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The Dominated Convergence Theorems for Sequences of Integrable Fuzzy Random Variables

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Abstract

Fuzzy random variable is a useful tool to describe twofold uncertainty. The purpose of this paper is to study the convergence modes of fuzzy random variables in equilibrium theory. We first introduce several convergence concepts for sequences of fuzzy random variables, such as convergence in equilibrium measure and convergence in equilibrium distribution. Then, we discuss the convergence criteria for the convergence modes. On the basis of the convergence criteria, we establish the convergence relations among the convergence modes. Finally, we define the integral of fuzzy random variable with respect to equilibrium measure, and establish the dominated convergence theorem and bounded convergence theorem for sequences of integrable fuzzy random variables.

Keywords: fuzzy random variable, convergence criterion, equilibrium expected value, dominated convergence theorem

1 Introduction

Fuzzy random variable was introduced by Kwakernaak [8] to describe the phenomena in which fuzziness and randomness appear simultaneously in a decision process. Since then, its variants as well as extensions were presented by a number of researchers, such as Kruse and Meyer [7], Puri and Ralescu [29] and Liu and Liu [17]. The interested reader may refer to [2, 23], where various measurability criteria for fuzzy random vectors were characterized by possibility and credibility measures. In this paper, we adopt the definition of fuzzy random variable presented in [17] for fuzzy random optimization [9, 24], including fuzzy random chance-constrained programming [4, 11, 16, 19, 20, 25, 30], fuzzy random dependent-chance programming [9, 12] and fuzzy random expected value model [3, 5, 14, 18, 31]. For other formulations of fuzzy random optimization models, the interested reader may refer to Luhandjula [26], Nematian [28] and Wang and Qiao [32].

Fuzzy random optimization problems include fuzzy random variable parameters described by possibility and probability distributions, when the uncertain parameters have continuous distributions, algorithms to solve such optimization problems usually rely on approximation schemes and heuristic algorithms. For example, Liu [14, 15] developed the approximation schemes for two-stage fuzzy random programming with recourse; Hao and Liu [5] and Liu et al. [25] studied fuzzy random portfolio optimization problems by employing the approximation methods to compute the variance and mean chance of fuzzy random variables; Qin and Liu [30, 31] combined the approximation method with heuristic algorithms to solve fuzzy random data envelopment analysis, and the convergence modes in mean theory were documented in Liu et al. [22]. To further develop the approximation schemes for fuzzy random optimization problems, the purpose of present paper is to study the convergence modes of fuzzy random variables in equilibrium theory [19], in which we discuss the convergence criteria of convergence modes, the convergence relations of convergence modes and the convergence theorems of integrable fuzzy random variables.

This paper is organized as follows. Section 2 recalls several concepts in fuzzy random theory, including fuzzy variable, fuzzy random variable, and the equilibrium measure of fuzzy random event. In Section 3, we first define some convergence concepts for sequences of fuzzy random variables, including convergence in

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equilibrium measure, and convergence in equilibrium distribution, then we discuss the convergence criteria for convergence almost uniform, convergence almost sure, and convergence in equilibrium measure. The convergence relationships among the convergence modes are also discussed in this section. In Section 4, we first define the integral of fuzzy random variable, then we deal with two main convergent results, the first is called the dominated convergence theorem for integrable fuzzy random variables, and the second is called the bounded convergence theorem for essentially bounded fuzzy random variables. Finally, Section 5 gives the conclusions.

2 Preliminaries

Given a universe Γ, an ample field $\mathcal{A}$ on Γ is a class of subsets of Γ that is closed under arbitrary unions, intersections and complement in Γ. If Pos is a set function defined on the ample field $\mathcal{A}$, then the set function Cr defined by

$$\mathrm{Cr}(A) = \frac{1}{2}\left(1 + \mathrm{Pos}(A) - \mathrm{Pos}(A^c)\right), A \in \mathcal{A} \tag{1}$$

is called a *credibility measure* [13]. It is easy to check that Cr is a self-dual set function in the sense that

$$\mathrm{Cr}(A) = 1 - \mathrm{Cr}(A^c), A \in \mathcal{A}.$$

The triplet $(\Gamma, \mathcal{A}, \mathrm{Cr})$ is called a credibility space, in which a fuzzy variable is defined as follows.

Definition 1 ([10]). *A fuzzy variable X is defined as a function from a credibility space $(\Gamma, \mathcal{A}, \mathrm{Cr})$ to the real line $\Re$ such that for every $t \in \Re$,*

$$\{\gamma \in \Gamma \mid X(\gamma) \leq t\} \in \mathcal{A}. \tag{2}$$

The possibility distribution of the fuzzy variable X is defined as

$$\mu_X(t) = \mathrm{Pos}(\{\gamma \in \Gamma \mid X(\gamma) = t\}), \quad t \in \Re.$$

In fuzzy random theory, we formally define a fuzzy random variable as follows.

