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Optimizing Multiproduct Multiconstraint Inventory Control Systems with Stochastic Period Length and Emergency Order

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Abstract

While the usual assumptions in multiperiodic inventory control problems are that the orders are placed either at the beginning of each period (periodic review) or depending on the inventory level they can happen at any time (continuous review), in this paper we assume the periods between two replenishments of several products are identical and independent random variables. For the problem at hand, the order quantities (decision variables) are of integer-type and there are two kinds of space and service level constraints for each product. A model of the problem is first developed in which a combination of backorder and emergency orders is considered for the shortages, and the costs are shortage, holding, purchasing, and transportation. Then, we show the model is of an integer-nonlinear-programming type and to solve it, two meta-heuristic algorithms of genetic algorithm and simulated annealing are employed. At the end, a numerical example is given to demonstrate the applicability of the proposed methodology.

Keywords: inventory control, stochastic replenishment, partial back-ordering, integer nonlinear programming, simulated annealing, genetic algorithm

1 Introduction and Literature Review

While in multiperiodic inventory control models the continuous review and the periodic review are the two major policies, the underlying assumptions of the proposed models restrict their correct utilization in real-world environments. In continuous review policy, the user has the freedom to act at anytime and replenish orders based upon the available inventory level. However, in the periodic review, the user is allowed to replenish the orders only in predetermined times.

Chiang [8] considered a periodic review model with discounts, in which the period was partly long and the costs were purchasing, holding, and fixed ordering. The important aspect of his study was to introduce emergency orders to prevent shortages. Bylka [5] investigated a model constrained on the amounts of orders and backorders, in which the lead-time was constant and demand was stochastic. By analyzing the changes in the lead-time and the ordering cost, they tried to derive the optimized ordering time. Ertogral and Rahim [10] analyzed a multiperiod inventory problem with independently and identically distributed (i.i.d.) replenishment intervals. In this closer to reality scheme, a supplier visits a retailer with random inter-arrival times and the retailer replenishes his inventories based on a replenish-up-to-level inventory control policy. They also assumed that only a certain fraction of the unmet demand was backordered and the rest was lost. In this setting under general distribution between replenishment epochs, they showed the concavity of the expected profit function and gave the condition that would hold for the optimal replenish-up-to-level. Later, Chiang [9] extended a periodic review inventory system in which the period length was not constant and followed a probability distribution. They assumed the supplier's visit-intervals were i.i.d. random variables.

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The periodic inventory control problems were investigated in depth in different research works. Chew at al. [7] extended a periodic review policy for a perishable product with a predetermined lifetime in which the price and the inventory allocation were jointly determined. They assumed a price-sensitive demand where the price would increase as the time goes on. Lee and Schwarz [20] developed a periodic inventory control system with one product and stochastic lead-time. They assumed the order would be delivered immediately or one period later.

Recently, Annadurai and Uthayakumar [4] considered a periodic review system in which the decision variables were the period length, the maximum inventory level, and the fraction of the shortage that was lost. The aim of their research was to minimize the lost-sale rate. Further, Yun and Xiaobo [31] developed a periodic review inventory system serving multiple demand classes that were differentiated by a treatment for shortages. They assumed shortages of some classes were treated according to lost sales. The aim of their research was to determine both the inventory replenishment decisions and the allocation for all classes. Chen and Yang [6] considered a periodic-review single-product inventory system with stochastic demand in which the pricing and the ordering decisions were made simultaneously over a finite horizon. Ray et al. [24] studied two periodic review inventory models that differed in terms of how backordering cost was charged. In the first model, the backordering cost was charged per unit backordered and was independent of the length of time. In the second model however, the backordering cost was charged based on the number of backorders that were time-dependent. Teunter et al. [30] proposed a new method for determining order-up-to levels of items in a periodic review system in which the lead-time demands followed a compound binomial pattern. Assuming the orders to be placed either none or at least as much as a minimum order quantity, Kiesmuller et al. [18] proposed a periodic review single-item single-stage inventory system with stochastic demand. Silver and Bischak [26] extended the periodic review base stock system under exact fill rate and normally stochastic demand. Taleizadeh et al. [27] developed a multiproduct multiconstraint stochastic inventory control model in which the period length was a random variable. Later, Taleizadeh et al. [28] extended their previous work by considering fuzzy cost factor and then fuzzy stochastic period length [29].

This paper provides an extension of a model of inventory control introduced in Ertogral and Rahim [10] to multiple inventory items, integral decision variables, multiple constraints, and emergency orders. As the derived optimization models are highly nonlinear, we propose to use either simulated annealing or genetic algorithms to solve for a near optimal solution.

Four main specifications of the proposed model of this research that makes it closer to real-life inventory environments and have led to its novelty are 1) to model both multiproducts and multiconstraints problems, 2) the fact that the decision variables are integer, 3) the existence of emergency orders, and 4) incorporating the transportation cost. By deploying these conditions simultaneously, the developed model is different from other models in the periodic review literature. Further, the model is helpful in situations in which a supplier visits a retailer with random inter-arrival times and the retailer replenishes his inventories based on a replenish-up-to-level inventory control policy. Moreover, due to some limitations on the production capacity, the supply of the raw materials, and the like, the period length may be uncertain and the goods may not be delivered on time. As an example, when demand increases and the production capacity is limited, in case of breakdowns or late receipts of imported raw materials (as they are delayed at customs) the lead-time and hence the cycle length are increased. The stochastic nature of these factors causes the period length to be stochastic.

