\min\{\alpha,\varepsilon\}, \min\{\alpha,\varepsilon\}, \varepsilon\} = \min\{\alpha,\varepsilon\}$, so $y \in F$.

Conversely, suppose that F is a filter of X, let $x, y \in X$.

case 1) If $x, x*y \in F$, then $y \in F$, thus

$$\mu(y) = \min\{\alpha,\varepsilon\} = \min\{\mu(x), \mu(x*y), \varepsilon\}.$$

case 2) If $x \notin F$ or $x*y \notin F$, then

$$\mu(y) \geq \beta = \min\{\mu(x), \mu(x*y), \varepsilon\}.$$

case 3) If $x \notin F$ and $x*y \in F$, then

$$\mu(y) \geq \beta = \min\{\mu(x), \mu(x*y), \varepsilon\} = \{\beta, \min\{\alpha,\varepsilon\}, \varepsilon\}.$$

case 4) If $x \in F$ and $x*y \notin F$, then

$$\mu(y) \geq \beta = \min\{\mu(x), \mu(x*y), \varepsilon\} = \{\min\{\alpha,\varepsilon\}, \beta, \varepsilon\}.$$

This shows that μ is an ε–generalized fuzzy filter. Moreover, we have

$$X_\mu := \{x \in X \mid \mu(x) = \min\{\mu(1), \varepsilon\}\} = \{x \in X \mid \mu(x) = \min\{\alpha,\varepsilon\}\} = F.$$

Theorem 9. *Each filter of X is a level filter(ideal) of a ε–generalized fuzzy filter(ideal) of X for some $\varepsilon \in (0,1]$.*

Proof. Let F be a filter of X, $\varepsilon \in (0,1]$ and μ be a fuzzy set defined by

$$\mu(x) = \begin{cases} \min\{\alpha, \varepsilon\} & \text{if } x \in F \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in [0,1]$. It is clear that $U(\mu; \min\{\alpha,\varepsilon\}) = F$. Let $x, y \in X$. We consider the following cases:
case 1) If $x, x * y \in F$, then $y \in F$. Therefore

$$\mu(y) = \min\{\alpha,\varepsilon\} = \min\{\min\{\alpha,\varepsilon\}, \min\{\alpha,\varepsilon\}, \varepsilon\} = \min\{\mu(x), \mu(x*y), \varepsilon\}.$$

case 2) If $x, x * y \notin F$, then $\mu(x) = 0 = \mu(x*y)$, and so

$$\mu(y) \geq 0 = \min\{0,0,\varepsilon\} = \min\{\mu(x), \mu(x*y), \varepsilon\}.$$

case 3) If $x \in F$ and $x * y \notin F$, then $\mu(x) = \min\{\alpha,\varepsilon\}$ and $\mu(x*y) = 0$. Thus

$$\mu(y) \geq 0 = \min\{\min\{\alpha,\varepsilon\}, 0, \varepsilon\} = \min\{\mu(x), \mu(x*y), \varepsilon\}.$$

case 4) If $x * y \in F$ and $x \notin F$, then by the same argument as in case 3, we can conclude that

$$\mu(y) \geq \min\{\mu(x), \mu(x*y), \varepsilon\}.$$

Therefore μ is a ε–generalized fuzzy filter of X.

Let μ be a fuzzy filter(ideal) and let f be a homomorphism on X. Then we define a new fuzzy filter(ideal) by $\mu_f(x) = \min\{\mu(f(x)), \varepsilon\}$, for all $x \in X$, and $\varepsilon \in (0,1]$.

Theorem 10. *Let $\varepsilon \in (0,1]$, f be a homomorphism on X. If μ is an ε–generalized fuzzy filter(ideal), then so is μ_f.*

Proof. Let $x \in X$. Then

$$\mu_f(x) = \min\{\mu(f(x)), \varepsilon\} \leq \mu(1) = \mu(f(1)) = \mu_f(1).$$

Now, let $x, y, z \in X$. Then

$$\begin{aligned} \mu_f(x) = \mu(f(y)) &\geq \min\{\mu(f(x)), \mu(f(x) * f(y)), \varepsilon\} \\ &= \min\{\mu_f(x), \mu(f(x*y)), \varepsilon\} \\ &= \min\{\mu_f(x), \mu_f(x*y), \varepsilon\}. \end{aligned}$$

Theorem 11. *Let $\varepsilon \in (0,1]$ and $f : X \to Y$ is an epimorphism of BE–algebras and let μ be ε–generalized fuzzy filter(ideal. Then $f(\mu)$ defined by $f(\mu)(y) = \inf\{\mu(x) : f(x) = y \text{ for all } y \in Y\}$ is a ε–generalized fuzzy filter(ideal) of Y.*

