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# Scrutiny Malmquist Productivity Index on Fuzzy Data by Credibility Theory with an Application to Social Security Organizations

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**Abstract**

The aim of this paper is to estimate the productivity change under fuzzy environment based on the credibility theory. Although productivity change has been widely studied, it has been neglected under uncertain environments and, particularly, fuzzy environments. Malmquist productivity index (MPI), which is one of the sub-branches of data envelopment analysis (DEA), provides a measure of productivity change over time. In this paper, the fuzzy DEA models were first defuzzified by applying the credibility theory. The obtained results were used for calculating MPI on fuzzy data. In a special case, when fuzzy numbers were considered to be in a trapezoidal form and a credibility level was available, the obtained models were transformed to linear programs. The advantages of the proposed method compared to the methods in the literature are discussed by a numerical example. One of the advantages of the proposed method is that it has no ambiguity in estimating the MPI and, also, the results obtained through this method support the imprecise nature of the data. Finally, an empirical example on social security institutions in Iran illustrates the performance as well as the effectiveness of the presented method.


**Keywords:** data envelopment analysis (DEA), malmquist productivity index (MPI), credibility theory, fuzzy number

## 1 Introduction

Productivity change is one of the major sources of economic development and a thorough understanding of the factors affecting productivity is very important. Malmquist Productivity Index (MPI), which is used for calculating the productivity change, is one of the sub branches of Data Envelopment Analysis (DEA) and was first introduced by Fare et al. [3] in 1992 under the title of FGLR. Fare et al. [4] extended the method suggested in [3] for determining the productivity growth in industrial countries.

Assume that there are $n$ decision making units (DMUs) to be evaluated, each DMU with $m$ inputs and $s$ outputs. We denote by $x_{ij}$ $(i = 1,...,m)$ and $y_{rj}$ $(r = 1,...,s)$ the values of inputs and outputs of $DMU_j$ $(j = 1,...,n)$ , which are all known and positive. In a famous model of DEA which is known as CCR model and was proposed by Charnes et al. [1], the efficiency of an evaluating DMU is measured as follows:

$$\begin{cases} \theta_o = \min & \theta \\ \quad\quad \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \quad i = 1,...,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1,...,s \\ & \lambda_j \geq 0, \quad j = 1,....,n. \end{cases} \tag{1}$$

Let the data be available in two periods of $a$ and $b$, where $x^a_{ij}$ $(i = 1,...,m)$ and $y^a_{rj}$ $(r = 1,...,s)$ are respectively the inputs and outputs of $DMU_j$ $(j = 1,...,n)$ in period $a$, and $x^b_{ij}$ $(i = 1,...,m)$ and $y^b_{rj}$ $(r = 1,...,s)$ are that of $DMU_j$ $(j = 1,...,n)$ in period $b$. MPI based on a definition from Fare et al. [4] measures the productivity change of $DMU_o$ in relation to the periods $a$ and $b$ through the following formula:

$$\text{MPI}_\text{o} \equiv (\frac{\theta^{a,b}_o}{\theta^{a,a}_o} \times \frac{\theta^{b,b}_o}{\theta^{b,a}_o})^{\frac{1}{2}} \tag{2}$$

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where $\theta_o^{a,b}$ is the efficiency of $DMU_o$ in period $b$ over period $a$ and is measured by:

$$\begin{cases} \theta_o^{a,b} = \min & \theta \\ \quad\quad \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij}^a \le \theta x_{io}^b, \quad i = 1, ..., m \\ & \sum_{j=1}^{n} \lambda_j y_{rj}^a \ge y_{ro}^b, \quad r = 1, ..., s \\ & \lambda_j \ge 0, \quad j = 1, ..., n. \end{cases} \tag{3}$$

Based on the definition of Fare [4], we have:

**Definition 1** *1. $MPI_o > 1$, increase productivity and observe progress.*
*2. $MPI_o < 1$, decrease productivity and observe regress.*
*3. $MPI_o = 1$, no change in productivity at time $a$ in comparison to $b$.*

After the presentation of DEA as a method for evaluation the performance of DMUs, variety of its sub branches, such as ranking, return to scale, resource allocation, MPI, were formed. Using the fuzzy data is proposed in almost all the mentioned subcategories. However, MPI with fuzzy data has not received much attention. The studies performed by Jahanshahloo et al. [6, 7] and Hatami-Marbini et al. [5] are the examples of the researches conducted on the fuzzy MPI. As a novel research, Emrouznejad et al. [2] have provided a method to overall profit MPI under fuzzy and interval environments. Based on this method, the presented paper studies MPI with fuzzy data.