Definition 2 ([17]). *Assume $(\Omega, \Sigma, \mathrm{Pr})$ is a probability space. A fuzzy random variable is a map $\xi : \Omega \to \mathcal{F}_v$ such that for any Borel subset B of $\Re$, the credibility function $\mathrm{Cr}\{\xi_\omega \in B\}$ is measurable with respect to ω, where $\mathcal{F}_v$ is a collection of fuzzy variables defined on a credibility space.*

The following definition is about the equilibrium chance of a fuzzy random event.

Definition 3 ([19]). *Let ξ be a fuzzy random variable, and B a Borel subset of $\Re$. Then the equilibrium (chance) measure of an event $\{\xi \in B\}$ is defined as*

$$\mathrm{Ch}^e\{\xi \in B\} = \bigvee_{0 \leq \alpha \leq 1} \left[\alpha \wedge \mathrm{Pr}\{\omega \mid \mathrm{Cr}\{\xi_\omega \in B\} \geq \alpha\}\right]. \tag{3}$$

Since $0 \leq \mathrm{Cr}\{\xi_\omega \in B\} \leq 1$, the integral in Eq. (3) is well-defined. For any Borel subset B of $\Re$, the equilibrium measure has the following dual relation

$$\mathrm{Ch}^e\left\{\xi \in B\right\} = 1 - \mathrm{Ch}^e\left\{\xi \in B^c\right\}. \tag{4}$$

We next recall three convergence concepts for sequences of fuzzy random variables.

Definition 4 ([22]). *Let $\{\xi_n\}$ be a sequence of fuzzy random variables. The sequence $\{\xi_n\}$ is said to converge uniformly to a fuzzy random variable ξ on $\Omega \times \Gamma$, denoted by $\xi_n \xrightarrow{u.} \xi$, if*

$$\lim_{n\to\infty} \sup_{(\omega,\gamma)\in\Omega\times\Gamma} |\xi_{n,\omega}(\gamma) - \xi_\omega(\gamma)| = 0.$$

Definition 5 ([22]). *Let $\{\xi_n\}$ be a sequence of fuzzy random variables. The sequence $\{\xi_n\}$ is said to converge almost uniformly to a fuzzy random variable ξ, denoted by $\xi_n \xrightarrow{a.u.} \xi$, if there exist two nonincreasing sequences $\{E_m\} \subset \Sigma$, $\{F_m\} \subset \mathcal{A}$ with $\lim_m \mathrm{Pr}(E_m) = \lim_m \mathrm{Cr}(F_m) = 0$ such that for each $m = 1, 2, \ldots,$ we have $\xi_n \xrightarrow{u.} \xi$ on $\Omega\backslash E_m \times \Gamma\backslash F_m$.*

Definition 6 ([22]). *Let $\{\xi_n\}$ be a sequence of fuzzy random variables. The sequence $\{\xi_n\}$ is said to converge almost surely to a fuzzy random variable ξ, denoted by $\xi_n \xrightarrow{a.s.} \xi$, if there exist $E \in \Sigma, F \in \mathcal{A}$ with* $\Pr(E) = \mathrm{Cr}(F) = 0$ *such that for every $(\omega, \gamma) \in \Omega \backslash E \times \Gamma \backslash F$, we have*

$$\lim_{n\to\infty} \xi_{n,\omega}(\gamma) \to \xi_\omega(\gamma).$$

3 Relationships among Convergence Modes

In this section, we first use the equilibrium measure to introduce some new concepts. Let ξ be a fuzzy random variable defined on a probability space. The equilibrium distribution function of ξ is defined as

$$G^e_\xi(t) = \mathrm{Ch}^e\{\xi \geq t\}, t \in \Re.$$

It is evident that G^e_ξ is a nonincreasing $[0,1]$-valued function.

Let $\{F_n\}$ and F be nonincreasing real-valued functions. The sequence $\{F_n\}$ is said to converge weakly to F, denoted by $F_n \xrightarrow{w.} F$, if $F_n(t) \to F(t)$ for all continuity points t of F.

As for the convergence modes with respect to the equilibrium measure, we have:

Definition 7. *Let $\{\xi_n\}$ be a sequence of fuzzy random variables. The sequence $\{\xi_n\}$ is said to converge in equilibrium measure to a fuzzy random variable ξ, denoted by $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathrm{Ch}^e\{|\xi_n - \xi| \geq \varepsilon\} = 0.$$

Definition 8. *Let $G^e_{\xi_n}$ be the equilibrium distribution functions of fuzzy random variables ξ_n, and G^e_ξ the equilibrium distribution function of fuzzy random variable ξ. The sequence $\{\xi_n\}$ is said to converge in equilibrium distribution to ξ, denoted by $\xi_n \xrightarrow{e.d.} \xi$, if $G^e_{\xi_n} \xrightarrow{w.} G^e_\xi$.*

3.1 Convergence Criteria

The following lemma deals with the criterion for convergence almost uniform.