The rest of the paper is organized as follows. In Section 2, the problem along with its assumptions is defined. In Section 3, we model the defined problem of Section 2. To do this we first introduce the parameters and the variables of the problem. Then, a single product problem is modeled, and finally the multiproduct problem is formulated. In the fourth section of the paper, we explain the ways to solve the model at hand and analyze it under special conditions. Incorporating a numerical example, the solution methods are investigated in Section 5. The conclusion and recommendations for future research come in Section 6.

2 Problem Definition

Consider a periodic inventory control problem for one provider in which the times required to order several products are stochastic in nature. Let the time-periods between two replenishments of the products be identical and independent random variables. The demands of the products are constants and in case of shortage, a fraction is considered backorder and a fraction lost-sale. In this case, the percentage of the number of customers that wait to receive their orders is known. To avoid lost sales, emergency orders can be placed. While the purchasing cost per unit of an emergency order is more than the purchasing cost per unit of an ordinary order, it is less than the shortage cost per unit of lost sales. For ordinary orders, the system costs are holding, backorder, and purchase. However, a transportation cost is added to the purchasing cost of an emergency order. Further, there is no difference in the selling price per unit of both ordinary and emergency orders and the lead-times of the emergency orders are zero. While all the purchased products are sold, the warehouse space and the service level of each product are considered constraints of the problem. Moreover, the decision variables are integers. We need to identify the inventory levels in each cycle such that the expected profit is maximized.

3 Modeling

For the problem at hand, since the time-periods between two replenishments are independent random variables, in order to maximize the expected profit of the planning horizon we need to consider only one period. Furthermore, since we assumed the costs associated with the inventory control system are holding and shortage (including emergency order and transportation costs), we need to calculate the expected inventory level and the expected required storage space in each period. Before modeling, we first define the parameters and the variables of the model.

3.1 The Parameters and the Variables of the Model

For $i = 1, 2, ..., n$, define the parameters and the variables of the model as

r_i : The maximum inventory level of the i^{th} product

T_i : A random variable denoting the time-period between two replenishments (cycle length) of the i^{th} product

$f_{T_i}(t_i)$: The Probability density function of T_i

h_i : The holding cost per unit inventory of the i^{th} product in each period

π_i : The backorder cost per unit demand of the i^{th} product

c_i : The purchasing cost per unit of the i^{th} product

c_i^E : The purchasing cost per unit of an emergency order of the i^{th} product

k_i : The transportation cost per unit of an order of the i^{th} product

v_i : The selling price per unit of the i^{th} product

d_i : The constant demand rate of the i^{th} product

t_{D_i} : The time at which the inventory level of the i^{th} product reaches zero

α_i : The percentage of unsatisfied demands of the i^{th} product that is backordered

I_i : The expected amount of the i^{th} product inventory per cycle

l_i : The expected amount of the i^{th} product lost-sale in each cycle

b_i : The expected amount of the i^{th} product backorder in each cycle

q_i : The expected amount of the i^{th} product order in each cycle

q_i^E : The expected amount of the i^{th} product emergency order in each cycle ($q_i^E = L_i$)

λ_i : The lower limit of the service level for the i^{th} product

f_i : The required warehouse space per unit of the i^{th} product

F : Total available warehouse space

m : Number of shipments

A : The constant cost of each shipment

CE_i : The expected purchase cost of an emergency order of the i^{th} product

CH_i : The expected holding cost per cycle of the i^{th} product

CB_i : The expected shortage cost in backorder state of the i^{th} product

CL_i : The expected shortage cost in lost-sale state of the i^{th} product

CP_i : The expected purchase cost of ordinary orders of the i^{th} product

CT_i : The expected transportation cost of the i^{th} product

ER : The expected revenue obtained from sales

Z : The expected profit obtained in each cycle

For sake of simplicity, in Section 3.3 we first consider a single-product problem. Then, we extend the modeling to the multi-product modeling in Section 3.4. However, we first introduce the pictorial representation of the single-product problem in Section 3.2.

3.2 Inventory Diagram

According to Ertogral and Rahim [10] and considering the fact that the time-periods between replenishments are stochastic variables, two cases may occur. In the first case, the time-period between replenishments is less than the amount of time required for the inventory level to reach zero (see Figure 1). In the second case, it is greater (see Figure 2). Figure 3 depicts the shortages in both cases.