One of the topics of interest to fuzzy researchers is selecting an appropriate method for solving a fuzzy programming problem. There are mainly three basic methods for solving a fuzzy model. One of them is $\alpha$-cut method [13] which converts fuzzy data to interval data and then the fuzzy model is transformed to an interval model. The obtained interval model is converted to two crisp models to solve. Interval numbers are obtained as a result of solving the corresponding models. The main disadvantage of this method can be considered as the computational burdensome, because this method provides two crisp problems corresponding to each $\alpha$-cut which must be solved. Since the obtained results of using $\alpha$-cut method are interval data, decision making based on them is misleading. This is another imperfection of the $\alpha$-cut method. Another method for dealing with fuzzy programming is the ranking function method where fuzzy constraints are converted to crisp constraints base on the ranking functions. A crisp model to solve is then gained. Although, the ranking function method was welcomed by many researchers, the results of solving the corresponding model do not protect the imprecise nature of the data. In other word, the data are fuzzy numbers, while the results are crisp numbers. The third method which was recently proposed for solving fuzzy models is using the measure theory. Two most popular introduced measures for solving fuzzy models are possibility and credibility measures. In the possibility theory, which was suggested by Zadeh [14], the results obtained from solving the corresponding model with each $\alpha$ possibility level are crisp numbers and, consequently, decision making in each level of $\alpha$ is done easily. Different results are usually gained from different possibility levels; therefore, results are imprecise which have consistency with the imprecise nature of the data. Although the possibility theory has been widely used for solving fuzzy models and overcomes the problems of using $\alpha$-cut and ranking function methods, it does not have the self-duality property. This means that the optimal solution of a fuzzy model by applying the possibility theory has no relation to the optimal solution of its dual. For more details, see [8]. The credibility theory has all the advantages of the possibility theory and is self-dual, as well. Liu and Liu [8, 10] introduced the concept of the credibility theory in order to present a self-dual measure. Liu [9] redefined this notion and revised some of its basic tools. Afterwards, this theory has been applied for evaluating and understanding lots of fuzzy phenomena. Wen and Li [11] have used this theory in order to solve the DEA model with fuzzy data. One year later, Wen et al. [12] provided a hybrid method for solving the fuzzy DEA model by applying the credibility theory. Based on these, We suggest the credibility theory for solving the fuzzy DEA models which are related to the estimation of MPI. An example is provided to compare the proposed method with the previous method for estimating the fuzzy MPI. An example of 19 Social Securities in Iran is also provided in order to state the ability of the proposed method.

The rest of this paper is organized as follows. The preliminaries of the credibility theory are listed in Section 2. Problem formulation is presented in Section 3. In this section, MPI with fuzzy numbers is calculated by the credibility theory. In special case, when data are considered as trapezoidal or triangular fuzzy numbers, the proposed models are transformed into parametric linear programs. In Section 4, two examples are provided, one to illustrate the proposed method and compare it with the methods in the literature, and the other is the empirical results of estimating MPI on 19 Social Securities in Iran. Summary and conclusions are given in the final section.

## 2 Credibility Theory

The $\text{Cr}\{A\}$, i.e., the credibility that $A$ will occur, is satisfied in the following principles. Let $\Theta$ be a nonempty set and $\mathcal{P}$ its power set, then
**1)** (Normality) $\text{Cr}\{\Theta\} = 1$.
**2)** (Monotonicity) $\text{Cr}\{A\} \leq \text{Cr}\{B\}$ whenever $A \subset B$.
**3)** (Self-Duality) $\text{Cr}\{A\}+\text{Cr}\{A^c\} = 1$ for any event $A$.
**4)** (Maximality) $\text{Cr}\{\bigcup_i A_i\} = \sup_i \text{Cr}\{A_i\}$ for any events $\{A_i\}$ with $\sup_i \text{Cr}\{A_i\} < 0.5$.
By considering the paper of Liu and Liu [9], we have the following definition:

**Definition 2** *The set function* Cr *is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms.*

Consider following fuzzy model as:

$$\begin{cases} \max & f(x,\xi) \\ \text{s.t.} & g_j(x,\xi) \leq 0, \quad j = 1,...,p \end{cases} \tag{4}$$

where $x$ is a decision vector and $\xi$ is a fuzzy vector, $f(x,\xi)$ is the objective function and $g_j(x,\xi)$, $(j = 1,\ldots,p)$ are the constraint functions in fuzzy programming (4). Since the objective function is subject to uncertainty, it can be expressed as max $\overline{f}$ and the constraint $f(x,\xi) \geq \overline{f}$ is added to the constraints. Therefore, the fuzzy model of (4) is expressed as follows:

$$\begin{cases} \max & \overline{f} \\ \text{s.t.} & f(x,\xi) \geq \overline{f} \\ & g_j(x,\xi) \leq 0, \quad j = 1,...,p. \end{cases} \tag{5}$$

Since the constraints of (5) does not produce a non-fuzzy feasible set, hence the confidence level $\alpha$ is determined as if the required fuzzy constraints be in it. Therefore, we have the following credibility constraints as:
$$\text{Cr}\{g_j(x,\xi) \leq 0,\} \geq \alpha, \; j = 1,\ldots,p.$$
Similarly, constraint f$(x,\xi) \geq \overline{f}$ can be converted to a credibility constraint as $\text{Cr}\{f(x,\xi) \geq \overline{f}\} \geq \alpha$.
Model (5) is divided into the two following models by implementing the credibility theory as:

$$\begin{cases} \max \; \max & \overline{f} \\ \qquad \text{s.t.} & \text{Cr}\{f(x,\xi) \geq \overline{f}\} \geq \alpha \\ & \text{Cr}\{g_j(x,\xi) \leq 0\} \geq \alpha, \quad j = 1,...,p \end{cases} \tag{6}$$

and

$$\begin{cases} \max \; \min & \overline{f} \\ \qquad \text{s.t.} & \text{Cr}\{f(x,\xi) \leq \overline{f}\} \geq \alpha \\ & \text{Cr}\{g_j(x,\xi) \leq 0\} \geq \alpha, \quad j = 1,...,p \end{cases} \tag{7}$$

wherein $\alpha$ is the predetermined confidence level. The models (6) and (7) are named optimistic and pessimistic models, respectively. As it can be observed, the credibility theory adds any objective function as a constraint to the main constraints.