Lemma 1. *Let $\{\xi_n\}$ and ξ be fuzzy random variables, and Ch^e the equilibrium measure. If $\xi_n \xrightarrow{a.u.} \xi$, then for every $\varepsilon > 0$, we have*

$$\lim_{m\to\infty} \mathrm{Ch}^e\left(\bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right) = 0. \tag{5}$$

Conversely, if ω is a finite discrete random variable, then Eq. (5) implies $\xi_n \xrightarrow{a.u.} \xi$.

Proof. If $\xi_n \xrightarrow{a.u.} \xi$, then it is easy to show that the limit

$$\xi_{n,\omega} \xrightarrow{a.u.} \xi_\omega$$

holds almost surely with respect to ω. As a consequence, we have

$$\limsup_{n\to\infty} \mathrm{Cr}\left(\bigcup_{n=m}^{\infty} \{\gamma \mid |\xi_{n,\omega}(\gamma) - \xi_\omega(\gamma)| \geq \varepsilon\}\right) \overset{a.s.}{=} 0.$$

Finally, according to the dominated convergence theorem of fuzzy integral, we obtain

$$\lim_{m\to\infty} \mathrm{Ch}^e\left(\bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right) = 0.$$

Conversely, if Eq. (5) is valid, then we have

$$\mathrm{Cr}\left(\bigcup_{n=m}^{\infty} \{|\xi_{n,\omega} - \xi_\omega| \geq \varepsilon\}\right) \xrightarrow{\Pr} 0.$$

By Riesz's theorem, there is a subsequence of $\{\mathrm{Cr}(\bigcup_{n=m}^{\infty}\{|\xi_{n,\omega}-\xi_\omega|\geq\varepsilon\})\}$ convergence almost sure. By the monotonicity of $\{\mathrm{Cr}(\bigcup_{n=m}^{\infty}\{|\xi_{n,\omega}-\xi_\omega|\geq\varepsilon\})\}$, we have

$$\lim_{m\to\infty}\mathrm{Cr}\left(\bigcup_{n=m}^{\infty}\{|\xi_{n,\omega}-\xi_\omega|\geq\varepsilon\}\right)\overset{\text{a.s.}}{=}0.$$

Since ω is a finite discrete random variable, By [22, Proposition 1], we have $\xi_n\xrightarrow{a.u.}\xi$. The proof of the lemma is complete. □

The criterion for convergence almost sure is given in the following lemma.

Lemma 2. *Let $\{\xi_n\}$ and ξ be fuzzy random variables, and Ch^e the equilibrium measure. Then $\xi_n\xrightarrow{a.s.}\xi$ if and only if for every $\varepsilon>0$,*

$$\mathrm{Ch}^e\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{|\xi_n-\xi|\geq\varepsilon\}\right)=0. \tag{6}$$

Proof. First, it is easy to check that $\xi_n\xrightarrow{a.s.}\xi$ if and only if the limit

$$\xi_{n,\omega}\xrightarrow{a.s.}\xi_\omega$$

holds almost surely with respect to ω, i.e., there exists $E\in\Sigma$ with $\Pr(E)=0$ such that for each $\omega\in\Omega\backslash E$, we have

$$\xi_{n,\omega}\xrightarrow{a.s.}\xi_\omega.$$

As a consequence, for every $\varepsilon>0$,

$$\mathrm{Cr}\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{\gamma\in\Gamma\mid|\xi_{n,\omega}(\gamma)-\xi_\omega(\gamma)|\geq\varepsilon\}\right)=0,\omega\in\Omega\backslash E,$$

which, by the property of fuzzy integral, is equivalent to

$$\mathrm{Ch}^e\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{|\xi_n-\xi|\geq\varepsilon\}\right)=0.$$

The proof of the lemma is complete. □

We now deal with the criterion for convergence in equilibrium measure.

Lemma 3. *Let $\{\xi_n\}$ and ξ be fuzzy random variables, and Ch^e the equilibrium measure. Then $\xi_n\xrightarrow{\mathrm{Ch}^e}\xi$ if and only if for every $\varepsilon>0$,*

$$\mathrm{Cr}\{\gamma\mid|\xi_{n,\omega}(\gamma)-\xi_\omega(\gamma)|\geq\varepsilon\}\xrightarrow{\Pr}0.$$

Proof. Assume that $\xi_n\xrightarrow{\mathrm{Ch}^e}\xi$, then for every $\varepsilon>0$ and $\alpha\in(0,1]$, one has

$$0\leq\alpha\wedge\Pr\{\omega\mid\mathrm{Cr}\{|\xi_{n,\omega}(\gamma)-\xi_\omega(\gamma)|\geq\varepsilon\}\geq\alpha\}\leq\mathrm{Ch}^e\{|\xi_n-\xi|\geq\varepsilon\}.$$