3.3 Deriving the Costs and the Profit of a Single-Product Model

In order to obtain the expected profit in each cycle, we need to evaluate all of the terms in Equation (1) [10]:

$$Z_i = ER_i - CP_i - CH_i - CB_i - CL_i = v_i q_i - c_i q_i - h_i I_i - \pi_i b_i - (v_i - c_i) l_i \,. \tag{1}$$

Based on Figure 3, the random lost sale quantity is $(1-\alpha_i)(d_i T_i - r_i)$, where its expected value is obtained using Equation (2)

$$l_i = (1-\alpha_i) \int_{t_{D_i}}^{T_{Max_i}} (d_i T_i - r_i) f_{T_i}(t_i) dt_i \qquad t_{D_i} < T_i \le T_{Max_i} \,. \tag{2}$$

Further, the random backordered quantity is $\alpha_i (d_i T_i - r_i)$ with an expected value of

$$b_i = \alpha_i \int_{t_{D_i}}^{T_{Max_i}} (d_i T_i - r_i) f_{T_i}(t_i) dt_i \qquad t_{D_i} < T_i \le T_{Max_i} \,. \tag{3}$$

Moreover, since the random inventory when the inventory level is positive is $r_i T_i - d_i T_i^2 / 2$, and $r_i^2 / (2d_i)$ during stock out periods, the expected inventory level is obtained using Equation (4)

$$I_i = \int_{T_{Min_i}}^{t_{D_i}} \left(r_i T_i - \frac{d_i T_i^2}{2} \right) f_{T_i}(t_i) dt_i + \int_{t_{D_i}}^{T_{Max_i}} \left(\frac{r_i^2}{2d_i} \right) f_{T_i}(t_i) dt_i \,. \tag{4}$$

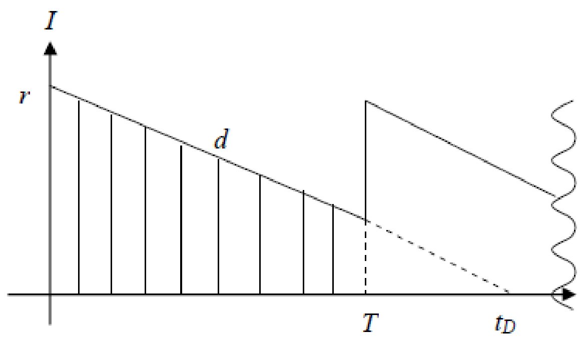

Figure 1: Presenting the inventory cycle when $T_{Min} \le T \le t_D$

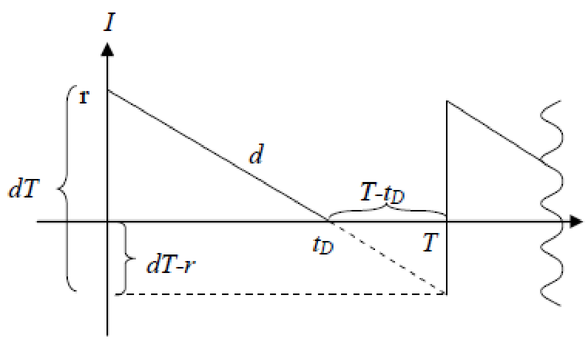

Figure 2: Presenting the inventory cycle when $t_D < T \le T_{Max}$

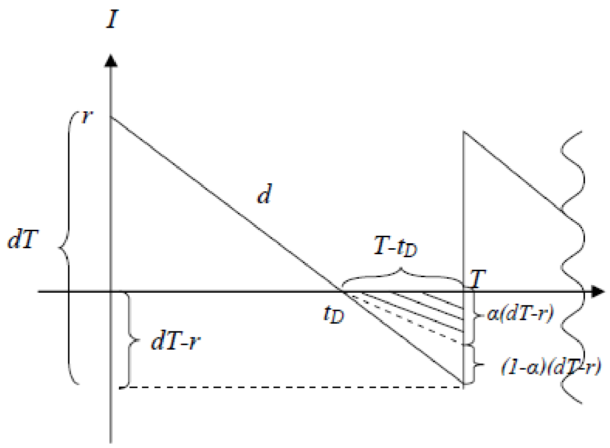

Figure 3: Presenting shortages in two cases of compact back order and lost sales

In order to calculate the expected order quantity, one needs to determine the required quantities during both stock and stock-out periods. Since they are respectively d_iT_i and $r_i+\alpha_i\left(d_iT_i-r_i\right)$, the expected order quantity will be

$$q_i=\int_{T_{Min_i}}^{t_{D_i}}\left(d_iT_i\right)f_{T_i}(t_i)dt_i+\int_{t_{D_i}}^{T_{Max_i}}\left(r_i+\alpha_i\left(d_iT_i-r_i\right)\right)f_{T_i}(t_i)dt_i. \tag{5}$$

In this paper, to avoid lost sales of Equation (2) we put an emergency order of size $q_i^E=l_i$. Since the purchasing cost per unit of an emergency order is c_i^E and that the selling price per unit is P_i, the cost of an emergency order is calculated as:

$$CE_i=c_i^Eq_i^E-P_iq_i^E=(c_i^E-v_i)\left[\left(1-\alpha_i\right)\int_{t_{D_i}}^{T_{Max_i}}(d_iT_i-r_i)f_{T_i}(t_i)dt_i\right] \tag{6}$$

where $c_i^E>v_i$.