Now we aim to solve the preceding fuzzy problems. One way is to convert the constraint $\text{Cr}\{g(x,\xi) \leq 0\} \geq \alpha$ into the deterministic form and solving the obtained non-fuzzy problem.

After providing the constraint of the problem into the form $\text{Cr}\{g(x,\xi) \leq 0\} \geq \alpha$, and by considering the data as fuzzy numbers, these constraints are converted into the deterministic constraints. The following theorem illustrates the way of converting these constraints into the deterministic form. In this theorem, the trapezoidal fuzzy numbers are used and the theorem gets done in this way.

**Theorem 1** *Assume that $g(x,\xi)$ is given in the following form:*

$$g(x,\xi) = h_1(x)\xi_1 + h_2(x)\xi_2 + \cdots + h_t(x)\xi_t + h_0(x), \tag{8}$$

*$h_k(x)$ $(k = 0, 1, ..., t)$ is a function of variable $x$ and $\xi_k$ $(k = 1, ..., t)$ is a trapezoidal fuzzy number in the form $(\xi_{k1}, \xi_{k2}, \xi_{k3}, \xi_{k4})$. By defining $h_k^+(x) = \max\{h_k(x), 0\}$ and $h_k^-(x) = -\min\{h_k(x) \wedge 0\}$, we obtain*
*1) If $\alpha < 1/2$, then $\mathrm{Cr}\{g(x, \xi) \leq 0\} \geq \alpha$ if and only if*

$$(1 - 2\alpha)\sum_{k=1}^{t}[\xi_{k1}h_k^+(x) - \xi_{k4}h_k^-(x)] + (2\alpha)\sum_{k=1}^{t}[\xi_{k2}h_k^+(x) - \xi_{k3}h_k^-] + h_0(x) \leq 0. \tag{9}$$

*2) If $\alpha \geq 1/2$, then $\mathrm{Cr}\{g(x, \xi) \leq 0\} \geq \alpha$ if and only if*

$$(2 - 2\alpha)\sum_{k=1}^{t}[\xi_{k3}h_k^+(x) - \xi_{k2}h_k^-(x)] + (2\alpha - 1)\sum_{k=1}^{t}[\xi_{k4}h_k^+(x) - \xi_{k1}h_k^-] + h_0(x) \leq 0. \tag{10}$$

**Proof:** See [8].

# 3 Main Results

In this work, the last definitions at the end of the previous section are used for calculating the MPI. Moreover, the CCR model is used for calculating the score of efficiency; i.e., $\theta$. Hence, the credibility theory is implemented and then its crisp equivalent will be obtained by topics discussed in the previous section.

The fuzzy DEA model is in the form of a fuzzy linear programming. Fuzzy CCR models cannot be solved like a crisp CCR model (because all the parameters are fuzzy sets). In this section, the fuzzy CCR model (FCCR) is written and its defuzzification is presented by considering the data as fuzzy numbers in the following form.

Let $\tilde{x}_{ij}$ $(i = 1, ..., m)$ and $\tilde{y}_{rj}$ $(r = 1, ..., s)$ be the fuzzy inputs and outputs related to $DMU_j$ $(j = 1, ..., n)$. Hence, the fuzzy version of the CCR model (1) gets the following form:

$$\begin{cases} \tilde{\theta}_o = \min & \theta \\ \quad \text{s.t.} & \sum_{j=1}^{n}\lambda_j\tilde{x}_{ij} \leq \theta\tilde{x}_{io}, \quad i = 1, ..., m \\ & \sum_{j=1}^{n}\lambda_j\tilde{y}_{rj} \geq \tilde{y}_{ro}, \quad r = 1, ..., s \\ & \lambda_j \geq 0, \quad j = 1, ..., n. \end{cases} \tag{11}$$

When fuzzy inputs and outputs for two periods $a$ and $b$ are given in which $\tilde{x}_{ij}^a$ $(i = 1, ..., m)$ and $\tilde{y}_{rj}^a$ $(r = 1, ..., s)$ are fuzzy numbers of $DMU_j$ $(j = 1, ..., n)$ for period $a$, $\tilde{x}_{ij}^b$ $(i = 1, ..., m)$ and $\tilde{y}_{rj}^b$ $(r = 1, ..., s)$ are that of $DMU_j$ $(j = 1, ..., n)$ for period $b$, , the fuzzy version of model (3) for estimating the MPI can be represented as:

$$\begin{cases} \tilde{\theta}_o^{a,b} = \min & \theta \\ \quad \text{s.t.} & \sum_{j=1}^{n}\lambda_j\tilde{x}_{ij}^a \leq \theta\tilde{x}_{io}^b, \quad i = 1, ..., m \\ & \sum_{j=1}^{n}\lambda_j\tilde{y}_{rj}^a \geq \tilde{y}_{ro}^b, \quad r = 1, ..., s \\ & \lambda_j \geq 0, \quad j = 1, ..., n. \end{cases} \tag{12}$$

