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Intuitionistic Fuzzy Number and Its Arithmetic Operation with Application on System Failure

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Abstract

We have introduced intuitionistic fuzzy number and its arithmetic operations based on extension principle of intuitionistic fuzzy sets. Here two types of intuitionistic fuzzy sets, namely triangular intuitionistic fuzzy number and trapezoidal intuitionistic fuzzy number is presented. We also present that the arithmetic operation of two or more intuitionistic fuzzy number is again an intuitionistic fuzzy number. The starting failure of an automobile system is presented by intuitionistic fuzzy system. Each components failure is represented by trapezoidal intuitionistic fuzzy number of the system failure model to compute the imprecise failure. Finally, the presented concepts are analyzed through suitable numerical example.

Keywords: intuitionistic fuzzy number, triangular intuitionistic fuzzy number, trapezoidal intuitionistic fuzzy number, extension principle, failure rate

1 Introduction

Intuitionistic fuzzy set (IFS) is one of the generalizations of fuzzy sets theory [22]. Out of several higher-order fuzzy sets, IFS first introduced by Atanassov [1] have been found to be compatible to deal with vagueness. The conception of IFS can be viewed as an appropriate/alternative approach in case where available information is not sufficient to define the impreciseness by the conventional fuzzy set. In fuzzy sets the degree of acceptance is considered only but IFS is characterized by a membership function and a non-membership function so that the sum of both values is less than one [2]. Presently IFSs are being studied and used in different fields of science. Among the research works on IFS we can mention Atanassov [2-6], Atanassov and Gargov [7], Szmidt and Kacprzyk [19], Buhaescu [9], Ban [8], Deschrijver and Kerre [13], Stoyanova [18]. With the best of our knowledge, Burillo et al. [10] proposed definition of intuitionistic fuzzy number (IFN) and studied perturbations of IFN and the first properties of the correlation between these numbers. Several researchers [15, 17, 21] considered the problem of ranking a set of IFNs to define a fuzzy rank and a characteristic vagueness factor for each IFN.

In the real world problems, the collected data or system parameters are often imprecise because of incomplete or non-obtainable information, and the probabilistic approach to the conventional reliability analysis is inadequate to account for such built-in uncertainties in data. Therefore concept of fuzzy reliability has been introduced and formulated either in the context of the possibility measures or as a transition from fuzzy success state to fuzzy failure state [11, 12]. Cheng and Mon [16] considered that components are with different membership functions, then interval arithmetic and α-cuts were used to evaluate fuzzy system reliability. Verma [20] presented the dynamic reliability evaluation of the deteriorating system using the concept of probist reliability as a triangular fuzzy number. Mahapatra and Roy [14] evaluate system reliability by considering reliability of components as triangular intuitionistic fuzzy number.

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In this paper, we have presented IFN according to the approach of fuzzy number presentation. Triangular intuitionistic fuzzy number (TIFN) and trapezoidal intuitionistic fuzzy number (TrIFN) are defined, and their arithmetic operations based on intuitoinistic fuzzy extension principle and (α, β)-cut method is presented. The grade of a membership function indicates a subjective degree of preference of a decision maker within a given tolerance and grade of a non-membership function indicates a subjective degree of negative response of a decision maker within a given tolerance. Here we consider failure of components of starting failure of an automobile system as TrIFN. Intuitionitic fuzzy fault tree analysis is presented for starting failure of the automobile system. Arithmetic operations of TrIFN are used to evaluate imprecise system failure.

2 Basic Concept of Intuitionistic Fuzzy Sets

Fuzzy set theory was first introduced by Zadeh [22] in 1965. Let X be universe of discourse defined by $X = \{x_1, x_2, ..., x_n\}$. The grade of membership of an element $x_i \in X$ in a fuzzy set is represented by real value in [0,1]. It does indicate the evidence for $x_i \in X$, but does not indicate the evidence against $x_i \in X$. Atanassov [1] presented the concept of IFS, an IFS $\overset{\sim i}{A}$ in X is characterized by a membership function $\mu_{\overset{\sim i}{A}}(x)$ and a non-membership function $\nu_{\overset{\sim i}{A}}(x)$. Here $\mu_{\overset{\sim i}{A}}(x)$ and $\nu_{\overset{\sim i}{A}}(x)$ are associated with each point in X, a real number in [0,1] with the values of $\mu_{\overset{\sim i}{A}}(x)$ and $\nu_{\overset{\sim i}{A}}(x)$ at X representing the grade of membership and non-membership of x in $\overset{\sim i}{A}$. When $\overset{\sim i}{A}$ is an ordinary (crisp) set, its membership function can take only two values zero and one. An IFS becomes a fuzzy set $\tilde{A}$ when $\nu_{\overset{\sim i}{A}}(x) = 0$ but $\mu_{\overset{\sim i}{A}}(x) \in [0,1] \forall x \in \overset{\sim i}{A}$.

Definition 2.1 Intuitionistic Fuzzy Set: Let a set X be fixed. An IFS $\overset{\sim i}{A}$ in X is an object having the form $\overset{\sim i}{A} = \left\{ < x, \mu_{\overset{\sim i}{A}}(x), \nu_{\overset{\sim i}{A}}(x) >: x \in X \right\}$, where the $\mu_{\overset{\sim i}{A}}(x)$: X → [0,1] and $\nu_{\overset{\sim i}{A}}(x)$:X→[0,1] define the degree of membership and degree of non-membership respectively, of the element x∈X to the set $\tilde{A}^{i}$, which is a subset of X, for every element of x∈X, $0 \le \mu_{\overset{\sim i}{A}}(x) + \nu_{\overset{\sim i}{A}}(x) \le 1$.

Definition 2.2 (α, β)-level Intervals or (α, β)-cuts: A set of (α, β)-cut, generated by an IFS $\overset{\sim i}{A}$, where $\alpha, \beta \in [0,1]$ are fixed numbers such that $\alpha + \beta \le 1$ is defined as

$$A_{\alpha,\beta} = \left\{ \left(x, \mu_{\overset{\sim i}{A}}(x), \nu_{\overset{\sim i}{A}}(x) \right) : x \in X, \mu_{\overset{\sim i}{A}}(x) \ge \alpha, \nu_{\overset{\sim i}{A}}(x) \le \beta, \ \alpha, \beta \in [0,1] \right\}.$$

We define (α, β)-level interval or (α, β)-cut, denoted by $A_{\alpha,\beta}$, as the crisp set of elements x which belong to $\overset{\sim i}{A}$ at least to the degree α and which belong to $\overset{\sim i}{A}$ at most to the degree β.

3 Presentation of Intuitionistic Fuzzy Numbers and Its Properties

Definition 3.1 Intuitionistic Fuzzy Number: An IFN $\overset{\sim i}{\mathrm{A}}$ is defined as follows:

i) an intuitionistic fuzzy subset of the real line

ii) normal, i.e., there is any $x_0 \in R$ such that $\mu_{\overset{\sim i}{\mathrm{A}}}(x_0) = 1$ (so $\nu_{\overset{\sim i}{\mathrm{A}}}(x_0) = 0$)

iii) a convex set for the membership function $\mu_{\overset{\sim i}{\mathrm{A}}}(x)$, i.e.,

$$\mu_{\overset{\sim i}{\mathrm{A}}}\left(\lambda x_1 + (1-\lambda) x_2\right) \ge \min\left(\mu_{\overset{\sim i}{\mathrm{A}}}(x_1), \mu_{\overset{\sim i}{\mathrm{A}}}(x_2)\right) \quad \forall x_1, x_2 \in R, \ \lambda \in [0,1]$$

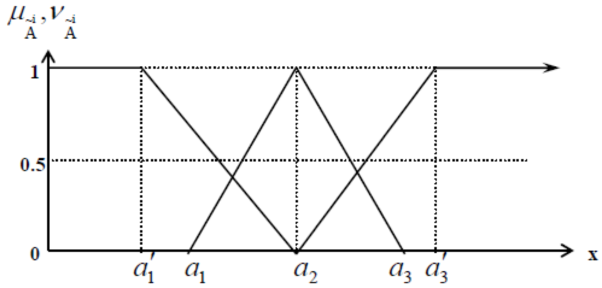
iv) a concave set for the non-membership function $\nu_{\overset{\sim i}{\mathrm{A}}}(x)$, i.e.,

$$\nu_{\tilde{A}^i}\left(\lambda x_1+(1-\lambda)x_2\right)\le \max\left(\nu_{\tilde{A}^i}(x_1),\nu_{\tilde{A}^i}(x_2)\right)\quad \forall x_1,x_2\in R,\ \lambda\in[0,1].$$

Definition 3.2 Triangular Intuitionistic Fuzzy Number: A TIFN $\tilde{A}^i$ is a subset of IFS in R with following membership function and non-membership function as follows:

$$\mu_{\tilde{A}^i}(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1} & \text{for } a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2} & \text{for } a_2\le x\le a_3\\ 0 & \text{otherwise}\end{cases}\quad \text{and}\quad \nu_{\tilde{A}^i}(x)=\begin{cases}\dfrac{a_2-x}{a_2-a_1'} & \text{for } a_1'\le x\le a_2\\ \dfrac{x-a_2}{a_3'-a_2} & \text{for } a_2\le x\le a_3'\\ 1 & \text{otherwise}\end{cases}$$

where $a_1'\le a_1\le a_2\le a_3\le a_3'$ and TIFN (Fig. 1) is denoted by $\tilde{A}^i_{\text{TIFN}}=\left(a_1,a_2,a_3;a_1',a_2,a_3'\right)$.