The transportation cost is calculated based on Equation (7), in which f_iq_i is the required space to ship the order from the supplier.

$$CT_i=\begin{cases}A+k_iq_i & 0<f_iq_i\le\hat{f}\\ 2A+k_iq_i & \hat{f}<f_iq_i\le2\hat{f}\\ \vdots & \vdots\\ mA+k_iq_i & (m-1)\hat{f}<f_iq_i\le m\hat{f}\end{cases} \tag{7}$$

By introducing binary variables $Y_j,\ j=1,2,...,m,$ the transportation cost can be incorporated with the mathematical model of the problem as

$$\begin{gathered}CT_i=k_iq_i+\sum_{j=1}^{m}jAY_j\\ 0<f_iq_i\le\hat{f}Y_1\\ \hat{f}Y_2<f_iq_i\le2\hat{f}Y_2\\ \vdots\\ (m-1)\hat{f}Y_m<f_iq_i\le m\hat{f}Y_m\\ Y_1+Y_2+\cdots+Y_m=1\\ Y_j=0,1.\end{gathered} \tag{8}$$

As the total available warehouse space is F, the space required for each unit of product i is f_i, and the upper limit for the inventory is r_i, the space constraint will be

$$f_ir_i\le F. \tag{9}$$

Since the shortages only occur when the cycle time is more than t_D and that the lower limit for the service level is λ, then

$$P(T_i > t_{D_i}) = \int_{\frac{r_i}{d_i}}^{T_{Max_i}} f_{T_i}(t_i)dt_i \le 1-\lambda_i. \tag{10}$$

In short, the complete mathematical model of the single product inventory is

$$\begin{aligned}
Max\ Z &= ER_i - CP_i - CH_i - CB_i - CE_i - CT_i \\
&= (v_i - c_i)\left[\int_{T_{Min_i}}^{\frac{r_i}{d_i}} \left(d_i T_i\right) f_{T_i}(t_i)dt_i + \int_{\frac{r_i}{d_i}}^{T_{Max_i}} \left(r_i + \beta_i(d_i T_i - r_i)\right) f_{T_i}(t_i)dt_i\right] \\
&\quad -\pi_i \alpha_i \left[\int_{t_{D_i}}^{T_{Max_i}} (d_i T_i - r_i) f_{T_i}(t_i)dt_i\right] - (c_i^E - v_i)\left[\left(1-\alpha_i\right)\int_{t_{D_i}}^{T_{Max_i}} (d_i T_i - r_i) f_{T_i}(t_i)dt_i\right] \\
&\quad -k_i\left[\int_{T_{Mini}}^{\frac{r_i}{d_i}} \left(d_i T_i\right) f_{T_i}(t_i)dt_i + \int_{\frac{r_i}{d_i}}^{T_{Max_i}} \left(r_i + \alpha_i(d_i T_i - r_i)\right) f_{T_i}(t_i)dt_i\right] - \sum_{j=1}^{m} jAY_j \\
s.t.\quad & f_i r_i \le F \\
& \int_{\frac{r_i}{d_i}}^{T_{Max_i}} f_{T_i}(t_i)dt_i \le 1-\lambda_i \\
& 0 < f_i q_i \le \hat{f}Y_1 \\
& \hat{f}Y_2 < f_i q_i \le 2\,\hat{f}Y_2 \\
& \vdots \\
& (m-1)\,\hat{f}Y_m < f_i q_i \le m\hat{f}Y_m \\
& Y_1 + Y_2 + \cdots + Y_m = 1 \\
& Y_j = 0,1 \\
& r \ge 0, Integer.
\end{aligned} \tag{11}$$

3.4 The Multiproduct Model

The single-product inventory model of Section 3.3 can be easily extended to a multiple-product model as follows:

$$\begin{aligned}
Max\ Z = &\sum_{i=1}^{n}[(v_i - c_i)q_i - h_i I_i - \pi_i b_i - (c_i^E - v_i)q_i^E - k_i q_i] - \sum_{j=1}^{m} jAY_j \\
s.t.\quad &\sum_{i=1}^{n} f_i r_i \le F \\
&0 < \sum_{i=1}^{n} f_i q_i \le \hat{f}Y_1 \\
&(j-1)\,\hat{f}Y_j < \sum_{i=1}^{n} f_i q_i \le j\hat{f}Y_j \qquad \forall\ j = 2,3,\ldots,m \\
&P(T_i > t_{D_i}) \le 1-\lambda_i \qquad \forall\ i = 1,2,\ldots,n \\
&\sum_{j=1}^{m} Y_j = 1 \\
&Y_j = 0,1 \qquad \forall\ j = 1,2,\ldots,m \\
&r_i \ge 0, Integer \qquad \forall\ i = 1,2,\ldots,n.
\end{aligned} \tag{12}$$

In what follows, we consider two probability density functions for T_i and hence we develop two models.