As a consequence, we have

$$\mathrm{Cr}\{\gamma\mid|\xi_{n,\omega}(\gamma)-\xi_\omega(\gamma)|\geq\varepsilon\}\xrightarrow{\Pr}0.$$

On the other hand, if

$$\mathrm{Cr}\{\gamma\mid|\xi_{n,\omega}(\gamma)-\xi_\omega(\gamma)|\geq\varepsilon\}\xrightarrow{\Pr}0,$$

then for every $\alpha\in(0,1]$, we have

$$\lim_{n\to\infty}\Pr\{\omega\mid\mathrm{Cr}\{|\xi_{n,\omega}(\gamma)-\xi_\omega(\gamma)|\geq\varepsilon\}\geq\alpha\}=0.$$

Finally, according to the dominated convergence theorem of fuzzy integral, we obtain

$$\lim_{n\to\infty}\mathrm{Ch}^e\{|\xi_n-\xi|\geq\varepsilon\}=0,$$

which completes the proof of the lemma. □

Proposition 1. *Let $\{\xi_n\}$ and ξ be fuzzy random variables, and Ch^e the equilibrium measure. If $\xi_{n,\omega} \xrightarrow{\mathrm{Cr}} \xi_\omega$ almost surely with respect to ω, then $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$.*

Proof. Assume that $\xi_{n,\omega} \xrightarrow{\mathrm{Cr}} \xi_\omega$ almost surely with respect to ω. Then for every $\varepsilon > 0$, the limit

$$\lim_{n\to\infty} \mathrm{Cr}\{\gamma \mid |\xi_{n,\omega}(\gamma) - \xi_\omega(\gamma)| \geq \varepsilon\} = 0$$

holds with probability 1. Since convergence a.s. implies convergence in probability, one has

$$\mathrm{Cr}\{\gamma \mid |\xi_{n,\omega}(\gamma) - \xi_\omega(\gamma)| \geq \varepsilon\} \xrightarrow{\mathrm{Pr}} 0,$$

which, by Lemma 3, implies $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$. □

3.2 Convergence Relations

First, the following theorem compares convergence almost uniform and convergence almost sure.

Theorem 1. *Let $\{\xi_n\}$ and ξ be fuzzy random variables. If $\xi_n \xrightarrow{a.u.} \xi$, then $\xi_n \xrightarrow{a.s.} \xi$.*

Proof. Assume $\xi_n \xrightarrow{a.u.} \xi$. By Lemma 1, we have

$$\lim_{m\to\infty} \mathrm{Ch}^e\left(\bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right) = 0.$$

By the monotonicity of equilibrium measure, we have

$$\mathrm{Ch}^e\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right) \leq \mathrm{Ch}^e\left(\bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right),$$

which implies

$$\mathrm{Ch}^e\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right) = 0.$$

It follows from Lemma 2 that $\xi_n \xrightarrow{a.s.} \xi$. The proof of the theorem is complete. □

The relation between convergence almost sure and convergence in equilibrium measure is discussed in the following theorem.

Theorem 2. *Let $\{\xi_n\}$ and ξ be fuzzy random variables, and Ch^e the equilibrium measure. If $\xi_n \xrightarrow{a.u.} \xi$, then $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$. Conversely, if ω is a finite discrete random variable, then $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$ implies $\xi_n \xrightarrow{a.u.} \xi$.*

Proof. Suppose $\xi_n \xrightarrow{a.u.} \xi$. By Lemma 1, we have

$$\lim_{m\to\infty} \mathrm{Ch}^e\left(\bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right) = 0.$$

According to the following inequality

$$\mathrm{Ch}^e\{|\xi_m - \xi| \geq \varepsilon\} \leq \mathrm{Ch}^e\left(\bigcup_{n=m}^{\infty} \{|\xi_n - \xi| \geq \varepsilon\}\right),$$

we obtain that

$$\lim_{m\to\infty} \mathrm{Ch}^e\{|\xi_m - \xi| \geq \varepsilon\} = 0,$$

i.e., $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$.

Assume ω is a discrete random variable with the following probability distribution

$$\omega \sim \begin{pmatrix} \omega_1, & \omega_2, & \ldots, & \omega_N \\ p_1, & p_2, & \ldots, & p_N \end{pmatrix},$$

where $p_i > 0$ and $\sum_{i=1}^N p_i = 1$. Let $\alpha_i = \mathrm{Cr}\left\{|\xi_{n,\omega_i}(\gamma) - \xi_{\omega_i}(\gamma)| \ge \varepsilon\right\}$. Then we have

$$\mathrm{Cr}\left\{|\xi_{n,\omega_i}(\gamma) - \xi_{\omega_i}(\gamma)| \ge \varepsilon\right\} \wedge p_i \le \mathrm{Ch}^e\left\{|\xi_n - \xi| \ge \varepsilon\right\}.$$