Figuer 1: Membership and non-membership functions of TIFN

Note 1 Here $\mu_{\tilde{A}^i}(x)$ increases with constant rate for $x\in[a_1,a_2]$ and decreases with constant rate for $x\in[a_2,a_3]$ but $\nu_{\tilde{A}^i}(x)$ decreases with constant rate for $x\in[a_1',a_2]$ and increases with constant rate for $x\in[a_2,a_3']$.

Definition 3.3 Trapezoidal Intuitionistic Fuzzy Number: A TrIFN (Fig. 2) $\tilde{A}^i$ is a subset of IFS in R with membership function and non-membership function as follows

$$\mu_{\tilde{A}^i}(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1} & \text{for } a_1\le x\le a_2\\ 1 & \text{for } a_2\le x\le a_3\\ \dfrac{a_4-x}{a_4-a_3} & \text{for } a_3\le x\le a_4\\ 0 & \text{otherwise}\end{cases}\quad \text{and}\quad \nu_{\tilde{A}^i}(x)=\begin{cases}\dfrac{a_2-x}{a_2-a_1'} & \text{for } a_1'\le x\le a_2\\ 0 & \text{for } a_2\le x\le a_3\\ \dfrac{x-a_3}{a_4'-a_3} & \text{for } a_3\le x\le a_4'\\ 1 & \text{otherwise}\end{cases}$$

where $a_1'\le a_1\le a_2\le a_3\le a_4\le a_4'$ and TrIFN is denoted by $\tilde{A}^i_{\text{TrIFN}}=\left(a_1,a_2,a_3,a_4;a_1',a_2,a_3,a_4'\right)$

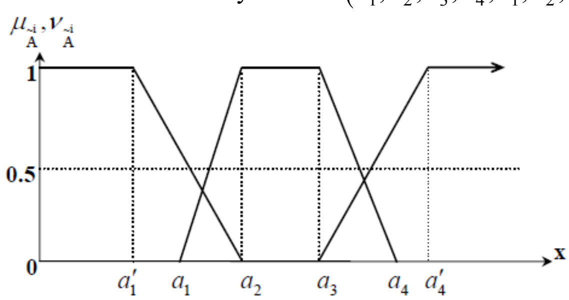


Figure 2: Membership and non-membership function of TrIFN

Note 2 Here $\mu_{\tilde{A}^i}(x)$ increases with constant rate for $x\in[a_1,a_2]$ and decreases with constant rate for $x\in[a_3,a_4]$ but $\nu_{\tilde{A}^i}(x)$ decreases with constant rate for $x\in[a_1',a_2]$ and increases with constant rate for $x\in[a_3,a_4']$.

Here we have presented a chart of IFN with transformation rule to fuzzy numbers and interval and real number.

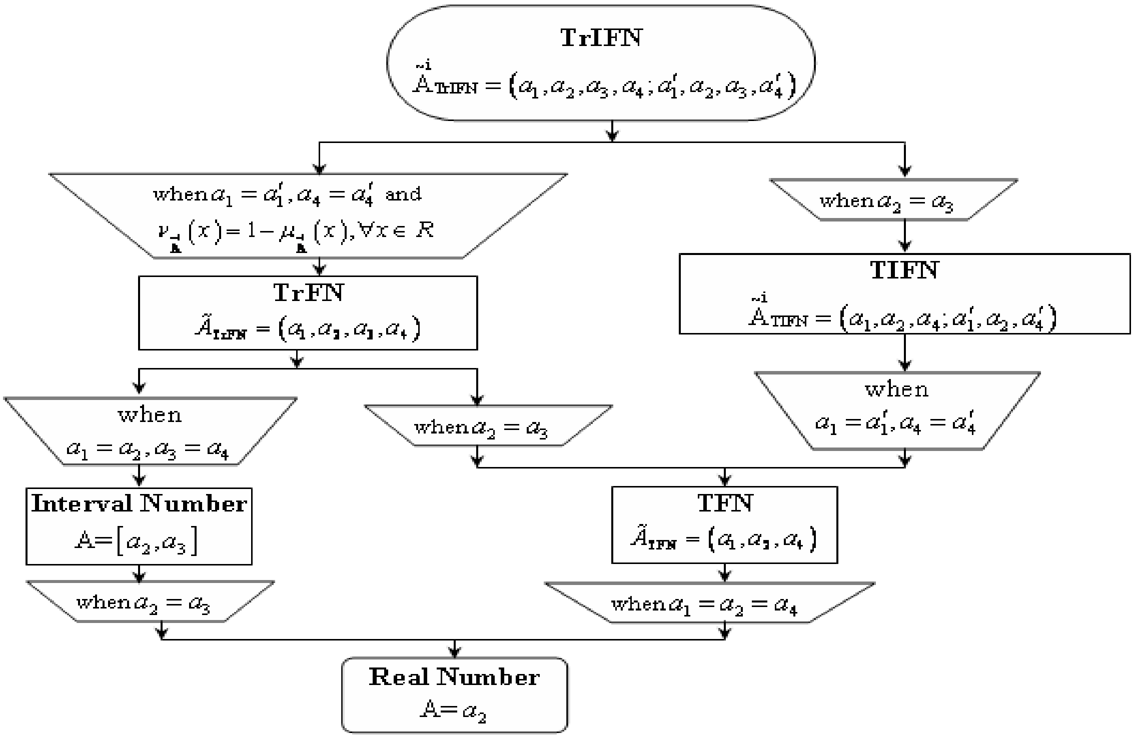


Figure 3: Chart of transformation rule on IFN

4 Extension Principle for Intuitionistic Fuzzy Sets

Let $f : X \to Y$ be a mapping from a set X to a set Y. then the extension principle allows us to define the IFS $\tilde{B}^i$ in Y induced by the IFS $\tilde{A}^i$ in X through f as follows

$$\tilde{B}^i = \left\{ < y, \mu_{\tilde{B}^i}(y), \nu_{\tilde{B}^i}(y) >: y = f(x), x \in X \right\}$$

with

$$\mu_{\tilde{B}^i}(y) = \begin{cases} \sup\limits_{y=f(x)} \mu_{\tilde{A}^i}(x) & : f^{-1}(y) \neq \phi \\ 0 & : f^{-1}(y) = \phi \end{cases} \quad \text{and} \quad \nu_{\tilde{B}^i}(y) = \begin{cases} \inf\limits_{y=f(x)} \nu_{\tilde{A}^i}(x) & : f^{-1}(y) \neq \phi \\ 0 & : f^{-1}(y) = \phi \end{cases}$$

where $f^{-1}(y)$ is the inverse image of y.

4.1 Cartesian Product of Intuitionistic Fuzzy Sets

Let $\tilde{A}^i_1, ..., \tilde{A}^i_n$ be IFSs in $X_1, ..., X_n$ with the corresponding membership functions $\mu_{\tilde{A}^i_1}(y), ..., \mu_{\tilde{A}^i_n}(y)$ and non-membership function $\nu_{\tilde{A}^i_1}(y), ..., \nu_{\tilde{A}^i_n}(y)$ respectively. Then the Cartesian product of the IFSs $\tilde{A}^i_1, ..., \tilde{A}^i_n$ denoted by $\tilde{A}^i_1 \times \cdots \times \tilde{A}^i_n$ is defined as IFS in $X_1 \times \cdots \times X_n$ whose membership functions and non-membership functions are expressed by

$$\mu_{\tilde{A}^i_1 \times \cdots \times \tilde{A}^i_1}(x_1, ..., x_n) = \min\left(\mu_{\tilde{A}^i_n}(x_1), ..., \mu_{\tilde{A}^i_n}(x_1) \right) \text{and}\, \nu_{\tilde{A}^i_1 \times \cdots \times \tilde{A}^i_1}(x_1, ..., x_n) = \max\left(\nu_{\tilde{A}^i_n}(x_1), ..., \nu_{\tilde{A}^i_n}(x_1) \right).$$

4.2 Extension Principle in Cartesian Space

Let $f : X_1 \times \cdots \times X_n \to Y$ be a mapping from $X_1 \times \cdots \times X_n$ to a set Y such that $y = f(x_1,...,x_n)$. Then the extension principle allows us to define the IFS B in Y induced by the IFS $\overset{\sim i}{\mathrm{A}}_1 \times \cdots \times \overset{\sim i}{\mathrm{A}}_n$ in $X_1 \times \cdots \times X_n$ through f as follows

$$\overset{\sim i}{\mathrm{B}} = \left\{ < y, \mu_{\overset{\sim i}{B}}(y), \nu_{\overset{\sim i}{B}}(y) >: y = f(x_1,...,x_n), (x_1,...,x_n) \in (x_1,...,x_n) \right\}$$

with

$$\mu_{\overset{\sim i}{\mathrm{B}}}(y) = \begin{cases} \sup\limits_{x_1,...,x_n \in X_1,...,X_n} \mu_{\mathrm{A}_1 \times \cdots \times \mathrm{A}_n}(x_1,...,x_n) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases}$$

and

$$\nu_{\overset{\sim i}{\mathrm{B}}}(y) = \begin{cases} \inf\limits_{x_1,...,x_n \in X_1,...,X_n} \nu_{\mathrm{A}_1 \times \cdots \times \mathrm{A}_n}(x_1,...,x_n) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases}$$

where $f^{-1}(y)$ is the inverse image of y.