3.4.1 T_i Follows a Uniform Distribution

In this case the probability density function of T_i is $f_{T_i}(t_i) = 1/(t_{\max_i} - t_{\min_i})$. Accordingly, (12) will change to

$$Max\ \ Z=\sum_{i=1}^{n}\left[\frac{h_i}{6d_i^2(t_{\max_i}-t_{\min_i})}\right]r_i^3-\sum_{i=1}^{n}\left[\frac{(c_i^E-c_i-k_i)(1-\alpha_i)+\pi_i\alpha_i+h_it_{\max_i}}{2d_i(t_{\max_i}-t_{\min_i})}\right]r_i^2$$
$$+\sum_{i=1}^{n}\left[\frac{2(c_i^E-c_i-k_i)(1-\alpha_i)t_{\max_i}+h_it_{\min_i}^2+2\pi_i\alpha_it_{\max_i}}{2\left(t_{\max_i}-t_{\min_i}\right)}\right]r_i$$
$$+\sum_{i=1}^{n}\left[\frac{-h_it_{\min_i}^3d_i+3(v_i-c_i-k_i)(\alpha_it_{\max_i}^2-t_{\min_i}^2)d_i-3t_{\max_i}^2d_i(\pi_i\alpha_i+(c_i^E-v_i)(1-\alpha_i))}{6(t_{\max_i}-t_{\min_i})}\right]-\sum_{j=1}^{m}jAY_j$$

$$s.t.\ \ \sum_{i=1}^{n}f_ir_i\le F \tag{13}$$
$$\frac{d_it_{Max_i}-r_i}{d_i(t_{Max_i}-t_{Min_i})}\le 1-\lambda_i \qquad \forall\ i=1,2,\ldots,n$$
$$0<\sum_{i=1}^{n}f_i\left(\frac{(\alpha_i-1)r_i^2+(2d_it_{Max_i}(1-\alpha_i))r_i+(\alpha_it_{Max_i}^2-t_{Min_i}^2)d_i^2}{2d_i(t_{Max_i}-t_{Min_i})}\right)\le \hat{f}Y_1$$
$$(j-1)\hat{f}Y_j<\sum_{i=1}^{n}f_i\left(\frac{(\alpha_i-1)r_i^2+(2d_it_{Max_i}(1-\alpha_i))r_i+(\alpha_it_{Max_i}^2-t_{Min_i}^2)d_i^2}{2d_i(t_{Max_i}-t_{Min_i})}\right)\le j\hat{f}Y_j\ ,\ \forall\ j=2,3,\ldots,m$$
$$\sum_{j=1}^{m}Y_j=1$$
$$Y_j=0,1 \qquad \forall\ j=1,2,\ldots,m$$
$$r_i\ge 0, Integer \qquad \forall\ i=1,2,\ldots,n.$$

3.4.2 T_i Follows an Exponential Distribution

If T_i follows an exponential distribution with parameter β_i , then the probability density function of T_i will be $f(t_i)=\beta_ie^{-\beta_it_i}$. In this case, the model becomes:

$$Max\ Z=\sum_{i=1}^{n}\left[\frac{1}{\beta_i}[d_i(1-\alpha_i)(c_i+k_i-c_i^E)-\pi_i\alpha_id_i]e^{-\left(\frac{r_i}{d_i}\right)\beta_i}+\frac{1}{\beta_i}\left[d_i(v_i-c_i-k_i)-h_ir_i\right]+\frac{h_id_i}{\beta_i^2}(1-e^{-\left(\frac{r_i}{d_i}\right)\beta_i})\right]$$

$$s.t.\ \ \sum_{i=1}^{n}f_ir_i\le F$$
$$e^{-\left(\frac{r_i}{d_i}\right)\beta_i}\le 1-\lambda_i \qquad \forall\ i=1,2,\ldots,n$$
$$0<\sum_{i=1}^{n}f_i\left(\frac{d_i}{\lambda_i}((\alpha_i-1)e^{-\left(\frac{r_i}{d_i}\right)\beta_i}+1)\right)\le \hat{f}Y_1 \tag{14}$$
$$(j-1)\hat{f}Y_j<\sum_{i=1}^{n}f_i\left(\frac{d_i}{\lambda_i}((\alpha_i-1)e^{-\left(\frac{r_i}{d_i}\right)\beta_i}+1)\right)\le j\hat{f}Y_j \qquad \forall\ j=2,3,\ldots,m$$
$$\sum_{j=1}^{m}Y_j=1$$
$$Y_j=0,1 \qquad \forall\ j=1,2,\ldots,m$$
$$r_i\ge 0, Integer \qquad \forall\ i=1,2,\ldots,n.$$

In the next section, we will introduce two meta-heuristic algorithms to solve the problem.

4 The Solution Algorithms

Since the models in (13) and (14) are integer-nonlinear in nature, reaching an analytical solution (if any) is difficult [11]. Many researchers have used meta-heuristic algorithms to solve complicated optimization problems in many fields of scientific and engineering disciplines. Some of these meta-heuristic algorithms are simulating annealing [27], Tabu search [15], genetic algorithms [22, 25], particle swarm optimization [23, 13 and 16], neural networks [2], harmony search [14, 17] and ant colony [3]. As a result, in this section two iterative-search algorithm of simulated annealing and population-base genetic algorithm are employed to solve the models.

4.1 Simulated Annealing

To solve complex optimization problems, Aarts and Korst [1] proposed a local search algorithm named simulated annealing (SA) that was inspired by physical annealing processes. SA is an efficient and effective method that produces good suboptimal solutions and has been used in many combinatorial optimization problems of different areas of sciences [19]. An SA algorithm follows search directions that improve the objective function value. While exploring solution space, SA offers the possibility of accepting worse neighbor solutions in a controlled manner in order to escape from local minima. The main steps in a SA algorithm are: (1) generating neighbor, (2) evaluating the objective function, (3) assigning an initial temperature, (4) changing the temperature, (5) cooling scheme, and (6) stopping.