By performing the credibility theory on the above model (12), we obtain:

$$\begin{cases} \theta_o^{a,b}(\alpha) = \min & \overline{f} \\ \quad \text{s.t.} & \mathrm{Cr}\{\overline{f} \geq \theta\} \geq \beta \\ & \mathrm{Cr}\{\sum_{j=1}^{n}\lambda_j\tilde{x}_{ij}^a \leq \theta\tilde{x}_{io}^b\} \geq \alpha, \quad i = 1, ..., m \\ & \mathrm{Cr}\{\sum_{j=1}^{n}\lambda_j\tilde{y}_{rj}^a \geq \tilde{y}_{ro}^b\} \geq \alpha, \quad r = 1, ..., s \\ & \lambda_j \geq 0, \quad j = 1, ..., n. \end{cases} \tag{13}$$

Based on the above model, the fuzzy MPI can be represented as:

$$MPI_o(\alpha) = \left(\frac{\theta_o^{a,b}(\alpha)}{\theta_o^{a,a}(\alpha)} \times \frac{\theta_o^{b,b}(\alpha)}{\theta_o^{b,a}(\alpha)}\right)^{\frac{1}{2}} \tag{14}$$

and we have following definition for the productivity change in fuzzy environment as:

**Definition 3** *1. $MPI_o(\alpha) > 1$, increase productivity in credibility level $\alpha$ and observe $\alpha$-progress.*
*2. $MPI_o(\alpha) < 1$, decrease productivity in credibility level $\alpha$ and observe $\alpha$-regress.*
*3. $MPI_o(\alpha) = 1$, no change in productivity at time $a$ in comparison to $b$ in credibility level $\alpha$.*

Now consider the inputs and outputs in two periods $a$ and $b$ are trapezoidal fuzzy numbers as:

$$\begin{aligned}\tilde{x}^a_{ij} &= (x^a_{ij1}, x^a_{ij2}, x^a_{ij3}, x^a_{ij4}), && i=1,...,m, \quad j=1,...,n,\\ \tilde{y}^a_{rj} &= (y^a_{rj1}, y^a_{rj2}, y^a_{rj3}, y^a_{rj4}), && r=1,...,s, \quad j=1,...,n. \qquad (15)\\ \tilde{x}^b_{ij} &= (x^b_{ij1}, x^b_{ij2}, x^b_{ij3}, x^b_{ij4}), && r=1,...,s, \quad j=1,...,n,\\ \tilde{y}^b_{rj} &= (y^b_{rj1}, y^b_{rj2}, y^b_{rj3}, y^b_{rj4}), && i=1,...,m, \quad j=1,...,n. \qquad (16)\end{aligned}$$

Using the mentioned theorem which shows how the fuzzy models are converted into the crisp forms, we have: If $\alpha < 1/2$, then

$$\mathrm{Cr}\{\overline{f} \geq \theta\} \geq \alpha \Rightarrow \mathrm{Cr}\{\theta - \overline{f} \leq 0\} \geq \alpha \Rightarrow (1-2\alpha)(\theta - \overline{f}) + (2\alpha)(\theta - \overline{f}) = \theta - \overline{f} \leq 0. \qquad (17)$$

Therefore, for each $\alpha$, the first constraint gets the form of $\theta - \overline{f} \leq 0$.
The second constraint is in the following form:

$$(1-2\alpha)\sum_{j=1}^{n} x^a_{ij1}\lambda_j + (2\alpha)\sum_{j=1}^{n} x^a_{ij2}\lambda_j - \theta((1-2\alpha)x^b_{io4} + (2\alpha)x^b_{io3}) \leq 0 \qquad (18)$$

and the third one, as follows

$$(1-2\alpha)y^b_{ro1} + (2\alpha)y^b_{ro2} - (1-2\alpha)\sum_{j=1}^{n} y^a_{rj4}\lambda_j - (2\alpha)\sum_{j=1}^{n} y^a_{rj3}\lambda_j \leq 0. \qquad (19)$$

So, we obtain the following linear program:

$$\begin{cases} \theta^{a,b}_o(\alpha) = \min \quad \overline{f} \\ \quad \text{s.t.} \quad \theta - \overline{f} \leq 0 \\ \qquad (1-2\alpha)\sum_{j=1}^{n} x^a_{ij1}\lambda_j + (2\alpha)\sum_{j=1}^{n} x^a_{ij2}\lambda_j - \theta((1-2\alpha)x^b_{io4} + (2\alpha)x^b_{io3}) \leq 0, \quad i=1,...,m \\ \qquad (1-2\alpha)y^b_{ro1} + (2\alpha)y^b_{ro2} - (1-2\alpha)\sum_{j=1}^{n} y^a_{rj4}\lambda_j - (2\alpha)\sum_{j=1}^{n} y^a_{rj3}\lambda_j \leq 0, \quad r=1,...,s \\ \qquad \lambda_j \geq 0, \quad j=1,...,n. \end{cases} \qquad (20)$$