5 Arithmetic Operations on Intuitionistic Fuzzy Numbers

In this section, we have presented arithmetic operations of IFNs based on intuitionistic fuzzy extension principle and approximation ((α, β)-cuts) method.

5.1 Arithmetic Operations of Intuitionistic Fuzzy Numbers based on Extension Principle

The arithmetic operation (*) of two IFNs is a mapping of an input vector $X = [x_1, x_2]^T$ define in the Cartesian product space $R \times R$ onto an output y define in the real space R. If $\overset{\sim i}{A}_1$ *and* $\overset{\sim i}{A}_2$ are IFN then their outcome of arithmetic operation is also a IFN determined with the formula

$$\left(\overset{\sim i}{A}_1 * \overset{\sim i}{A}_2\right)(y) = \left\{ \left(y, \sup_{y = x_1 * x_2} \left[\min\left(\mu_{\overset{\sim i}{\mathrm{A}}_1}(x_1), \mu_{\overset{\sim i}{\mathrm{A}}_2}(x_2) \right) \right], \inf_{y = x_1 * x_2} \left[\max\left(\nu_{\overset{\sim i}{\mathrm{A}}_1}(x_1), \nu_{\overset{\sim i}{\mathrm{A}}_2}(x_2) \right) \right] \right) \forall x_1, x_2, y \in R \right\}$$

to calculate the arithmetic operation of IFNs it is sufficient to determine the membership function and non-membership function as follows

$$\mu_{\overset{\sim i}{A}_1 * \overset{\sim i}{A}_2}(y) = \sup_{y = x_1 * x_2} \left[\min\left(\mu_{\overset{\sim i}{\mathrm{A}}_1}(x_1), \mu_{\overset{\sim i}{\mathrm{A}}_2}(x_2) \right) \right] \text{and } \nu_{\overset{\sim i}{A}_1 * \overset{\sim i}{A}_2}(y) = \inf_{y = x_1 * x_2} \left[\max\left(\nu_{\overset{\sim i}{\mathrm{A}}_1}(x_1), \nu_{\overset{\sim i}{\mathrm{A}}_2}(x_2) \right) \right].$$

5.2 Arithmetic Operations of Intuitionistic Fuzzy Numbers based on (α, β)-cuts Method

If $\overset{\sim i}{A}$ is an IFN, then (α, β)-level interval or (α, β)-cut is given by

$$A_{\alpha,\beta} = \begin{cases} \left[A_1(\alpha), A_2(\alpha) \right] \text{ for degree of acceptance } \alpha \in [0,1] \\ \left[A_1'(\beta), A_2'(\beta) \right] \text{ for degree of rejection } \beta \in [0,1] \end{cases} \text{ with } \alpha + \beta \leq 1.$$

Here (i) $\dfrac{dA_1(\alpha)}{d\alpha} > 0, \dfrac{dA_2(\alpha)}{d\alpha} < 0 \ \ \forall \alpha \in (0,1),\ A_1(1) \leq A_2(1)$ and (ii) $\dfrac{dA_1'(\beta)}{d\beta} < 0, \dfrac{dA_2'(\beta)}{d\beta} > 0 \ \ \forall \beta \in (0,1)$, $A_1'(0) \leq A_2'(0)$.

It is expressed as $A_{\alpha,\beta} = \left\{ \left[A_1(\alpha), A_2(\alpha) \right]; \left[A_1'(\beta), A_2'(\beta) \right] \right\},\ \alpha + \beta \leq 1,\ \alpha, \beta \in [0,1]$.

For instance, if $\overset{\sim i}{A} = (a_1, a_2, a_3, a_4; a_1', a_2, a_3, a_4')$ is a TrIFN, then (α, β)-level intervals or (α, β)-cuts is

$$A_{\alpha,\beta} = \{[A_1(\alpha), A_2(\alpha)]; [A_1'(\beta), A_2'(\beta)]\},\ \alpha+\beta\le 1,\ \alpha,\beta\in[0,1]$$

where $A_1(\alpha) = a_1 + \alpha(a_2 - a_1), A_2(\alpha) = a_4 - \alpha(a_4 - a_3); A_1'(\beta) = a_2 - \beta(a_2 - a_1'), A_2'(\beta) = a_3 + \beta(a_4' - a_3)$.

Property 5.1 (a) If TrIFN $\overset{\sim i}{A} = (a_1, a_2, a_3, a_4; a_1', a_2, a_3, a_4')$ and $y = ka$ (with k>0), then $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ is a TrIFN $(ka_1, ka_2, ka_3, ka_4; ka_1', ka_2, ka_3, ka_4')$.

(b) If $y = ka$ (with k<0), then $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ is a TrIFN $(ka_4, ka_3, ka_2, ka_1; ka_4', ka_3, ka_2, ka_1')$.

Proof: (a) When k>0, with the transformation $y = ka$, we can find the membership function for membership (acceptance) function of TrIFN $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ by α-cut method.

Left-hand and right-hand α-cut of $\overset{\sim i}{A}$ is $\mu_{\overset{\sim i}{A}}(x) \ge \alpha \Rightarrow [a_1 + \alpha(a_2 - a_1), a_4 - \alpha(a_4 - a_3)]$ for any $\alpha\in[0,1]$, i.e., $x \in [a_1 + \alpha(a_2 - a_1), a_4 - \alpha(a_4 - a_3)]$. So, $y(=ka) \in [ka_1 + \alpha(ka_2 - ka_1), ka_4 - \alpha(ka_4 - ka_3)]$.

Thus, we get the membership function of $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ as

$$\mu_{\overset{\sim i}{y}}(y) = \begin{cases} \dfrac{y - ka_1}{ka_2 - ka_1} & \text{for } ka_1 \le y \le ka_2 \\ 1 & \text{for } ka_2 \le y \le ka_3 \\ \dfrac{ka_4 - y}{ka_4 - ka_3} & \text{for } ka_3 \le y \le ka_4 \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Hence the rule is proved for membership function.

For non-membership function, β-cut of $\overset{\sim i}{A}$ is $\nu_{\overset{\sim i}{A}}(x) \le \beta \Rightarrow [a_2 - \beta(a_2 - a_1'), a_3 + \beta(a_4' - a_3)]$ for any $\beta\in[0,1]$, i.e., $x \in [a_2 - \beta(a_2 - a_1'), a_3 + \beta(a_4' - a_3)]$. So, $y(=ka) \in [ka_2 - \beta(ka_2 - ka_1'), ka_3 + \beta(ka_4' - ka_3)]$.

Thus, we get the non-membership function of $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ as

$$\nu_{\overset{\sim i}{A}}(y) = \begin{cases} \dfrac{ka_2 - y}{ka_2 - ka_1'} & \text{for } ka_1' \le y \le ka_2 \\ 0 & \text{for } ka_2 \le y \le ka_3 \\ \dfrac{y - ka_3}{ka_4' - ka_3} & \text{for } ka_3 \le y \le ka_4' \\ 1 & \text{otherwise.} \end{cases} \tag{5.2}$$

Hence rule is proved for non-membership function.

Thus we have $\overset{\sim i}{Y} = k\overset{\sim i}{A} = (ka_1, ka_2, ka_3, ka_4; ka_1', ka_2, ka_3, ka_4')$ is a TrIFN.

(b) Similarly we can proof that, if $y = ka$ and k<0 , then

$$\mu_{\overset{\sim i}{y}}(y) = \begin{cases} \dfrac{y - ka_4}{ka_3 - ka_4} & \text{for } ka_4 \le y \le ka_3 \\ 1 & \text{for } ka_3 \le y \le ka_2 \\ \dfrac{ka_1 - y}{ka_1 - ka_2} & \text{for } ka_2 \le y \le ka_1 \\ 0 & \text{otherwise.} \end{cases} \text{ and } \nu_{\overset{\sim i}{y}}(y) = \begin{cases} \dfrac{ka_3 - y}{ka_3 - ka_4'} & \text{for } ka_4' \le y \le ka_3 \\ 0 & \text{for } ka_3 \le y \le ka_2 \\ \dfrac{y - ka_2}{ka_1' - ka_2} & \text{for } ka_2 \le y \le ka_1' \\ 1 & \text{otherwise.} \end{cases} \tag{5.3}$$

Property 5.2 If $\overset{\sim i}{A} = (a_1, a_2, a_3, a_4; a_1', a_2, a_3, a_4')$ and $\overset{\sim i}{B} = (b_1, b_2, b_3, b_4; b_1', b_2, b_3, b_4')$ are two TrIFNs, then $\overset{\sim i}{C} = \overset{\sim i}{A} \oplus \overset{\sim i}{B}$ is also TrIFN $\overset{\sim i}{A} \oplus \overset{\sim i}{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; a_1' + b_1', a_2 + b_2, a_3 + b_3, a_4' + b_4')$.