The neighbor generation is an important component of SA. In this paper, the initial solutions are generated in two different ways. In the first way, they are randomly selected among a feasible solution space and in the second, they are generated using the best solutions obtained by the genetic algorithm described in Section 4.2.

When a solution is generated, it should be evaluated by its objective function value. In the maximization models of this research, if the objective function of the new solution (*j*) becomes bigger than the objective function of the previous solution (*i*), then (*i*) will be replaced by (*j*). Otherwise, by generating a random number the better solution is selected.

One of the important parameters of the SA algorithm is its initial temperature. The initial temperature has a significant effect on the possibility of selecting a bad solution. On the one hand, if a high value assumed for the initial temperature, a solution with a bad objective function value has a high chance of being accepted. On the other hand, low value of the initial temperature makes the probability of the solution to be a local optimum high. In this paper, different large values of 1000, 1500, and 2000 are chosen for the initial temperatures.

The range of temperature changes in a SA algorithm is also one of the primary aspects of the annealing process. In this paper, we change the temperature of the SA algorithm based on a geometric function given in Equation (15) with $\theta = 0.9$, 0.95, and 0.99.

$$T^n = \theta T^{n-1}\ ;\ n = 1, 2, \ldots \qquad 0 < \theta < 1. \tag{15}$$

Analyzing the equilibrium state after a couple of renitence in a specific temperature of a SA algorithm is important and necessary as well. This step should be performed to make sure if the annealing process needs to continue in its current temperature or it should be stopped and transferred to the next temperature. In this research, reaching to the pre-defined final temperature T_F is used as the stopping criterion. Furthermore, different values of 50, 100, and 200 are employed for $N(t)$ (number of iteration in each temperature).

In short, the steps involved in the proposed SA algorithm are shown in Figure 4.

1. *Choosing an initial solution i from the group of feasible solutions S*
2. *Choosing the initial temperature* $T_0 > 0$
3. *Selecting the number of iterations* $N(t)$ *at each temperature*
4. *Selecting the final temperature* T_F
5. *Determining the process of the temperature reduction until it reaches* T_F
6. *Setting the temperature exchange counter* n *to zero for each temperature*
7. *Creating the j solution at the neighborhood of the i solution*
8. *Evaluating the objective function at any temperature and calculate* $\Delta = z(j) - z(i)$
9. *Accepting the solution j, if* $\Delta < 0$. *Else, generating a random number* $RN \sim U[0,1]$.
   If $RN < e^{\left(\frac{-\Delta}{T_0}\right)}$ *then select solution j*
10. *Setting* $n = n+1$. *If* n *is equal to* $N(t)$ *then go to 12. Otherwise, go to 7*
11. *Reducing the temperature. If it reaches* T_F *then stop. Otherwise, go to 6*

Figure 4: The steps of the proposed SA algorithm

4.2 Genetic Algorithm

Holland [12] was the first who introduced the fundamental principal of genetic algorithms (GA). GA, as a population-based meta-heuristic algorithm, was inspired by the concept of survival of the fittest chromosomes. A chromosome is a string of genes that are considered the coded figure of a possible solution. In an optimization application of the GA, a variable is considered a gene and a solution vector containing several gens is a chromosome. In this paper, the chromosomes are strings of the inventory levels of the products (r_j).

A GA operates through a simple cycle of stages including 1) creation of a “population” of strings, 2) evaluation of the strings, 3) selection of the best strings, and 4) genetic manipulation to create new population of strings.

A group of chromosomes is called population. One of the main characteristics of a GA is working on a set of chromosomes (solutions), instead of focusing on a single solution (or one chromosome). The number of population in a generation is the population size and is denoted by N. Creation of a population is usually performed by random generation over feasible or infeasible solution spaces of the on hand problem. Moreover, some hints on choosing a proper population size are given by Man et al. [21]. In this research, the feasible solution space is considered to generate populations of different sizes of 10, 100, and 1000.

A solution is evaluated based on its objective function value. In the maximization problem of this research, the chromosomes with higher objective function values are accepted the best known. Further, the fitness proportional selection assigns a selection probability to each solution.