By $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$, we deduce that

$$\xi_{n,\omega_i} \xrightarrow{\mathrm{Cr}} \xi_{\omega_i}, i = 1, 2, \ldots, N.$$

Then for each positive integer $k = 1, 2, \ldots,$ and $i = 1, 2, \ldots, N$, we have

$$\lim_{n\to\infty} \mathrm{Cr}\left\{\gamma \in \Gamma \mid |\xi_{n,\omega_i}(\gamma) - \xi_{\omega_i}(\gamma)| \ge 1/k\right\} = 0.$$

Thus, for each m, there exists N_{km} such that for $i = 1, 2, \ldots, N$, one has

$$\mathrm{Cr}\left\{\gamma \in \Gamma \mid |\xi_{n,\omega_i}(\gamma) - \xi_{\omega_i}(\gamma)| \ge 1/k\right\} < 1/2m$$

whenever $n \ge N_{km}$. Letting

$$E_m = \bigcup_{i=1}^{N} \bigcup_{k=1}^{\infty} \bigcup_{n\ge N_{km}} \{\gamma \in \Gamma \mid |\xi_{n,\omega_i}(\gamma) - \xi_{\omega_i}(\gamma)| \ge 1/k\},$$

then we have

$$\mathrm{Cr}(E_m) \le \sup_i \sup_k \sup_{n\ge N_{km}} \mathrm{Pos}\left\{\gamma \mid |\xi_{n,\omega_i}(\gamma) - \xi_{\omega_i}(\gamma)| \ge 1/k\right\} < 1/m.$$

It is easy to show that $\{\xi_{n,\omega_i}\}$ converges to ξ_{ω_i} uniformly on each $\Gamma \backslash E_m$. Thus, $\xi_n \xrightarrow{a.u.} \xi$. The proof of the theorem is complete. □

By Theorems 1 and 2, we obtain the following result about the relation between convergence in equilibrium measure and convergence almost sure.

Theorem 3. *Let $\{\xi_n\}$ and ξ be fuzzy random variables, and Ch^e the equilibrium measure. If ω is a finite discrete random variable, then $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$ implies $\xi_n \xrightarrow{a.s} \xi$.*

Finally, the following theorem discusses the relation between convergence in equilibrium measure and convergence in equilibrium distribution.

Theorem 4. *Let $\{\xi_n\}$ and ξ be fuzzy random variables, and Ch^e the equilibrium measure. If $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$, then $\xi_n \xrightarrow{e.d.} \xi$.*

Proof. For every $\omega \in \Omega$, let $G_{n,\omega}$ and G_ω be the credibility distribution functions of $\xi_{n,\omega}$ and ξ_ω, respectively. Then for every $t \in \Re, \varepsilon > 0$ and integer n, one has

$$\begin{aligned} \mathrm{Cr}\{\xi_{n,\omega} \ge t\} &\le \mathrm{Cr}\{\xi_{n,\omega} \ge t, |\xi_{n,\omega} - \xi_\omega| < \varepsilon\} + \mathrm{Cr}\{\xi_{n,\omega} \ge t, |\xi_{n,\omega} - \xi_\omega| \ge \varepsilon\} \\ &\le \mathrm{Cr}\{\xi_\omega \ge t - \varepsilon\} + \mathrm{Cr}\{|\xi_{n,\omega} - \xi_\omega| \ge \varepsilon\}. \end{aligned}$$

That is,

$$G_{n,\omega}(t) \le G_\omega(t - \varepsilon) + \mathrm{Cr}\{|\xi_{n,\omega} - \xi_\omega| \ge \varepsilon\}.$$

Letting $n \to \infty$, and then $\varepsilon \to 0$, we obtain

$$\limsup_{n\to\infty} G_{n,\omega}(t) \le G_\omega(t - 0).$$

On the other hand, according to the following inequality

$$\begin{aligned} \mathrm{Cr}\{\xi_\omega \ge t + \varepsilon\} &\le \mathrm{Cr}\{\xi_\omega \ge t + \varepsilon, |\xi_{n,\omega} - \xi_\omega| < \varepsilon\} + \mathrm{Cr}\{\xi_\omega \ge t + \varepsilon, |\xi_{n,\omega} - \xi_\omega| \ge \varepsilon\} \\ &\le \mathrm{Cr}\{\xi_{n,\omega} \ge t\} + \mathrm{Cr}\{|\xi_{n,\omega} - \xi_\omega| \ge \varepsilon\}, \end{aligned}$$

we have

$$G_\omega(t+\varepsilon) \leq G_{n,\omega}(t) + \mathrm{Cr}\{|\xi_{n,\omega} - \xi_\omega| \geq \varepsilon\},$$

and

$$\liminf_{n\to\infty} G_{n,\omega}(t) \geq G_\omega(t+0).$$

Therefore, $G_{n,\omega} \xrightarrow{w.} G_\omega$ almost sure with respect to ω. By the dominated convergence theorem of fuzzy integral, we have $G_n^e \xrightarrow{w.} G^e$, i.e., $\xi_n \xrightarrow{e.d.} \xi$. □

4 Dominated Convergence Theorems

In equilibrium theory, we define the equilibrium expected value of a fuzzy random variable as follows.