Proof: With the transformation z=x+y, we can find the membership function of acceptance (membership) IFS $\overset{\sim i}{C} = \overset{\sim i}{A} \oplus \overset{\sim i}{B}$ by α-cut method.

α-cut for membership function of $\overset{\sim i}{A}$ is $\left[a_1 + \alpha\left(a_2 - a_1\right), a_4 - \alpha\left(a_4 - a_3\right)\right] \forall \alpha \in [0,1]$, i.e.,

$$x \in \left[a_1 + \alpha\left(a_2 - a_1\right), a_4 - \alpha\left(a_4 - a_3\right)\right].$$

α-cut for membership function of $\overset{\sim i}{B}$ is $\left[b_1 + \alpha\left(b_2 - b_1\right), b_4 - \alpha\left(b_4 - b_3\right)\right] \forall \alpha \in [0,1]$, i.e. ,

$$y \in \left[b_1 + \alpha\left(b_2 - b_1\right), b_4 - \alpha\left(b_4 - b_3\right)\right].$$

So,

$$z\ (=x+y) \in \left[a_1 + b_1 + \alpha\left(\left(a_2 - a_1\right) + \left(b_2 - b_1\right)\right), a_4 + b_4 - \alpha\left(\left(a_4 - a_3\right) + \left(b_4 - b_3\right)\right)\right].$$

So, the membership (acceptance) function of $\overset{\sim i}{C} = \overset{\sim i}{A} \oplus \overset{\sim i}{B}$ is

$$\mu_{\overset{\sim i}{C}}(z) = \begin{cases} \dfrac{z - a_1 - b_1}{\left(a_2 - a_1\right) + \left(b_2 - b_1\right)} & \text{for } a_1 + b_1 \le z \le a_2 + b_2 \\ 1 & \text{for } a_2 + b_2 \le z \le a_3 + b_3 \\ \dfrac{a_4 + b_4 - z}{\left(a_4 - a_3\right) + \left(b_4 - b_3\right)} & \text{for } a_3 + b_3 \le z \le a_4 + b_4 \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Hence additions rule is proved for membership function.

For non-membership function, β-cut of $\overset{\sim i}{A}$ is $\left[a_2 - \beta\left(a_2 - a_1'\right), a_3 + \beta\left(a_4' - a_3\right)\right] \forall \beta \in [0,1]$, i.e.,

$$x \in \left[a_2 - \beta\left(a_2 - a_1'\right), a_3 + \beta\left(a_4' - a_3\right)\right].$$

β-cut for non-membership function of $\overset{\sim i}{B}$ is $\left[b_2 - \beta\left(b_2 - b_1'\right), b_3 + \beta\left(b_4' - b_3\right)\right] \forall \beta \in [0,1]$, i.e.,

$$y \in \left[b_2 - \beta\left(b_2 - b_1'\right), b_3 + \beta\left(b_4' - b_3\right)\right].$$

So, z (=x+y) $\in \left[a_2 + b_2 - \beta\left(\left(a_2 - a_1'\right) + \left(b_2 - b_1'\right)\right), a_3 + b_3 - \beta\left(\left(a_4' - a_3\right) + \left(b_4' - b_3\right)\right)\right]$.

So, the non-membership (rejection) function of $\overset{\sim i}{C} = \overset{\sim i}{A} \oplus \overset{\sim i}{B}$ is

$$\nu_{\overset{\sim i}{C}}(z) = \begin{cases} \dfrac{a_2 + b_2 - z}{\left(a_2 - a_1'\right) + \left(b_2 - b_1'\right)} & \text{for } a_1' + b_1' \le z \le a_2 + b_2 \\ 0 & \text{for } a_2 + b_2 \le z \le a_3 + b_3 \\ \dfrac{z - a_3 - b_3}{\left(a_4' - a_3\right) + \left(b_4' - b_3\right)} & \text{for } a_3 + b_3 \le z \le a_4' + b_4' \\ 1 & \text{otherwise.} \end{cases} \tag{5.5}$$

Hence additions rule is proved for non-membership function.

Thus we have $\overset{\sim i}{A} \oplus \overset{\sim i}{B} = \left(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; a_1' + b_1', a_2 + b_2, a_3 + b_3, a_4' + b_4'\right)$ is a TrIFN.

Note 3 If we have the transformation $\overset{\sim i}{C} = k_1 \overset{\sim i}{A} + k_2 \overset{\sim i}{B}$ (k_1, k_2 are (not all zero) real numbers), then the IFS $\overset{\sim i}{C} = k_1 \overset{\sim i}{A} + k_2 \overset{\sim i}{B}$ is the following TrIFN:

(i) $\left(k_1a_1 + k_2b_1, k_1a_2 + k_2b_2, k_1a_3 + k_2b_3, k_1a_4 + k_2b_4; k_1a_1' + k_2b_1', k_1a_2 + k_2b_2, k_1a_3 + k_2b_3, k_1a_4' + k_2a_4'\right)$ if $k_1 > 0, k_2 \ge 0$ or $k_1 \ge 0,\ k_2 > 0$,

(ii) $\left(k_1a_1 + k_2b_4, k_1a_2 + k_2b_3, k_1a_3 + k_2b_2, k_1a_4 + k_2b_1; k_1a_1' + k_2b_4', k_1a_2 + k_2b_3, k_1a_3 + k_2b_2, k_1a_4' + k_2b_1'\right)$ if $k_1 > 0, k_2 \le 0$ or $k_1 \ge 0, k_2 < 0$,

(iii) $\left(k_1a_4 + k_2b_1, k_1a_3 + k_2b_2, k_1a_2 + k_2b_3, k_1a_1 + k_2b_4; k_1a_4' + k_2b_1', k_1a_3 + k_2b_2, k_1a_2 + k_2b_3, k_1a_1' + k_2b_4'\right)$ if $k_1 < 0, k_2 \ge 0$, or $k_1 \le 0, k_2 > 0$

(iv) $\left(k_1a_4+k_2b_4,k_1a_3+k_2b_3,k_1a_2+k_2b_2,k_1a_1+k_2b_1;k_1a_4'+k_2b_4',k_1a_3+k_2b_3,k_1a_2+k_2b_2,k_1a_1'+k_2b_1'\right)$ if $k_1<0,k_2\le 0$ or $k_1\le 0,k_2<0$.

Property 5.3 If $\overset{\sim i}{A}=\left(a_1,a_2,a_3,a_4;a_1',a_2,a_3,a_4'\right)$ and $\overset{\sim i}{B}=\left(b_1,b_2,b_3,b_4;b_1',b_2,b_3,b_4'\right)$ are two TrIFN, then $\overset{\sim i}{P}=\overset{\sim i}{A}\otimes\overset{\sim i}{B}$ is approximated TrIFN $\overset{\sim i}{A}\otimes\overset{\sim i}{B}=\left(a_1b_1,a_2b_2,a_3b_3,a_4b_4;a_1'b_1',a_2b_2,a_3b_3,a_4'b_4'\right)$.

Proof: With the transformation z=x×y, we can find the membership function of acceptance (membership) IFS $\overset{\sim i}{P}=\overset{\sim i}{A}\otimes\overset{\sim i}{B}$ by α-cut method.

α-cut for membership function of $\overset{\sim i}{A}$ is $\mu_{\overset{\sim i}{A}}(x)\ge\alpha\Rightarrow\left[a_1+\alpha(a_2-a_1),a_4-\alpha(a_4-a_3)\right]\forall\alpha\in[0,1]$, i.e.,

$$x\in\left[a_1+\alpha(a_2-a_1),a_4-\alpha(a_4-a_3)\right].$$

α-cut for membership function of $\overset{\sim i}{B}$ is $\mu_{\overset{\sim i}{B}}(x)\ge\alpha\Rightarrow\left[b_1+\alpha(b_2-b_1),b_4-\alpha(b_4-b_3)\right]\forall\alpha\in[0,1]$, i.e.,

$$y\in\left[b_1+\alpha(b_2-b_1),b_4-\alpha(b_4-b_3)\right].$$

So, z (=x×y)$\in\left[\left(a_1+\alpha(a_2-a_1)\right)\left(b_1+\alpha(b_2-b_1)\right),\left(a_4-\alpha(a_4-a_3)\right)\left(b_4-\alpha(b_4-b_3)\right)\right]$.