At the end, the creation of the new population is performed by re-combinations of two types; mutation and crossover. The probabilities of the crossover (P_c) and mutation (P_m) are the parameters of the genetic algorithms. In this research, we test the single point, the two points, and the uniform crossovers shown in Figures 5 to 7 with the crossover probabilities of 0.85, 0.90, and 0.95, where r_j shows the chromosome containing the inventory levels of the products. Further, in the mutation operation of this research, we create a random number *RN* between (0,1) for each gene. If *RN* is less than a predetermined mutation probability P_m, then the mutation, performed based on the uniform function over the specific range of the variable, occurs in the gene. Otherwise, the mutation operation is not performed in that gene. Figure 8 depicts a mutation operation in which P_m is chosen 0.1.

r_i	135	65	232	95	35	210	94	35
r'_i	215	94	210	80	59	247	85	76

r_i	135	65	232	95	35	210	85	76
r'_i	215	94	210	80	59	247	94	35

Figure 5: The single-point crossover operation with *M*=7

r_i	135	65	232	95	35	210	94	35
r'_i	105	75	210	80	59	247	85	76

r_i	135	65	210	80	59	247	94	35
r'_i	105	75	232	95	35	210	85	76

Figure 6: The two-points crossover operation with *M*=3 and 7

r_i	135	65	232	95	35	210	94	35
r'_i	215	94	210	80	59	247	85	76

r_i	135	65	210	95	59	210	94	76

Figure 7: The uniform crossover operation with *M*=3 and 7

r_i	135	65	232	95	35	210	94	90
RN	0.623	0.245	0.325	0.845	0.256	0.354	0.012	0.756

r_i	135	65	232	95	35	210	94	84

Figure 8: A sample of the mutation operation

In this paper, 0.078, 0.088, and 0.1 are employed as different values of the P_m parameter. Further, the steps of the GA used in this paper are shown in Figure 9.

1. Setting the parameters P_c , P_m and N
2. Initializing the population randomly
3. Evaluating the objective function for all chromosomes based on objective function
4. Selecting individual for mating pool
5. Applying the crossover operation for each pair of chromosomes with probability P_c
6. Applying mutation operation for each chromosome with probability P_m
7. Replacing the current population by the resulting mating pool
8. Evaluating the objective function
9. If stopping criterion is met, then stop. Otherwise, go to step 5

Figure 9: The steps of the GA

In order to demonstrate the proposed SA and GA algorithms and to evaluate their performances, in the next section we bring a numerical example used in Ertogral and Rahim [10] with some modifications. In this example, two cases of the uniform and the exponential distributions for the time-period between two replenishments are investigated.

5 Numerical Examples

Consider a multiproduct inventory control problem involving eight products and general data given in Table (1). Tables (2) and (3) show the parameters of the uniform and the exponential distributions used for the time-period between two replenishments, respectively. The total available warehouse space is 18,000 and the available space for each shipment is 5,000 with a constant cost of 500 per shipment. Tables (4) and (5) show different values of the parameters of the SA and the GA methods, respectively. In this research all the possible combinations of the parameters in SA ($N(t)$, T_0 and α) and GA (P_c, P_m and N) methods are employed and using the *max(max)* criterion the best combination of the parameters has been selected. Furthermore, the single-point crossover had better performances than both the two-points and uniform crossover operations. Table (6) shows the best result. The best combinations of the SA and the GA algorithms are shown in Tables (7) and (8), respectively. Moreover, the convergence paths of the objective function values of the SA algorithm in uniform and exponential distributions are shown in Figures 10 and 11. These graphs for the GA method are shown in Figures 12 and 13. From the results, we see that the best solution of the GA method is better than the one obtained by the SA algorithm.

Table 1: General data

Product	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
h_i	2	2	2	2	2	2	2	2
π_i	5	5	5	5	5	5	5	5
v_i	100	100	100	100	150	150	150	150
c_i	65	65	65	65	70	70	70	70
c_i^E	105	105	105	105	155	155	155	155
k_i	3	3	3	3	3	3	3	3
f_i	3	3	3	3	6	6	6	6
d_i	10	10	10	10	10	10	10	10
λ_i	0.5	0.6	0.6	0.5	0.5	0.6	0.6	0.5
α_i	0.5	0.9	0.9	0.5	0.5	0.9	0.9	0.5

Table 2: Data for uniform distribution

Product	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
t_{Min_i}	20	20	50	50	20	20	50	50
t_{Max_i}	40	40	70	70	40	40	70	70

Table 3: Data for exponential distribution

Product	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
β_i	1/30	1/30	1/60	1/60	1/30	1/30	1/60	1/60

Table 4: The parameters of the SA algorithm

$N(t)$	T_0	θ
50	1000	0.9
100	1500	0.95
200	2000	0.99

Table 5: The parameters of the GA method

P_c	P_m	N
0.85	0.078	10
0.90	0.088	100
0.95	0.1	1000

Table 6: The best result for r_i

Distribution	Approach	Product								Z
		1	2	3	4	5	6	7	8	
Uniform	GA	301	321	621	601	300	320	621	610	4243
	SA	301	326	628	600	301	324	625	604	2307.7
Exponential	GA	209	276	550	417	208	275	550	417	65760
	SA	213	275	551	421	212	280	552	417	65123

Table 7: The best combination of the SA parameters

Numerical Example with	$N(t)$	T_0	θ
Uniform Distribution	200	2000	0.95
Exponential Distribution	200	1000	0.99

Table 8: The best combination of the GA parameters

Numerical Example with	P_c	P_m	N
Uniform Distribution	0.9	0.078	1000
Exponential Distribution	0.9	0.1	1000