The other models can be derived in the same way, by performing the desired periods of time.
Now if $\alpha \geq 1/2$, we get:

$$(2-2\alpha)(\theta - \overline{f}) + (2\alpha - 1)(\theta - \overline{f}) = \theta - \overline{f} \leq 0, \ \forall \ \alpha \qquad (21)$$

which looks like the first constraint when $\alpha < 1/2$.
The formulation of the second and third the constraint is as follows:

$$(2-2\alpha)\sum_{j=1}^{n} x^a_{ij3}\lambda_j + (2\alpha-1)\sum_{j=1}^{n} x^a_{ij4}\lambda_j - \theta((2-2\alpha)x^b_{io2} + (2\alpha-1)x^b_{io1}) \leq 0, \qquad (22)$$

$$(2-2\alpha)y^b_{ro3} + (2\alpha-1)y^b_{ro4} - (2-2\alpha)\sum_{j=1}^{n} y^a_{rj2}\lambda_j - (2\alpha-1)\sum_{j=1}^{n} y^a_{rj1}\lambda_j \leq 0. \qquad (23)$$

So, we have the following linear program:

$$\begin{cases} \theta^{a,b}_o(\alpha) = \min \quad \overline{f} \\ \text{s.t.} \quad \theta - \overline{f} \leq 0 \\ \quad (2-2\alpha)\sum_{j=1}^{n} x^a_{ij3}\lambda_j + (2\alpha-1)\sum_{j=1}^{n} x^a_{ij4}\lambda_j - \theta((2-2\alpha)x^b_{io2} + (2\alpha-1)x^b_{io1}) \leq 0, \quad i=1,...,m \\ \quad (2-2\alpha)y^b_{ro3} + (2\alpha-1)y^b_{ro4} - (2-2\alpha)\sum_{j=1}^{n} y^a_{rj2}\lambda_j - (2\alpha-1)\sum_{j=1}^{n} y^a_{rj1}\lambda_j \leq 0, \quad r=1,...,s \\ \quad \lambda_j \geq 0, \quad j=1,...,n. \end{cases} \qquad (24)$$

Just like the previous case, other models for desired periods of time can be obtained similarly.

**Theorem 2** *By applying the credibility theory on a fuzzy DEA model (20) with trapezoidal fuzzy data, when the confidence level of $\alpha$ is increased, the score of technical efficiency is increased, as well.*

**Proof:** Consider model (20). Let $\alpha_1 < \alpha_2$, then $\alpha_2 - \alpha_1 > 0$, and so, $\sum_{j=1}^{n} x^a_{ij1}\lambda_j \le \sum_{j=1}^{n} x^a_{ij2}\lambda_j$, therefore

$$(\alpha_2-\alpha_1)\sum_{j=1}^{n} x^a_{ij1}\lambda_j \le (\alpha_2-\alpha_1)\sum_{j=1}^{n} x^a_{ij2}\lambda_j \Rightarrow (2\alpha_2-2\alpha_1)\sum_{j=1}^{n} x^a_{ij1}\lambda_j \le (2\alpha_2-2\alpha_1)\sum_{j=1}^{n} x^a_{ij2}\lambda_j$$

$$\Rightarrow (1-2\alpha_1-1+2\alpha_2)\sum_{j=1}^{n} x^a_{ij1}\lambda_j \le (2\alpha_2-2\alpha_1)\sum_{j=1}^{n} x^a_{ij2}\lambda_j$$

$$\Rightarrow (1-2\alpha_1)\sum_{j=1}^{n} x^a_{ij1}\lambda_j + 2\alpha_1\sum_{j=1}^{n} x^a_{ij2}\lambda_j \le (1-2\alpha_2)\sum_{j=1}^{n} x^a_{ij1}\lambda_j + 2\alpha_2\sum_{j=1}^{n} x^a_{ij2}\lambda_j.$$

We know that $x^b_{io3} \le x^b_{io4}$, thus,

$$(\alpha_2-\alpha_1)x^b_{io3} \le (\alpha_2-\alpha_1)x^b_{io4} \Rightarrow (2\alpha_2-2\alpha_1)x^b_{io3} \le (1+2\alpha_2-1-2\alpha_1)x^b_{io4},$$

hence,

$$(1-2\alpha_2)x^b_{io4} + 2\alpha_2 x^a_{io3} \le (1-2\alpha_1)x^b_{io4} + 2\alpha_1 x^b_{io3}.$$

In this condition,

$$(1-2\alpha_1)\sum_{j=1}^{n} x^a_{ij1}\lambda_j + 2\alpha_1\sum_{j=1}^{n} x^a_{ij2}\lambda_j - \theta((1-2\alpha_1)x^b_{io4} + 2\alpha_1 x^b_{io3})$$

$$\le \sum_{j=1}^{n} x^a_{ij1}\lambda_j + 2\alpha_1\sum_{j=1}^{n} x^a_{ij2}\lambda_j(1-2\alpha_2) - \theta((1-2\alpha_2)x^b_{io4} + 2\alpha_2 x^b_{io3}).$$

Similarly, it is proven that

$$(1-2\alpha_1)y^b_{ro1} + 2\alpha_1 y^b_{ro2} - (1-2\alpha_1)\sum_{j=1}^{n} y^a_{rj4}\lambda_j - 2\alpha_1\sum_{j=1}^{n} y^a_{rj3}\lambda_j$$

$$\le (1-2\alpha_2)y^b_{ro1} + 2\alpha_2 y^b_{ro2} - (1-2\alpha_2)\sum_{j=1}^{n} y^a_{rj4}\lambda_j - 2\alpha_2\sum_{j=1}^{n} y^a_{rj3}\lambda_j.$$

According to the two above mentioned relations and the obtained models for $\alpha_1$ and $\alpha_2$, it is observed that the generated feasible region by applying $\alpha_1$ on the DEA model (20) includes the generated feasible region by applying $\alpha_2$ on the DEA model (20) and ,as a result, its objective function value will not be worse. Subsequently,

$$\alpha_1 \le \alpha_2 \Rightarrow \theta^{a,b}_o(\alpha_1) \le \theta^{a,b}_o(\alpha_2).$$

# 4 Numerical Examples

## 4.1 Illustrative Example

We illustrate our proposed method by a numerical example. Consider an example of two inputs and two outputs for four DMUs, in which all data are trapezoidal fuzzy numbers. The data for the two periods $a$ and $b$ are presented in Tables 1 and 2, respectively.