So, the membership (acceptance) function of $\overset{\sim i}{P}=\overset{\sim i}{A}\otimes\overset{\sim i}{B}$ is

$$\mu_{\overset{\sim i}{P}}(z)=\begin{cases}\dfrac{-B_1+\sqrt{B_1^2-4A_1(a_1b_1-z)}}{2A_1} & \text{for } a_1b_1\le z\le a_2b_2\\ 1 & \text{for } a_2b_2\le z\le a_3b_3\\ \dfrac{B_2-\sqrt{B_2^2-4A_2(a_4b_4-z)}}{2A_2} & \text{for } a_3b_3\le z\le a_4b_4\\ 0 & \text{otherwise}\end{cases}\tag{5.6}$$

where $A_1=(a_2-a_1)(b_2-b_1)$, $B_1=b_1(a_2-a_1)+a_1(b_2-b_1)$, $A_2=(a_4-a_3)(b_4-b_3)$ and $B_2=-\left(b_4(a_4-a_3)+a_4(b_4-b_3)\right)$.

For non-membership function, β-cut of $\overset{\sim i}{A}$ is $\nu_{\overset{\sim i}{A}}(x)\le\beta\Rightarrow\left[a_2-\beta(a_2-a_1'),a_3+\beta(a_4'-a_3)\right]\forall\beta\in[0,1]$, i.e., $x\in\left[a_2-\beta(a_2-a_1'),a_2+\beta(a_3'-a_2)\right]$, β-cut of $\overset{\sim i}{B}$ is $\nu_{\overset{\sim i}{B}}(x)\le\beta\Rightarrow\left[b_2-\beta(b_2-b_1'),b_3+\beta(b_4'-b_3)\right]\forall\beta\in[0,1]$, i.e., $y\in\left[b_2-\beta(b_2-b_1'),b_2+\beta(b_3'-b_2)\right]$.

So, z(=x×y) $\in\left[\left(a_2-\beta(a_2-a_1')\right)\left(b_2-\beta(b_2-b_1')\right),\left(a_3+\beta(a_4'-a_3)\right)\left(b_3+\beta(b_4'-b_3)\right)\right]$. So, the non-membership (rejection) function of $\overset{\sim i}{P}=\overset{\sim i}{A}\otimes\overset{\sim i}{B}$ is

$$\nu_{\overset{\sim i}{P}}(z)=\begin{cases}1-\dfrac{-B_1'+\sqrt{B_1'^2-4A_1'(a_1'b_1'-z)}}{2A_1'} & \text{for } a_1'b_1'\le z\le a_2b_2\\ 0 & \text{for } a_2b_2\le z\le a_3b_3\\ 1-\dfrac{B_2'-\sqrt{B_2'^2-4A_2'(a_4'b_4'-z)}}{2A_2'} & \text{for } a_3b_3\le z\le a_4'b_4'\\ 1 & \text{otherwise}\end{cases}\tag{5.7}$$

where $A_1'=(a_2-a_1')(b_2-b_1')$, $B_1'=b_1'(a_2-a_1')+a_1'(b_2-b_1')$, $A_2'=(a_4'-a_3)(b_4'-b_3)$ and $B_2'=-\left(b_4'(a_4'-a_3)+a_4'(b_4'-b_3)\right)$.

So $\overset{\sim i}{P}=\overset{\sim i}{A}\otimes\overset{\sim i}{B}$ represented by (5.6) and (5.7) is a trapezoidal shaped IFN. It can be approximated to a TrIFN $\overset{\sim i}{A}\otimes\overset{\sim i}{B}=\left(a_1b_1,a_2b_2,a_3b_3,a_4b_4;a_1'b_1',a_2b_2,a_3b_3,a_4'b_4'\right)$ (shown in Fig.3 with **- - - -** line).

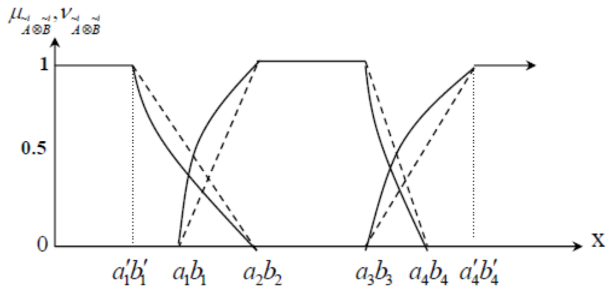


Figure 4: Membership and non-membership functions for product of two TrIFN

Theorem 1 The divergences due to approximation of trapezoidal shaped IFN to TrIFN for multiplication of two TrIFN are as follows:

(a) Maximum left divergence for membership (non-membership) function = $1/4$ product of left spread of TrIFN $\overset{\sim i}{A}$ and $\overset{\sim i}{B}$ for membership (non-membership) function.

(b) Maximum right divergence for membership (non-membership) function $= 1/4$ product of right spread of TrIFN $\overset{\sim i}{A}$ and $\overset{\sim i}{B}$ for membership (non-membership) function.

Proof: $\overset{\sim i}{A} \otimes \overset{\sim i}{B}$ represented by (5.6) and (5.7) is a trapezoidal shaped IFN. It can be approximated to a TrIFN $\overset{\sim i}{P} = \left(a_1b_1, a_2b_2, a_3b_3, a_4b_4; a_1'b_1', a_2b_2, a_3b_3, a_4'b_4'\right)$.

Now consider $(\alpha,\beta)-$cut of the above approximated TrIFN, which is given by $P_{\alpha,\beta} = \left\{\left[P_1(\alpha), P_2(\alpha)\right], \left[P_1'(\beta), P_2'(\beta)\right]\right\}$ where $P_1(\alpha) = a_1b_1 + \alpha\left(a_2b_2 - a_1b_1\right)$, $P_2(\alpha) = a_4b_4 - \alpha\left(a_4b_4 - a_3b_3\right)$, $P_1'(\beta) = a_2b_2 - \beta\left(a_2b_2 - a_1'b_1'\right)$, $P_2'(\beta) = a_3b_3 + \beta\left(a_4'b_4' - a_3b_3\right)$.

The corresponding left divergent for membership function $\left(\varepsilon_{lm}(\alpha)\right)$ is

$$\varepsilon_{lm}(\alpha) = \left(a_1 + \alpha\left(a_2 - a_1\right)\right)\left(b_1 + \alpha\left(b_2 - b_1\right)\right) - \left(a_1b_1 + \alpha\left(a_2b_2 - a_1b_1\right)\right).$$

To find out the optimum divergence $\frac{d\varepsilon_{lm}(\alpha)}{d\alpha} = 0$ gives $\alpha^* = 0.5 \in [0,1]$. The maximum left divergence for membership is $\varepsilon_{lm}\left(\alpha^*\right) = -\left(a_2 - a_1\right)\left(b_2 - b_1\right)/4$.

Again the corresponding right divergent for membership function $\left(\varepsilon_{rm}(\alpha)\right)$ is

$$\varepsilon_{rm}(\alpha) = \left(a_4 - \alpha\left(a_4 - a_3\right)\right)\left(b_4 - \alpha\left(b_4 - b_3\right)\right) - \left(a_4b_4 - \alpha\left(a_4b_4 - a_3b_3\right)\right).$$

To find out the optimum divergence $\frac{d\varepsilon_{rm}(\alpha)}{d\alpha} = 0$ gives $\alpha^* = 0.5 \in [0,1]$. The maximum right divergence for membership is $\varepsilon_{lm}\left(\alpha^*\right) = -\left(a_4 - a_3\right)\left(b_4 - b_3\right)/4$.

The corresponding left divergent for non-membership function $\varepsilon_{ln}(\beta)$ is

$$\varepsilon_{ln}(\beta) = \left(a_2 - \beta\left(a_2 - a_1'\right)\right)\left(b_2 - \beta\left(b_2 - b_1'\right)\right) - \left(a_2b_2 - \beta\left(a_2b_2 - a_1'b_1'\right)\right).$$

To find out the optimum divergence $\frac{d\varepsilon_{ln}(\beta)}{d\beta} = 0$ gives $\beta^* = 0.5 \in [0,1]$. The maximum left divergence for non-membership is $\varepsilon_{ln}\left(\beta^*\right) = -\left(a_2 - a_1'\right)\left(b_2 - b_1'\right)/4$.

Again the corresponding right divergent for non-membership function $\varepsilon_{rn}(\beta)$ is

$$\varepsilon_{rn}(\beta) = \left(a_3 + \beta\left(a_4' - a_3\right)\right)\left(b_3 + \beta\left(b_4' - b_3\right)\right) - \left(a_3b_3 + \beta\left(a_4'b_4' - a_3b_3\right)\right).$$

To find out the optimum divergence $\frac{d\varepsilon_{rn}(\beta)}{d\beta}=0$ gives $\beta^*=0.5\in[0,1]$. The maximum right divergence for non-membership is $\varepsilon_{rn}(\beta^*)=-(a_4'-a_3)(b_4'-b_3)/4$.

Note 4 It may conclude that when spreads are increasing, divergences due to approximation for product of two TrIFNs are also increasing. Divergences are insignificant for very small spreads. In such a situation product of two TrIFNs can directly be written as approximated TrIFN.