Definition 9. *Let ξ be a fuzzy random variable. Then the (equilibrium) expected value of ξ is defined as*

$$E^e[\xi] = \int_0^\infty \mathrm{Ch}^e\{\,\xi \geq r\,\}\,\mathrm{d}r - \int_{-\infty}^0 \mathrm{Ch}^e\{\,\xi \leq r\,\}\,\mathrm{d}r \tag{7}$$

provided one of the two integrals is finite.

Let $f : I \to \bar{\Re} = \Re \cup \{\pm\infty\}$ be a nonincreasing function on an interval $I \subset \bar{\Re}$, and the interval $J = [\inf_{x\in I} f(x), \sup_{x\in I} f(x)] \subset \bar{\Re}$. Then a map $\breve{f} : J \to \bar{I}$ is called the *pseudo-inverse function* of f if

$$a \vee \sup\{x \mid f(x) > y\} \leq \breve{f}(y) \leq a \vee \sup\{x \mid f(x) \geq y\},$$

where $a = \inf I$. It is easy to check that the function $\breve{f}$ is nonincreasing and its pseudo-inverse $(\breve{f})\breve{}$ equals f except on an at most countable set, and denoted e.c. for short, i.e.,

$$(\breve{f})\breve{} \stackrel{e.c.}{=} f. \tag{8}$$

The improper Riemann integrals of f and $\breve{f}$ have the following connections (see, [1]):

(i) For a nonincreasing function $f : \bar{\Re}_+ \to \bar{\Re}_+$ and any pseudo-inverse $\breve{f}$ of f, one has

$$\int_0^\infty f(x)\mathrm{d}x = \int_0^\infty \breve{f}(y)\mathrm{d}y. \tag{9}$$

(ii) For a nonincreasing function $f : [0,b] \to \bar{\Re}, 0 < b < \infty$ and any pseudo-inverse $\breve{f}$ of f, one has

$$\int_0^b f(x)\mathrm{d}x = \int_0^\infty \breve{f}(x)\mathrm{d}x + \int_{-\infty}^0 (\breve{f}(y) - b)\mathrm{d}y. \tag{10}$$

Next, we define the equilibrium fractile function of a fuzzy random variable:

Definition 10. *Let ξ be a fuzzy random variable with the equilibrium distribution function G_ξ^e. Then the equilibrium fractile function of ξ is defined by*

$$\mathrm{V@R}_\alpha^e(\xi) = \sup\{t \mid G_\xi^e(t) \geq \alpha\}, \alpha \in [0,1]. \tag{11}$$

The equilibrium distribution function G_ξ^e is nonincreasing, its equilibrium fractile function $\mathrm{V@R}_\alpha^e(\xi)$ is also nonincreasing with respect to α, and it is a pseudo-inverse function of G_ξ^e.

The following lemma gives an equivalent representation for the equilibrium expected value of fuzzy random variable:

Lemma 4. *If ξ is a fuzzy random variable, then its equilibrium expected value is equivalent to the following integral of the equilibrium fractile $\mathrm{V@R}_\alpha^e(\xi)$,*

$$E^e[\xi] = \int_0^1 \mathrm{V@R}_\alpha^e(\xi)\mathrm{d}\alpha.$$

Proof. In Eq. (10), let $f(\alpha) = \mathrm{V@R}^e_\alpha(\xi)$, and $b = 1$. Then, by Eq. (8), we have

$$\breve{f} \stackrel{e.c.}{=} G^e_\xi.$$

Therefore, according to Eq. (4), one has

$$\int_0^1 \mathrm{V@R}^e_\alpha(\xi)\mathrm{d}\alpha = \int_0^\infty G^e_\xi(t)\mathrm{d}t + \int_{-\infty}^0 (G^e_\xi(t) - 1)\mathrm{d}t = E^e[\xi],$$

which completes the proof of the lemma. □

Definition 11. *Let $\{\xi_n\}$ and ξ be fuzzy random variables, and their equilibrium fractile functions are $\{\mathrm{V@R}^e_\alpha(\xi_n)\}$ and $\mathrm{V@R}^e_\alpha(\xi)$, respectively. If $\mathrm{V@R}^e_\alpha(\xi_n) \xrightarrow{w.} \mathrm{V@R}^e_\alpha(\xi)$, then the sequence $\{\xi_n\}$ is said to converge in equilibrium fractile to ξ, and denoted by $\xi_n \xrightarrow{e.f.} \xi$.*

For sequences of fuzzy random variables, we have the following relation between the convergence in equilibrium distribution and the convergence in equilibrium fractile:

Lemma 5. *Let $\{\xi_n\}$ and ξ be fuzzy random variables. Then $\xi_n \xrightarrow{e.d.} \xi$ is equivalent to $\xi_n \xrightarrow{e.f.} \xi$.*

Proof. We first prove the sufficiency. Suppose $\xi_n \xrightarrow{e.d.} \xi$, we next prove $\xi_n \xrightarrow{e.f.} \xi$.