The results of the proposed method is compared with two alternative methods suggested by Jahanshahloo et al. [6] and Jahanshahloo et al. [7]. Table 3 shows the productivity change of DMUs based on the $\alpha$-cut and ranking function methods, and the results of applying the proposed method are provided in Table 4. To compare the $\alpha$-cut method and the proposed method in this article, we need $\alpha$-cut levels as well as $\alpha$ credibility levels. Hence, we use numbers 0, 0.25, 0.50, 0.75, and 1.

Table 1: Data in period a

| DMU | Input1 | Input2 | Output1 | output2 |
|---|---|---|---|---|
| 1 | (35, 39, 41, 42) | (19, 20, 22, 24) | (24, 26, 28, 31) | (38, 41, 43, 44) |
| 2 | (29, 29, 31, 33) | (14, 15, 17, 19) | (22, 22, 24, 26) | (33, 35, 37, 39) |
| 3 | (44, 48, 49, 51) | (22, 25, 26, 29) | (27, 32, 34, 36) | (43, 51, 54, 56) |
| 4 | (34, 39, 41, 43) | (21, 22, 24, 28) | (25, 29, 31, 34) | (51, 57, 58, 59) |

Table 2: Data in period b

| DMU | Input1 | Input2 | Output1 | output2 |
|---|---|---|---|---|
| 1 | (37, 38, 39, 4) | (21, 23, 24, 25) | (26, 28, 29, 31) | (41, 44, 45, 46) |
| 2 | (27, 29, 30, 31) | (12, 13, 14, 15) | (19, 21, 22, 24) | (31, 32, 33, 35) |
| 3 | (42, 45, 46, 47) | (20, 22, 23, 25) | (25, 27, 29, 32) | (23, 26, 27, 31) |
| 4 | (36, 37, 39, 41) | (23, 25, 26, 27) | (28, 31, 32, 33) | (31, 35, 36, 39) |

Table 3 shows that the MPI determined based on the $\alpha$-cut method is interval number, which includes both less and more than unit numbers. Therefore, for a given $\alpha$-cut, we cannot opine about the productivity change of DMUs. In other words, we cannot state that DMUs are progressive or regressive or have constant productivity. On the other hand, the method proposed in this paper determines a crisp number for each $\alpha$ credibility level in order to estimate the productivity change. Thus, we can judge about the productivity change of DMUs for each given $\alpha$ credibility level. For example, based on the method mentioned in this paper in which $\alpha$ is equal to 0.25, DMUs 1 and 2 have progress, while DMUs 3 and 4 have regress in productivity. In other hands, the productivity changes of DMUs based on the ranking function method, which are stated in the last column of Table 3, are crisp numbers while the data are fuzzy numbers which shows that this method does not give realistic results. Moreover, although the provided results are coincided with those of the credibility method with $\alpha = 0$ and 0.25, differ from them with $\alpha$= 0.5, 0.75, and 1. For example, with $\alpha$=0.75 and 1, DMU1 has progress in productivity based on the ranking function method, while it has regress in productivity by using the credibility method,. As a pluralization, the method mentioned in this paper gives crisp numbers in order to determine the productivity change of DMUs which leads to precise decision making about the productivity of DMUs for each given $\alpha$ credibility level. In addition, the status of productivity change may be varied from an $\alpha$ credibility level to another which is the result of the imprecise nature of the data.

Table 3: The results of $\alpha$-cut and ranking function methods

| DMU | 0.25 | 0.5 | 0.75 | 1 | R function |
|---|---|---|---|---|---|
| 1 | [0.6801,1.6012] | [0.7587,1.4720] | [0.8431,1.3655] | [0.9213,1.2752] | 1.0665 |
| 2 | [0.7498,1.6088] | [0.7966,1.4589] | [0.8422,1.3321] | [0.8939,1.2133] | 1.0270 |
| 3 | [0.5533,1.7212] | [0.6060,1.5113] | [0.6579,1.3289] | [0.7246,1.1396] | 0.9292 |
| 4 | [0.6262,1.2239] | [0.7034,1.1484] | [0.7870,1.1073] | [0.8455,1.0761] | 0.9280 |

Table 4: The results of the proposed method in this paper

| DMU | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| 1 | 1.1254 | 1.0185 | 0.9658 | 0.9222 |
| 2 | 1.0124 | 1.0085 | 1.0482 | 1.0769 |
| 3 | 0.9002 | 0.9225 | 0.9459 | 0.9525 |
| 4 | 0.8812 | 1.0402 | 0.9285 | 0.8314 |