Property 5.4 If $\overset{\sim i}{A}=(a_1,a_2,a_3,a_4;a_1',a_2,a_3,a_4')$ and $\overset{\sim i}{B}=(b_1,b_2,b_3,b_4;b_1',b_2,b_3,b_4')$ are two TrIFN, then $\overset{\sim i}{D}=\overset{\sim i}{A}\div\overset{\sim i}{B}$ is approximated TrIFN

$$\overset{\sim i}{A}\div\overset{\sim i}{B}=\left(\frac{a_1}{b_4},\frac{a_2}{b_3},\frac{a_3}{b_2},\frac{a_4}{b_1};\frac{a_1'}{b_4'},\frac{a_2}{b_3},\frac{a_3}{b_2},\frac{a_4'}{b_1'}\right).$$

Proof: With the transformation z=x÷y, we can find the membership function of acceptance (membeship) IFS $\overset{\sim i}{D}=\overset{\sim i}{A}\div\overset{\sim i}{B}$ by α-cut method.

α-cut for membership function of $\overset{\sim i}{A}$ is $\mu_{\overset{\sim i}{A}}(x)\ge\alpha\Rightarrow[a_1+\alpha(a_2-a_1),a_4-\alpha(a_4-a_3)]\ \forall\alpha\in[0,1]$, i.e., $x\in[a_1+\alpha(a_2-a_1),a_4-\alpha(a_4-a_3)]$. α-cut for membership function of $\overset{\sim i}{B}$ is $\mu_{\overset{\sim i}{B}}(x)\ge\alpha\Rightarrow[b_1+\alpha(b_2-b_1),b_4-\alpha(b_4-b_3)]\ \forall\alpha\in[0,1]$, i.e., $y\in[b_1+\alpha(b_2-b_1),b_4-\alpha(b_4-b_3)]$. So,

$$\text{z (=x÷y)}\in\left[\frac{a_1+\alpha(a_2-a_1)}{b_4-\alpha(b_4-b_3)},\frac{a_4-\alpha(a_4-a_3)}{b_1+\alpha(b_2-b_1)}\right].$$

So, we have the membership (acceptance) function of $\overset{\sim i}{D}=\overset{\sim i}{A}\div\overset{\sim i}{B}$ as

$$\mu_{\overset{\sim i}{D}}(z)=\begin{cases}\dfrac{b_4z-a_1}{(a_2-a_1)+z(b_4-b_3)} & \text{for}\ \frac{a_1}{b_1}\le z\le\frac{a_2}{b_3}\\ 1 & \text{for}\ \frac{a_2}{b_3}\le z\le\frac{a_3}{b_2}\\ \dfrac{a_1-b_1z}{(a_4-a_3)+z(b_2-b_1)} & \text{for}\ \frac{a_3}{b_2}\le z\le\frac{a_4}{b_1}\\ 0 & \text{otherwise.}\end{cases}\tag{5.8}$$

For non-membership function, β-cut of $\overset{\sim i}{A}$ is $\nu_{\overset{\sim i}{A}}(x)\le\beta\Rightarrow[a_2-\beta(a_2-a_1'),a_3+\beta(a_4'-a_3)]\forall\beta\in[0,1]$, i.e., $x\in[a_2-\beta(a_2-a_1'),a_2+\beta(a_3'-a_2)]$. β-cut of $\overset{\sim i}{B}$ is $\nu_{\overset{\sim i}{B}}(x)\le\beta\Rightarrow[b_2-\beta(b_2-b_1'),b_3+\beta(b_4'-b_3)]\forall\beta\in[0,1]$, i.e., $y\in[b_2-\beta(b_2-b_1'),b_2+\beta(b_3'-b_2)]$. So,

$$\text{z (=x÷y)}\in\left[\frac{a_2-\beta(a_2-a_1')}{b_3+\beta(b_4'-b_3)},\frac{a_3+\beta(a_4'-a_3)}{b_2-\beta(b_2-b_1')}\right].$$

So, we have the non-membership (rejection) function of $\overset{\sim i}{D}=\overset{\sim i}{A}\div\overset{\sim i}{B}$ as follows

$$\nu_{\overset{\sim i}{D}}(z)=\begin{cases}\dfrac{a_2-b_3z}{(a_2-a_1')+z(b_4'-b_3)} & \text{for}\ \frac{a_1'}{b_4'}\le z\le\frac{a_2}{b_3}\\ 0 & \text{for}\ \frac{a_2}{b_3}\le z\le\frac{a_3}{b_2}\\ \dfrac{b_2z-a_3}{(a_4'-a_3)+z(b_2-b_1')} & \text{for}\ \frac{a_3}{b_2}\le z\le\frac{a_4'}{b_1'}\\ 1 & \text{otherwise.}\end{cases}\tag{5.9}$$

So $\overset{\sim i}{D} = \overset{\sim i}{A} \div \overset{\sim i}{B}$ represented by (5.8) and (5.9) is a trapezoidal shaped IFN. It can be approximated to TrIFN

$$\overset{\sim i}{A} \div \overset{\sim i}{B} = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}; \frac{a_1'}{b_4'}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4'}{b_1'} \right).$$

Theorem 2 Divergences due to approximation of trapezoidal shaped IFN to TrIFN for division of two TrIFNs $\overset{\sim i}{A} = (a_1, a_2, a_3, a_4; a_1', a_2, a_3, a_4')$ and $\overset{\sim i}{B} = (b_1, b_2, b_3, b_4; b_1', b_2, b_3, b_4')$ are as follows:

(a) Maximum left and right divergences for membership function are

$$\frac{\sqrt{b_4} - \sqrt{b_3}}{\sqrt{b_4} + \sqrt{b_3}} \left(\frac{a_2}{b_3} - \frac{a_1}{b_4} \right) \text{and} \frac{\sqrt{b_2} - \sqrt{b_1}}{\sqrt{b_2} + \sqrt{b_1}} \left(\frac{a_4}{b_1} - \frac{a_3}{b_2} \right)$$

respectively.

(b) Maximum left and right divergences for non-membership function are

$$\frac{\sqrt{b_4'} - \sqrt{b_3}}{\sqrt{b_4'} + \sqrt{b_3}} \left(\frac{a_2}{b_3} - \frac{a_1'}{b_4'} \right) \text{ and } \frac{\sqrt{b_2} - \sqrt{b_1'}}{\sqrt{b_2} + \sqrt{b_1'}} \left(\frac{a_4'}{b_1'} - \frac{a_3}{b_2} \right)$$

respectively.

Proof: Let $\overset{\sim i}{D} = \overset{\sim i}{A} \div \overset{\sim i}{B}$ represented by (5.8) and (5.9) is a trapezoidal shaped IFN. It can be approximated to a TrIFN

$$\overset{\sim i}{A} \div \overset{\sim i}{B} = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}; \frac{a_1'}{b_4'}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4'}{b_1'} \right).$$

Now consider $(\alpha, \beta)-$cut of the above approximated TrIFN, which is given by

$$D_{\alpha,\beta} = \left\{ \left[D_1(\alpha), D_2(\alpha) \right], \left[D_1'(\beta), D_2'(\beta) \right] \right\}$$

where $D_1(\alpha) = \frac{a_1}{b_4} + \alpha\left(\frac{a_2}{b_3} - \frac{a_1}{b_4} \right)$, $D_2(\alpha) = \frac{a_4}{b_1} - \alpha\left(\frac{a_4}{b_1} - \frac{a_3}{b_2} \right)$, $D_1'(\beta) = \frac{a_2}{b_3} - \beta\left(\frac{a_2}{b_3} - \frac{a_1'}{b_4'} \right)$, $D_2'(\beta) = \frac{a_3}{b_2} + \beta\left(\frac{a_4'}{b_1'} - \frac{a_3}{b_2} \right)$.

The corresponding left divergent for membership function $(\varepsilon_{lm}(\alpha))$ is $\varepsilon_{lm}(\alpha) = \frac{a_1 + \alpha(a_2 - a_1)}{b_4 - \alpha(b_4 - b_3)} - \left(\frac{a_1}{b_4} + \alpha\left(\frac{a_2}{b_3} - \frac{a_1}{b_4} \right) \right)$. To find out the optimum divergence $\frac{d\varepsilon_{lm}(\alpha)}{d\alpha} = 0$ gives $\alpha^* = \frac{\sqrt{b_4}}{\sqrt{b_4} + \sqrt{b_3}} \in [0,1]$. The maximum left divergence for membership is $\varepsilon_{lm}(\alpha^*) = -\frac{\sqrt{b_4} - \sqrt{b_3}}{\sqrt{b_4} + \sqrt{b_3}} \left(\frac{a_2}{b_3} - \frac{a_1}{b_4} \right)$.

Again the corresponding right divergent for membership function $(\varepsilon_{rm}(\alpha))$ is

$$\varepsilon_{rm}(\alpha) = \frac{a_4 - \alpha(a_4 - a_3)}{b_1 + \alpha(b_2 - b_1)} - \left(\frac{a_4}{b_1} - \alpha\left(\frac{a_4}{b_1} - \frac{a_3}{b_2} \right) \right).$$

To find out the optimum divergence $\frac{d\varepsilon_{rm}(\alpha)}{d\alpha} = 0$ gives $\alpha^* = \frac{\sqrt{b_1}}{\sqrt{b_2} + \sqrt{b_1}} \in [0,1]$. The maximum right divergence for membership is $\varepsilon_{lm}(\alpha^*) = -\frac{\sqrt{b_2} - \sqrt{b_1}}{\sqrt{b_2} + \sqrt{b_1}} \left(\frac{a_4}{b_1} - \frac{a_3}{b_2} \right)$.