In fact, suppose $\alpha \in (0, 1)$ is such that there is at most one value t having $G^e_\xi(t) = \alpha$. Denote $z = \mathrm{V@R}^e_\alpha(\xi)$.

On the one hand, we have $G^e_\xi(t) > \alpha$ for $t < z$. Thus $G^e_{\xi_n}(t) > \alpha$ for $n \geq N_t$ (some positive integer), provided that $t < z$ is a continuity point of G^e_ξ. Hence $\mathrm{V@R}^e_\alpha(\xi_n) \geq t$, provided that $t < z$ is a continuity point of G^e_ξ. Therefore

$$\liminf_{n\to\infty} \mathrm{V@R}^e_\alpha(\xi_n) \geq t.$$

Since there is an increasing sequence $\{t_n\}$ of continuity points of G^e_ξ converging to z, we have

$$\liminf_{n\to\infty} \mathrm{V@R}^e_\alpha(\xi_n) \geq z.$$

On the other hand, as $t > z$, we have $G^e_\xi(t) < \alpha$. Thus $G^e_{\xi_n}(t) < \alpha$ for $n \geq N'_t$ (some positive integer), provided that $t > z$ is a continuity point of G^e_ξ. Hence $\mathrm{V@R}^e_\alpha(\xi_n) \leq t$, provided that $t > z$ is a continuity point of G^e_ξ. Therefore

$$\limsup_{n\to\infty} \mathrm{V@R}^e_\alpha(\xi_n) \leq t.$$

Since there is a decreasing sequence $\{t_n\}$ of continuity points converging to z, we have

$$\limsup_{n\to\infty} \mathrm{V@R}^e_\alpha(\xi_n) \leq z.$$

Therefore, $\mathrm{V@R}^e_\alpha(\xi_n) \to \mathrm{V@R}^e_\alpha(\xi)$ for all except at most countably infinite number of α's, i.e., for all except those α's that have many values of t such that $G^e_\xi(t) = \alpha$, which correspond to the heights of flat spots of G^e_ξ, and these flat spot height α's are exactly the discontinuity points of $\mathrm{V@R}^e_\alpha(\xi_n)$. That is, $\xi_n \xrightarrow{e.f.} \xi$.

Note that G^e_ξ is the pseudo-inverse function of fractile function $\mathrm{V@R}^e_\alpha(\xi)$, the proof of the necessity is similar to that of the sufficiency. □

For sequence of integrable fuzzy random variables, we have the following general convergence theorem:

Theorem 5. *Let $\{\xi_n\}$ be a sequence of fuzzy random variables, and η and ζ integrable fuzzy random variables such that*

$$G^e_\eta \stackrel{e.c.}{\leq} G^e_{\xi_n} \stackrel{e.c.}{\leq} G^e_\zeta.$$

If $\xi_n \xrightarrow{e.d.} \xi$ or $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$, then we have

$$\lim_{n\to\infty} E^e[\xi_n] = E^e[\xi].$$

Proof. By Lemma 4, we have

$$E^e[\xi] = \int_0^1 \mathrm{V@R}^e_\alpha(\xi)\mathrm{d}\alpha, \quad E^e[\xi_n] = \int_0^1 \mathrm{V@R}^e_\alpha(\xi_n)\mathrm{d}\alpha$$

for $n = 1, 2, \ldots$. Since $G^e_\eta \overset{e.c.}{\le} G^e_{\xi_n} \overset{e.c.}{\le} G^e_\zeta$, by the definition of fractile function, we deduce

$$\mathrm{V@R}^e_\alpha(\eta) \overset{e.c.}{\le} \mathrm{V@R}^e_\alpha(\xi_n) \overset{e.c.}{\le} \mathrm{V@R}^e_\alpha(\zeta), n = 1, 2, \ldots.$$