## 4.2 Applicable Example

Social security is a public right which should be implemented for all individuals of the society by government. Article 29 of the constitution obliges the government to provide the conditions of utilizing the social security services for all the population in the form of insurance of non-insurance system. Making use of social security benefits, such as retirement, unemployment, old age, disability, loss of caretaker, helplessness, accidents and injuries, requiring insurance or non-insurance medical and health care services, is a public right. According to the law, the government is committed to finance the aforementioned services and protections by using the general revenues and the income of public collaborations. From the beginning of the Islamic Republic, many organizations and offices were established and developed with insurance, protective, and assistance structures. Although all of these institutions and organizations have always been the source of prominent assistances and services, considering the perspective of establishing and leading a comprehensive social security and welfare system, these organizations have not been effective due to the lack of an extensive organizational structure and efficient management system which could not provide the possibility of continuous and coordinate policy making, managing, monitoring, and evaluating. Therefore, the specialists, authorities, and managers of social and economical domains accomplished planning and administrating a social security and welfare system to implement the comprehensive insurance coverage for the whole population which is now being executed in the form of a ministry under the title of Social Security and Welfare. Among these organizations, Social Security Organization (SSO), as the central organization of comprehensive social security and welfare system, has got the particular importance and status. The SSO is a social insurer organization with the major mission on compulsory coverage of wage-earners and salaried workers as well as voluntary coverage of self-employed individuals. In accordance with Article 29 of I.R. of Iran's Constitution and paragraph 2 and 4 of Article 21 of the Constitution, in order to create the macro - integration of welfare policies for developing social justice and protecting the whole population of the country against social, economical, and natural contingencies as well as their subsequences, regarding the aforementioned conditions and articles, a comprehensive social security system is established in order to provide the following services:

a) Retirement, disability and death;
b) Unemployment;
c) Old-age;
d) Helplessness, loss of caretaker and social vulnerabilities;
e) Accidents and injuries;
f) Physical, mental, and psychological disabilities;
g) Health care and medical insurance;
h) Protecting mothers especially during the maternity period and child-rearing;
i) Protecting orphan children and unprotected women;
j) Planning particular insurance system for widows, old women and self-dependent women;
k) Poverty and inequity alleviation.

In this section, considering the proposed method in the last section, the MPI is illustrated for evaluating 19 branches of Social Security Organization in Tehran in two periods of time. Each branch (DMU) uses two inputs to produce two outputs as Table 5. The inputs and outputs in the two periods $a$ and $b$ are given in Tables 6, 7, 8 and 9, respectively.

Table 5: The items which are tested in the empirical example

| Inputs | Outputs |
|---|---|
| Number of staffs | All individuals under insurance |
| Total number of computers | Sum of insured individuals' contracts |

Table 6: Inputs for 19 DMUs in the first period a

| DMU | Input 1 | Input 2 |
|---|---|---|
| 1 | (96,97.41,99.41,99.99) | (86,86.25,86.75,87) |
| 2 | (75,76.83,79.83,80.99) | (88,88.416,89.41,89.99) |
| 3 | (77,77.75,79.25,80) | (85,86,88,89) |
| 4 | (91,91.66,93.16,93.99) | (93,93.75,95.25,96) |
| 5 | (89,89.83,91.33,91.99) | (83,83,83,83) |
| 6 | (102,102.5,104,105) | (97,97,97,97) |
| 7 | (96,96.66,98.66,99.99) | (90,90.5,91.5,92) |
| 8 | (85,86,88.5,90) | (92,92,92,92) |
| 9 | (106,107.25,110.25,112) | (84,84.16,90.16,91.99) |
| 10 | (107,107.91,109.91,110.99) | (95,95,95,95) |
| 11 | (94,95.58,99.08,100.99) | (78,78,78,78) |
| 12 | (78,78.41,78.91,78.99) | (89,89,89,89) |
| 13 | (102,102,102,102) | (107,107.33,109.33,110.99) |
| 14 | (82,83,86.5,88) | (92,92.58,93.58,93.99) |
| 15 | (89,89.25,90.25,91) | (85,85,85,85) |
| 16 | (84,86.25,89.25,90) | (104,104,104,104) |
| 17 | (94,97.91,104.91,107.99) | (91,91.33,91.83,91.99) |
| 18 | (97,99,102,103) | (95,95.08,95.58,95.99) |
| 19 | (82,83,85.5,87) | (99.99,100.08,100.58,100.99) |