The corresponding left divergent for non-membership function $\varepsilon_{ln}(\beta)$ is $\varepsilon_{ln}(\beta) = \frac{a_2 - \beta(a_2 - a_1')}{b_3 + \beta(b_4' - b_3)} - \left(\frac{a_2}{b_3} - \beta\left(\frac{a_2}{b_3} - \frac{a_1'}{b_4'} \right) \right)$. To find out the optimum divergence $\frac{d\varepsilon_{ln}(\beta)}{d\beta} = 0$ gives $\beta^* = \frac{\sqrt{b_3}}{\sqrt{b_4'} + \sqrt{b_3}} \in [0,1]$. The maximum left divergence for non-membership is $\varepsilon_{ln}(\beta^*) = -\frac{\sqrt{b_4'} - \sqrt{b_3}}{\sqrt{b_4'} + \sqrt{b_3}} \left(\frac{a_2}{b_3} - \frac{a_1'}{b_4'} \right)$.

Again the corresponding right divergent for non-membership function $\varepsilon_{rn}(\beta)$ is $\varepsilon_{rn}(\beta) = \frac{a_3 + \beta(a_4' - a_3)}{b_2 - \beta(b_2 - b_1')} - \left(\frac{a_3}{b_2} + \beta\left(\frac{a_4'}{b_1'} - \frac{a_3}{b_2} \right) \right)$. To find out the optimum divergence $\frac{d\varepsilon_{rn}(\beta)}{d\beta} = 0$ gives $\beta^* = \frac{\sqrt{b_2}}{\sqrt{b_2} + \sqrt{b_1'}} \in [0,1]$. The maximum right divergence for non-membership is $\varepsilon_{rn}(\beta^*) = -\frac{\sqrt{b_2} - \sqrt{b_1'}}{\sqrt{b_2} + \sqrt{b_1'}} \left(\frac{a_4'}{b_1'} - \frac{a_3}{b_2} \right)$.

Note 5 When spreads increases, divergences due to approximation for division of two TrIFNs also increases. Divergences are insignificant for very small spreads. In such a situation division of two TrIFNs can directly be written as approximated TrIFN.

6 Numerical Exposure of Arithmetic Operation on Intuitionistic Fuzzy Number

Here we have presented numerical example of arithmetic operations of IFNs based on intuitionistic fuzzy extension principle and on approximation ((α, β)-cuts) method.

6.1 Addition of Two TrIFN by Intuitionistic Fuzzy Extension Principle

Let $\overset{\sim i}{A}=(1.5,2,3.5,4;0.5,2,3.5,4.5)$, $\overset{\sim i}{B}=(2,3,4,5.5;1.5,3,4,6)$ and $\overset{\sim i}{C}=\overset{\sim i}{A}\oplus\overset{\sim i}{B}$ (Fig.5). Then

$$\mu_{\overset{\sim i}{C}}(z)=\sup\left\{\min\left(\mu_{\overset{\sim i}{A}}(x),\mu_{\overset{\sim i}{B}}(y)\right):x+y=z\right\} \text{ and } \nu_{\overset{\sim i}{C}}(z)=\inf\left\{\max\left(\nu_{\overset{\sim i}{A}}(x),\nu_{\overset{\sim i}{B}}(y)\right):x+y=z\right\}.$$

Let us exhibit the computational procedure involve in above equation for membership function, first pick a value for z, then evaluate $\min\left(\mu_{\overset{\sim i}{A}}(x),\mu_{\overset{\sim i}{B}}(y)\right)$ for x and y which add up to z=4.5. We have done this for certain values of x and y as shown in Table 1. It appear that the max occurs for x=2 and y=2.5, therefore $\nu_{\overset{\sim i}{C}}(4.5)=0.5$. Now do this for other values of z. Similarly for non-membership function, evaluate $\max\left(\nu_{\overset{\sim i}{A}}(x),\nu_{\overset{\sim i}{B}}(y)\right)$ for x and y which add up to z=4.5. We have done this for certain values of x and y as shown in Table 2. The min occurs for x=1.75 and y=2.75 so that $\nu_{\overset{\sim i}{C}}(4.5)=0.16666$. Now do this for other values of z. Finally, we get $\overset{\sim i}{C}=(3.5,5,7.5,9.5;2,5,7.5,10.5)$ a TrIFN.

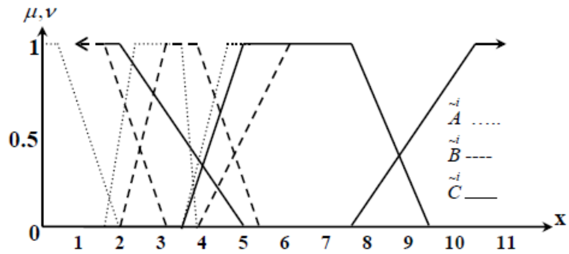


Figure 5: Addition of two TrIFN based on IF Extension principle

Table 1: Finding membership function of sum of two TrIFN

x	$\mu_{\overset{\sim i}{A}}(x)$	y	$\mu_{\overset{\sim i}{B}}(y)$	min(col#2, col#4)
1.5	0	3	1	0
1.75	0.5	2.75	0.75	0.25
2	1	2.5	0.5	0.5
2.25	1	2.25	0.25	0.25
2.5	1	2	0	0

6.2 Addition of Two TrIFN by (α, β)-cut Method

Let us consider two TrIFN $\overset{\sim i}{P}=(1.5,2,3.5,4;0.5,2,3.5,4.5)$ and $\overset{\sim i}{Q}=(2,3,4,5.5;1.5,3,4,6)$. Using Eq.(5.4) and Eq.(5.5), the addition of these two TrIFN is defined by $\overset{\sim i}{P}\oplus\overset{\sim i}{Q}=(3.5,5,7.5,9.5;2,5,7.5,10.5)$ with membership and non-membership function as follows

$$\mu_{\tilde{P}^i \oplus \tilde{Q}^i}(x)=\begin{cases}\frac{x-3.5}{1.5} & \text{if } 3.5\le x\le 5\\ 1 & \text{if } 5\le x\le 7.5\\ \frac{9.5-x}{2} & \text{if } 7.5\le x\le 9.5\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \nu_{\tilde{P}^i \oplus \tilde{Q}^i}(x)=\begin{cases}\frac{5-x}{3} & \text{if } 2\le x\le 5\\ 0 & \text{if } 5\le x\le 7.5\\ \frac{x-7.5}{3} & \text{if } 7.5\le x\le 10.5\\ 1 & \text{otherwise.}\end{cases}$$

Table 2: Finding non-membership function of sum of two TrIFN

x	$\nu_{\tilde{A}^i}(x)$	y	$\nu_{\tilde{B}^i}(y)$	max(col#2, col#4)
0.5	1	4	0	1
0.75	0.83333	3.75	0	0.83333
1	0.66666	3.5	0	0.66666
1.25	0.5	3.25	0	0.5
1.5	0.33333	3	0	0.33333
1.75	0.16666	2.75	0.16666	0.16666
2	0	2.5	0.33333	0.33333
2.25	0	2.25	0.5	0.5
2.5	0	2	0.66666	0.66666
2.75	0	1.75	0.83333	0.83333
3	0	1.5	1	1

6.3 Multiplication of Two TrIFN by (α, β)-cut Method

Let us consider two TrIFN $\tilde{P}^i=(1.5,2,3.5,4;0.5,2,3.5,4.5)$ and $\tilde{Q}^i=(2,3,4,5.5;1.5,3,4,6)$. Using Eq.(5.6) and Eq.(5.7), the multiplication of these two TrIFN is defined by the approximate TrIFN $\tilde{P}^i\otimes\tilde{Q}^i=(3,6,14,22;0.75,6,14,27)$ with membership and non-membership function as follows

$$\mu_{\tilde{P}^i \otimes \tilde{Q}^i}(x)=\begin{cases}\frac{x-3}{3} & \text{if } 3\le x\le 6\\ 1 & \text{if } 6\le x\le 14\\ \frac{22-x}{8} & \text{if } 14\le x\le 22\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \nu_{\tilde{P}^i \otimes \tilde{Q}^i}(x)=\begin{cases}\frac{6-x}{5.25} & \text{if } 0.75\le x\le 6\\ 0 & \text{if } 6\le x\le 14\\ \frac{x-14}{13} & \text{if } 14\le x\le 27\\ 1 & \text{otherwise.}\end{cases}$$

The corresponding left, right divergence for membership and non-membership functions are 0.25, 0.1875 and 0.5625, 0.5 respectively.