Suppose $\xi_n \xrightarrow{e.d.} \xi$, according to Lemma 5, we have

$$\xi_n \xrightarrow{e.f.} \xi,$$

i.e., $\mathrm{V@R}^e_\alpha(\xi_n) \xrightarrow{w.} \mathrm{V@R}^e_\alpha(\xi)$. Since η and ζ are integrable, i.e,

$$E^e[\eta] = \int_0^1 \mathrm{V@R}^e_\alpha(\eta))\mathrm{d}\alpha, \quad E^e[\zeta] = \int_0^1 \mathrm{V@R}^e_\alpha(\zeta)\mathrm{d}\alpha$$

are finite, by Lebesgue dominated convergence theorem, one has

$$\lim_{n\to\infty} \int_0^1 \mathrm{V@R}^e_\alpha(\xi_n)\mathrm{d}\alpha = \int_0^1 \mathrm{V@R}^e_\alpha(\xi)\mathrm{d}\alpha,$$

i.e.,

$$\lim_{n\to\infty} E^e[\xi_n] = E^e[\xi].$$

On the other hand, if $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$, then Theorem 4 implies that

$$\xi_n \xrightarrow{e.d.} \xi.$$

Therefore, we also have the desired result. The proof of the theorem is complete. □

Definition 12. *A fuzzy random variable ξ is said to be essentially bounded with respect to equilibrium measure if there is a positive number a such that*

$$G^e_\xi(-a) = 1, \text{ and } G^e_\xi(a) = 0.$$

A sequence $\{\xi_k\}$ of fuzzy random variables is said to be uniformly essentially bounded with respect to equilibrium measure if there is a positive number a such that for each $k = 1, 2, \ldots$, we have $G^e_{\xi_k}(-a) = 1$, and $G^e_{\xi_k}(a) = 0$.

For essentially bounded fuzzy random variables, we have the following result:

Theorem 6. *Let $\{\xi_n\}$ and ξ be fuzzy random variables. If $\{\xi_n\}$ is uniformly essentially bounded, and $\xi_n \xrightarrow{e.d.} \xi$ or $\xi_n \xrightarrow{\mathrm{Ch}^e} \xi$, then we have*

$$\lim_{n\to\infty} E^e[\xi_n] = E^e[\xi].$$

Proof. Since $\{\xi_n\}$ is uniformly essentially bounded fuzzy random variables, there exist a positive number a such that for any n,

$$\mathrm{Ch}^e\{\xi_n \ge -a\} = 1, \text{ and } \mathrm{Ch}^e\{\xi_n \ge a\} = 0.$$

By the self-duality of equilibrium chance Ch^e, for any n, we have

$$\mathrm{Ch}^e\{\xi_n < -a\} = 0, \text{ and } \mathrm{Ch}^e\{\xi_n < a\} = 1.$$

By the definition of equilibrium chance, for any n, the following equalities

$$\mathrm{Cr}\{\xi_{n,\omega} \ge a\} = \mathrm{Cr}\{\xi_{n,\omega} < -a\} = 0$$

hold almost sure with respect to ω.

Letting $\eta = -a$, and $\zeta = a$, then $G^e_\eta \le G^e_{\xi_n} \le G^e_\zeta$.

In fact, for every $t \in \Re$, by the subadditivity of Cr, the inequality

$$\mathrm{Cr}\{\xi_{n,\omega} \ge t\} \le \mathrm{Cr}\{\xi_{n,\omega} \ge t, \xi_{n,\omega} \ge \zeta_\omega\} + \mathrm{Cr}\{\xi_{n,\omega} \ge t, \xi_{n,\omega} < \zeta_\omega\} \le \mathrm{Cr}\{\zeta_\omega \ge t\}$$

holds almost sure with respect to ω. By the monotonicity of equilibrium chance, we obtain

$$\mathrm{Ch}^e\{\xi_n \ge t\} \le \mathrm{Ch}^e\{\zeta \ge t\},$$

i.e., $G^e_{\xi_n} \le G^e_\zeta$. Similarly, by

$$\mathrm{Cr}\{\eta_\omega \ge t\} \le \mathrm{Cr}\{\eta_\omega \ge t, \xi_{n,\omega} \ge \eta_\omega\} + \mathrm{Cr}\{\eta_\omega \ge t, \xi_{n,\omega} < \eta_\omega\} \le \mathrm{Cr}\{\xi_{n,\omega} \ge t\},$$

we have $G^e_\eta \le G^e_{\xi_n}$. It follows from Theorem 5 that

$$\lim_{n\to\infty} E^e[\xi_n] = E^e[\xi].$$

The proof of the theorem is complete. □

5 Conclusions

In equilibrium theory, we studied the convergence modes of fuzzy random variables, and obtained the following major new results:

(i) We introduced some new convergence modes, including convergence in equilibrium measure, convergence in equilibrium distribution, and convergence in equilibrium fractile for sequences of fuzzy random variables.

(ii) We discussed the convergence criteria about convergence almost uniform, convergence almost sure and convergence in equilibrium measure.

(iii) On the basis of convergence criteria, we established the convergence relations among convergence almost sure, convergence almost uniform, convergence in equilibrium measure and convergence in equilibrium distribution.

(iv) After introducing the integral of fuzzy random variable with respect to the equilibrium measure, we established the dominated convergence theorem and bounded convergence theorem for sequences of integrable fuzzy random variables.

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