Proof. Let μ be an ε–generalized fuzzy filter and let $u, v \in Y$. Since f is an epimorphism, then there exists $x, y \in X$, such that $f(x) = u$ and $f(y) = v$.

$$\begin{aligned} f(\mu)(v) = \inf_{y \in f^{-1}(v)} \mu(y) &\geq \inf_{y \in f^{-1}(v), x*y \in f^{-1}(u*v)} \{\min\{\mu(x), \mu(x*y), \varepsilon\}\} \\ &= \min\{\inf_{x \in f^{-1}(u)} \mu(x), \inf_{x*y \in f^{-1}(u*v)} \mu(x*y), \varepsilon\} \\ &= \min\{f(\mu)(u), f(\mu)(v), \varepsilon\}. \end{aligned}$$

Let μ and ν be ε–generalized fuzzy filter(ideal) of X and $\varepsilon \in (0,1]$. The ε–cartesian product μ and ν is defined by $(\mu \times \nu)(x,y) = \min\{\mu(x), \nu(y), \varepsilon\}$, for all $x, y \in X$.

Theorem 12. *Let μ and ν be ε–generalized fuzzy filter(ideal) of X and $\varepsilon \in (0,1]$. Then ε–cartesian product μ and ν is so.*

Proof. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}(\mu \times \nu)(1,1) = \min\{\mu(1), \nu(1), \varepsilon\} &\geq \min\{\min\{\mu(x), \varepsilon\}, \min\{\nu(x), \varepsilon\}, \varepsilon\} \\ &= \min\{\mu(x), \nu(x), \varepsilon\} = \min\{(\mu \times \nu)(x, x), \varepsilon\},\end{aligned}$$

and

$$\begin{aligned}(\mu \times \nu)(u, v) = \min\{\mu(u), \nu(v), \varepsilon\} &\geq \min\{\min\{\mu(x), \mu(x * u), \varepsilon\}, \min\{\nu(y), \nu(y * v), \varepsilon\}, \varepsilon\} \\ &= \min\{\min\{\mu(x), \nu(y), \varepsilon\}, \min\{\mu(x * u), \nu(y * v), \varepsilon\}, \varepsilon\} \\ &= \min\{(\mu \times \nu)(x, y), (\mu \times \nu)((x, y) * (u, v))), \varepsilon\}.\end{aligned}$$

Theorem 13. *Let μ be ε–generalized fuzzy filter(ideal) of X and $\varepsilon \in (0, 1]$. Then μ^+ is so.*

Proof. Let μ be ε–generalized fuzzy filter of X, $x, y \in X$ and $\varepsilon \in (0, 1]$. Since $1 - \mu(1) \geq 0$, we have $\varepsilon + 1 - \mu(1) \geq \varepsilon$, then

$$\begin{aligned}\mu^+(1) &= \mu(1) + 1 - \mu(1) \geq \min\{\mu(x), \varepsilon\} + 1 - \mu(1) \\ &= \min\{\mu(x) + 1 - \mu(1), \varepsilon + 1 - \mu(1)\} \\ &\geq \min\{\mu(x) + 1 - \mu(1), \varepsilon\} \\ &= \min\{\mu^+(x), \varepsilon\}\end{aligned}$$

and

$$\begin{aligned}\mu^+(y) &= \mu(y) + 1 - \mu(1) \geq \min\{\mu(x), \mu(x * y), \varepsilon\} + 1 - \mu(1) \\ &= \min\{\mu(x) + 1 - \mu(1), \mu(x * y) + 1 - \mu(1), \varepsilon + 1 - \mu(1)\} \\ &\geq \min\{\mu^+(x), \mu^+(x * y), \varepsilon\}\end{aligned}$$

Hence μ^+ is an ε–generalized fuzzy filter.

Definition 15. *Let μ be a fuzzy subset of X and $\delta \in [0, 1]$. A fuzzy δ–multiplication of μ, denoted by μ_δ, is defined by $\mu_\delta(x) = \mu(x).\delta$.*

Example 6: Let $\delta = 0.2$ and μ be fuzzy subset in Example 2, then $\mu_{0.2}(1) = 0.12$, $\mu_{0.2}(a) = 0.08$ and $\mu_{0.2}(b) = 0.06$.

Note. If $\lambda > \delta$, then $U(\mu_\delta; \lambda) = \emptyset$ and if $\lambda < \delta$, then $U(\mu_\delta; \lambda) = U(\mu; \lambda\delta^{-1})$. Also if $\lambda \geq \delta$, then $U(\mu_\delta; >, \lambda) = \emptyset$ and if $\lambda < \delta$, then $U(\mu_\delta; >, \lambda) = U(\mu; >, \lambda\delta^{-1})$.