6.4 Division of Two TrIFN by (α, β)-cut Method

Let us consider two TrIFN $\tilde{P}^i=(2,5,6,8;1.5,5,6,9)$ and $\tilde{Q}^i=(1,3,4,5;0.5,3,4,6)$. Using Eq.(5.8) and Eq.(5.9), the division of TrIFN is defined by $\tilde{P}^i\div\tilde{Q}^i=(0.4,1.25,2,8;0.25,1.25,2,18)$ with membership and non-membership function as follows

$$\mu_{\tilde{P}^i \div \tilde{Q}^i}(x) = \begin{cases} \dfrac{x-0.4}{0.85} & \text{if } 0.4 \le x \le 1.25 \\ 1 & \text{if } 1.25 \le x \le 2 \\ \dfrac{8-x}{6} & \text{if } 2 \le x \le 8 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{\tilde{P}^i \div \tilde{Q}^i}(x) = \begin{cases} \dfrac{1.25-x}{1} & \text{if } 0.25 \le x \le 1.25 \\ 0 & \text{if } 1.25 \le x \le 2 \\ \dfrac{x-2}{16} & \text{if } 2 \le x \le 18 \\ 1 & \text{otherwise} \end{cases}$$

The corresponding left and right divergence for membership are 0.047368 for α=0.527864 and 1.607695 for α=0.633974 respectively. Left and right divergence for non-membership are 0.10102 for β=0.55051 and 6.723266 for β=0.710102 respectively.

7 Application of System Failure using Intuitionistic Fuzzy Number

Starting failure of an automobile depends on different facts. The facts are battery low charge, ignition failure and fuel supply failure. There are two sub-factors of each of the facts. The fault-tree of failure to start of the automobile is shown in the Fig.6.

$\tilde{F}^i_{fs}$ represents the system failure to start of automobile.

$\tilde{F}^i_{blc}$ represents the failure to start of automobile due to Battery Low Charge.

$\tilde{F}^i_{if}$ represents the failure to start of automobile due to Ignition Failure.

$\tilde{F}^i_{fsf}$ represents the failure to start of automobile due to Fuel Supply Failure.

$\tilde{F}^i_{lbf}$ represents the failure to start of automobile due to Low Battery Fluid.

$\tilde{F}^i_{bis}$ represents the failure to start of automobile due to Battery Internal Short.

$\tilde{F}^i_{whf}$ represents the failure to start of automobile due to Wire Harness Failure.

$\tilde{F}^i_{spf}$ represents the failure to start of automobile due to Spark Plug Failure.

$\tilde{F}^i_{fif}$ represents the failure to start of automobile due to Fuel Injector Failure.

$\tilde{F}^i_{fpf}$ represents the failure to start of automobile due to Fuel Pump Failure.

The intuitonistic fuzzy failure to start of an automobile can be calculated when the failures of the occurrence of basic fault events are known. Failure to start of an automobile can be evaluated by using the following steps:

Step 1.

$$\tilde{F}^i_{blc} = 1\,\Theta(1\,\Theta\,\tilde{F}^i_{lbf})(1\,\Theta\,\tilde{F}^i_{bis}) \tag{7.1}$$

$$\tilde{F}^i_{if} = 1\,\Theta(1\,\Theta\,\tilde{F}^i_{whf})(1\,\Theta\,\tilde{F}^i_{spf})$$

$$\tilde{F}^i_{fsf} = 1\,\Theta(1\,\Theta\,\tilde{F}^i_{fif})(1\,\Theta\,\tilde{F}^i_{fpf})$$

Step 2.

$$\tilde{F}^i_{fs} = 1\,\Theta(1\,\Theta\,\tilde{F}^i_{blc})(1\,\Theta\,\tilde{F}^i_{if})(1\,\Theta\,\tilde{F}^i_{fsf}) \tag{7.2}$$

Here we present numerical explanation of starting failure of the automobile using fault tree analysis with intuitionistic fuzzy failure rate. The components failure rates as TrIFN are given by

$\tilde{F}^i_{lbf} = (0.02, 0.05, 0.06, 0.07; 0.01, 0.05, 0.06, 0.08)$, $\tilde{F}^i_{bis} = (0.02, 0.03, 0.05, 0.06; 0.01, 0.03, 0.05, 0.07)$,

$\tilde{F}^i_{whf} = (0.03, 0.04, 0.05, 0.07; 0.01, 0.04, 0.05, 0.09)$, $\tilde{F}^i_{spf} = (0.02, 0.04, 0.06, 0.08; 0.01, 0.04, 0.06, 0.09)$,

$\tilde{F}^i_{fif} = (0.03, 0.05, 0.07, 0.08; 0.02, 0.05, 0.07, 0.09)$ and $\tilde{F}^i_{fpf} = (0.04, 0.06, 0.07, 0.08; 0.03, 0.06, 0.07, 0.09)$.

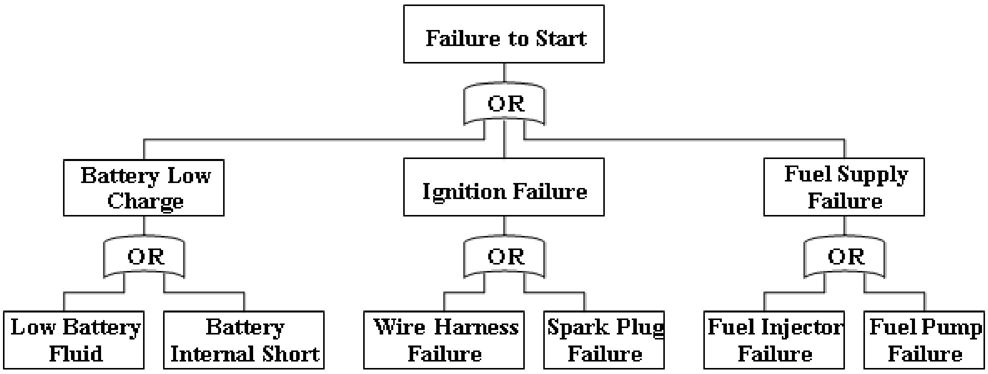


Figure 6: Fault-tree of failure to start of an automobile

So in the step1 (using Eq.7.1) the results are as follows

$$\overset{\sim i}{F}_{blc} = (0.0396, 0.0785, 0.107, 0.1258; 0.0199, 0.0785, 0.107, 0.1444),$$

$$\overset{\sim i}{F}_{if} = (0.0494, 0.0784, 0.107, 0.144; 0.0199, 0.0784, 0.107, 0.1719),$$

$$\overset{\sim i}{F}_{fsf} = (0.0688, 0.107, 0.1351, 0.1536; 0.0494, 0.107, 0.1351, 0.1719).$$

In the step 2 (using Eq.7.2), we obtain the failure to start of the automobile. The fuzzy failure to start of an automobile (Fig.6) is represented by the following TrIFN

$$\overset{\sim i}{F}_{fs} = (0.149855, 0.241615, 0.310286, 0.366922; 0.086857, 0.241615, 0.310286, 0.413272).$$

So the failure to start of the automobile is about an interval [0.241615, 0.310286] with tolerance level of acceptance is [0.149855, 0.366922] and tolerance level of rejection is [0.086857, 0.413272].

Here the left and right divergences are not significant since elements of these TrIFN are very small. Therefore, we consider TrIFN instead of trapezoidal shaped IFN to compute system failure.

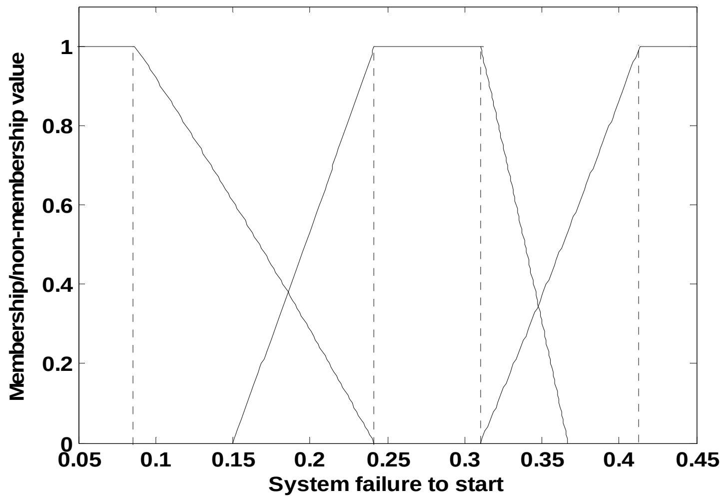


Figure 7: TrIFN representing the system failure to start of an automobile

8 Conclusion

In this paper, we have proposed a definition of IFN according to the fuzzy number presentation approach. Also arithmetic operations of proposed TrIFN are evaluated based on intuitionistic fuzzy extension principle and (α, β)-cuts method. The major advantage of using IFSs instead of fuzzy sets is that IFSs separate the positive and the negative evidence for the membership of an element in a set. We discuss the fault-tree of failure to start of an automobile with components having failure rates as trapezoidal intuitionistic fuzzy numbers. Finally, our approaches and computational procedures are efficient and simple to implement for calculation in intuitionistic fuzzy environment for all field of engineering and sciences where vagueness is occur.

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