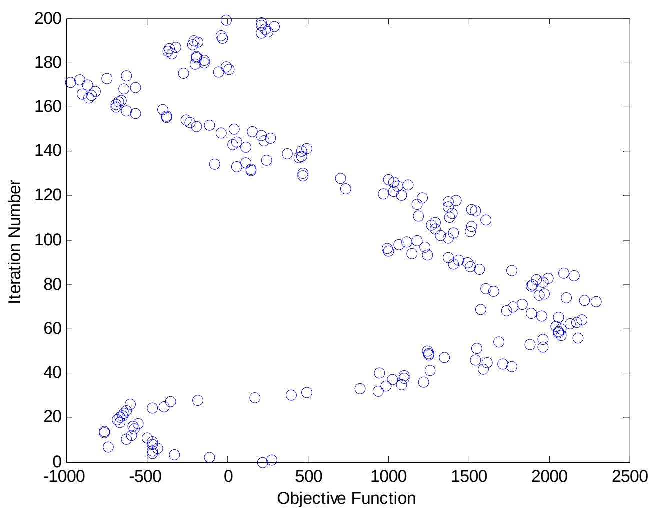

Figure 10: The convergence path of the best result in uniform example of SA

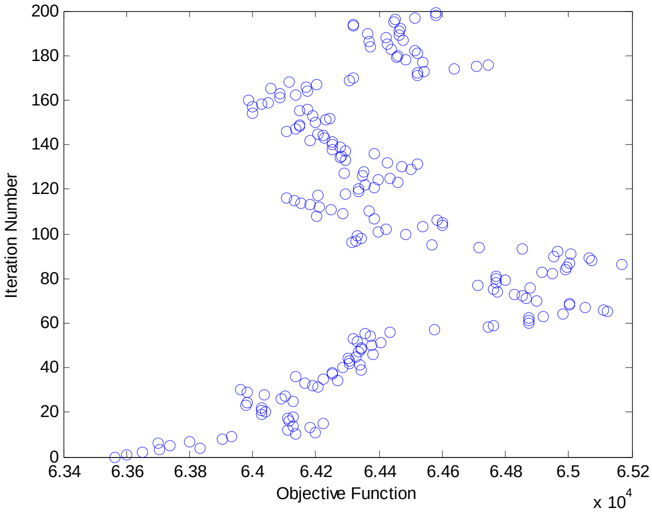

Figure 11: The Convergence path of the best result in exponential example of SA

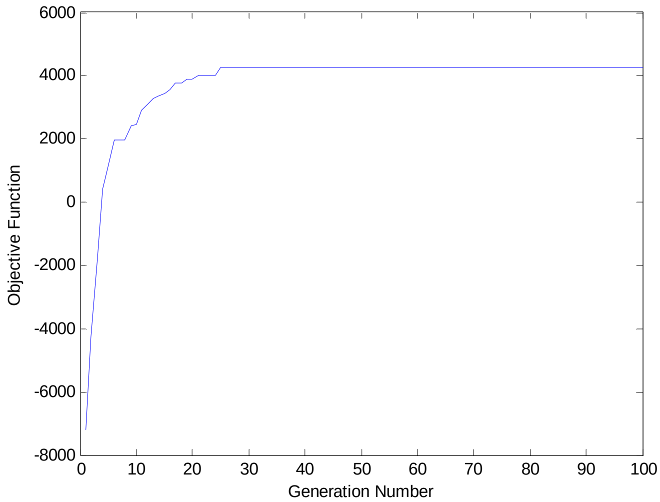

Figure 12: The convergence paths of the best result in uniform example of GA

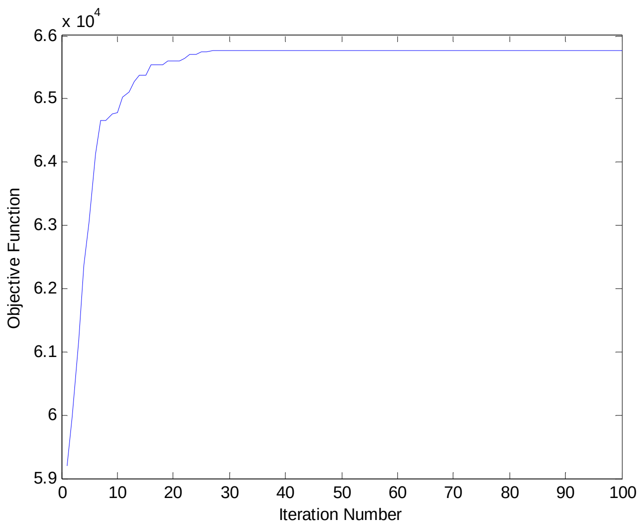

Figure 13: The convergence path of the best result in exponential example of GA

6 Conclusion and Recommendation for Future Research

In this paper, a stochastic replenishment multiproduct inventory model with partial backordering and emergency order under the service level and space constraints was investigated. Two mathematical modeling for two cases of uniform and exponential distribution of the time between two replenishments have been developed and shown to be integer-nonlinear programming. Then, two meta-heuristic solution algorithms of SA and GA were proposed to solve the models. Finally, based upon the results of two numerical examples it was shown that the best solution of the GA algorithm was better than the one in the SA algorithm.

Fuzzy parameters, some other probability distribution functions for the period length, and deterioration rate for the stock inventory can be considered in future works.

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