Theorem 14. *If μ is a fuzzy filter(ideal) of X, then μ_δ is so, for all $\delta \in [0, 1]$.*

Proof. Let μ be a fuzzy filter of X and $\delta \in [0, 1]$. Then $\mu_\delta(1) = \mu(1).\delta \geq \mu(x).\delta = \mu_\delta$. Now, let $x, y \in X$. $\mu_\delta(y) = \mu(y).\delta \geq \min\{\mu(x), \mu(x * y)\}.\delta = \min\{\mu(x).\delta, \mu(x * y).\delta\}$. Hence μ_δ is a fuzzy filter.

Proposition 8. *Let μ be a fuzzy subset and μ_δ be a fuzzy filter of X for some $\delta \in (0, 1]$. Then μ is a fuzzy filter.*

Proof. Let μ be a fuzzy subset and μ_δ be a fuzzy filter, for some $\delta \in (0, 1]$ and $x, y \in X$. Then $\mu(1).\delta = \mu_\delta(1) \geq \mu_\delta(x) = \mu(x).\delta$. and $\mu(y).\delta = \mu_\delta(y) \geq \min\{\mu_\delta(x), \mu_(x * y)\} = \min\{\mu(x).\delta, \mu(x * y).\delta\} = \min\{\mu(x), \mu(x * y)\}.\delta$.
Hence $\mu(1) \geq \mu(x)$ and $\mu(y) \geq \{\mu(x), \mu(x * y)\}$, therefore μ is a fuzzy filter.

Remark. If μ is an ε–generalized fuzzy filter of X, then μ_ε may be not a fuzzy filter. For example, let μ be ε–generalized fuzzy filter in Example 2, then $\mu_{0.2}(1) = 0.14$, $\mu_{0.2}(a) = 0.06$ and $\mu_{0.2}(a) = \mu_{0.2}(b) = 0.08$. Then μ_ε is not a fuzzy filter because

$$\mu_{0.2}(a) = 0.06 \ngeq \min\{\mu_{0.2}(c), \mu_{0.2}(c * a)\} = \min\{0.08, 0.08\} = 0.08.$$

Theorem 15. *For any fuzzy subset μ of BE–algebra X. Then the following conditions are equivalent:*

(i) μ is a fuzzy filter(ideal).

(ii) μ_δ is a fuzzy filter(ideal), for all $\delta \in (0, 1]$.

Proof. Necessity follows from Theorem 14. Let $\delta \in (0,1]$ be such that μ_δ is a fuzzy filter and $x, y \in X$. $\mu(1).\delta = \mu_\delta(1) \geq \mu_\delta(x) = \mu(x).\delta$ and

$$\mu(y).\delta = \mu_\delta(y) \geq \min\{\mu_\delta(x), \mu_\delta(x*y)\} = \min\{\mu(x).\delta, \mu(x*y).\delta\} = \min\{\mu(x), \mu(x*y)\}.\delta.$$

Now, since $\delta \neq 0$, it follows that $\mu(1) \geq \mu(x)$ and $\mu(y) \geq \min\{\mu(x), \mu(x*y)\}$, for all $x, y \in X$. Hence μ is a fuzzy filter.

Theorem 16. *Let μ be ε–generalized fuzzy filter(ideal) of X and $\varepsilon, \delta \in (0,1]$. Then μ_δ is $\varepsilon.\delta$–generalized fuzzy filter(ideal).*

Proof. Let μ be ε–generalized fuzzy filter(ideal) such that $\varepsilon, \delta \in (0,1]$ and $x, y \in X$. Then $\mu_\delta(1) = \mu(1).\delta \geq \min\{\mu(x), \varepsilon\}.\delta = \min\{\mu(x).\delta, \varepsilon.\delta\}$ and

$$\mu_\delta(y) = \mu(y).\delta \geq \min\{\mu(x), \mu(x*y), \varepsilon\}.\delta = \min\{\mu(x).\delta, \mu(x*y).\delta, \varepsilon.\delta\} = \min\{\mu_\delta(x), \mu_\delta(x*y), \varepsilon.\delta\}.$$

Hence μ_δ is a $\varepsilon.\delta$–generalized fuzzy filter.

3 Conclusion

BE-algebras studied by researchers and some classification is given. It is well known that the fuzzy filters(ideals) with special properties play an important role in the structure of the logic system. In this paper, we introduced the concept of ε–generalized fuzzy filter(ideals) and δ–multiplication of μ, on BE–algebras and investigated some of their useful properties of this structure. We hope above work would serve as a foundation for further on study the structure of BE–algebras and develop corresponding many-valued logical system and these concepts can be further generalized.

Acknowledgments

The authors are very indebted to the referees for valuable suggestions that improved the readability of the paper.

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