Table 7: Outputs for 19 DMUs in the first period a

| DMU | Output 1 | Output 2 |
|---|---|---|
| 1 | (55830,56200.33,56944.33,57317.99) | (29.99,33.41,40.91,44.99) |
| 2 | (36740,36770.25,36826.25,36852) | (0,7.83,18.83,21.99) |
| 3 | (38004,38225.08,38614.58,38782.99) | (11,14.33,22.33,26.99) |
| 4 | (35469,35577.08,35851.08,36016.99) | (10,20.16,42.66,54.99) |
| 5 | (52127,53398,54343,54817) | (9,18.08,35.08,42.99) |
| 6 | (70254,71350.16,75510.16,78573.99) | (7,9.66,15.66,18.99) |
| 7 | (32585,34220.66,36649.66,37442.99) | (47,70.08,111.08,128.99) |
| 8 | (42900,43963.83,46148.83,47269.99) | (11,14.16,22.16,26.99) |
| 9 | (85399,85810.25,86720.75,87220) | (43,50.58,77.58,96.99) |
| 10 | (46924,47033.25,47229.25,47316) | (9,17.66,31.16,35.99) |
| 11 | (36652,38567.16,42390.16,44297.99) | (81,129.75,210.25,242) |
| 12 | (39582,39588.25,39607.25,39620) | (11,15.66,25.66,30.99) |
| 13 | (56144,56814.41,58150.41,58815.99) | (30,36,49.5,57) |
| 14 | (87716,88307.25,89574.25,90250) | (28,31.83,39.33,42.99) |
| 15 | (47727,48152.41,49033.41,49488.99) | (15,18.66,26.16,29.99) |
| 16 | (52923,53011.25,53174.25,53249) | (15,17.5,24,28) |
| 17 | (78550,80892.75,86173.25,89111) | (13,16.83,22.83,24.99) |
| 18 | (46154,46325.41,46643.91,46790.99) | (13,14.58,18.58,20.99) |
| 19 | (27978,28475.91,30958.41,32942.99) | (29,56.14,204.41,324.99) |

Table 8: Inputs for 19 DMUs in the second period b

| DMU | Input 1 | Input 2 |
|---|---|---|
| 1 | (93,93.16,95.91,96.99) | (84,84.25,85.75,87) |
| 2 | (75,76,78,79) | (91,92,94,95) |
| 3 | (75,75.75,77.25,78) | (87,87,87,87) |
| 4 | (92,92.416,93.41,93.99) | (93,93,93,93) |
| 5 | (88,89.16,91.16,91.99) | (83,84.33,86.33,86.99) |
| 6 | (101,101.66,103.66,104.99) | (97,97,97,97) |
| 7 | (94,94.25,94.75,95) | (90,90.25,90.75,91) |
| 8 | (83,84.66,87.86,88.99) | (92,92.16,92.66,92.99) |
| 9 | (102,102.41,104.41,105. 99) | (92,92,92,92) |
| 10 | (102,102.25,102.75,103) | (95,95.66,96.66,96.99) |
| 11 | (93,94.25,95.75,96) | (79,79,79,79) |
| 12 | (76,77.08,78.58,78.99) | (91,91,91,91) |
| 13 | (103,103.83,105.83,106.99) | (103,103.66,104.66,104.99) |
| 14 | (86,87.25,89.25,90) | (95,95,95,95) |
| 15 | (87,88,90,91) | (85,85.16,85.66,85.99) |
| 16 | (90,90.75,92.25,93) | (104,104,104,104) |
| 17 | (111,112.66,115.66,116.99) | (92,93.16,94.66,94.99) |
| 18 | (94,95.16,97.66,98.99) | (98,98,98,98) |
| 19 | (85,85.83,87.33,87.99) | (101,101,101,101) |

Table 9: Outputs for 19 DMUs in the second period b

| DMU | Output 1 | Output 2 |
|---|---|---|
| 1 | (57668,58058.66,58.516.41,59093.24) | (32,40.58,60.08,70.99) |
| 2 | (36922,36983.41,39550.16,39617.24) | (14,17.91,28.41,34.99) |
| 3 | (25360,29899.16,34623.91,37129.24) | (20,25.83,39.33,46.99) |
| 4 | (36247,36449.41,41498.16,41683.74) | (21,31,50,59) |
| 5 | (36371,41380.16,46987.5,51833.83) | (28,34.25,44.75,49) |
| 6 | (69071,70439.83,72244.33,72842.49) | (0,7.91,24.41,32.99) |
| 7 | (37476,38071.83,39645.08,40080.74) | (73,88.91,115.41,125.99) |
| 8 | (48094,49141.58,52412.41,53389.82) | (12,16,25.5,31) |
| 9 | (84531,86420.25,90433.5,92656.75) | (0,34.08,89.58,110.99) |
| 10 | (46957,48133,50993.5,53117.5) | (19,24.58,35.08,39.99) |
| 11 | (31554,33763,40131.91,41816.41) | (170,191.33,210.33,237.99) |
| 12 | (27012,29151.58,32064.91,36224.82) | (21,23.5,29,32) |
| 13 | (59581,60492.75,63215.16,63988.91) | (31,39,62,77) |
| 14 | (80425,85372.83,91481.25,93291.91) | (36,40.91,55.91,65.99) |
| 15 | (39797,41769.25,46409.66,49466.41) | (26,28.75,34.25,37) |
| 16 | (53599,65592.83,80853.83,83521.99) | (14,19,31,38) |
| 17 | (72553,75921.83,86140.66,89407.82) | (19,23.66,35.16,41.99) |
| 18 | (46897,60280,74386.91,81236.91) | (13,18.16,28.66,33.99) |
| 19 | (28855,30522.08,32389.25,33113.16) | (23,41.33,87.83,115.99) |

Tables 10, 11, 12 and 13 show the amount of technical efficiency of each DMU in the time periods $a$ and $b$ in relation to time periods $a$ and $b$, with various confidence levels $\alpha$, $0 \leq \alpha \leq 1$.

As we can see in Tables 10, 11, 12 and 13 the amount of technical efficiency of each unit per time period is increases by increasing the confidence level of $\alpha$ (based on Theorem(3.2)). For example the amount of $\theta_1^{a,a}$, for $DMU_1$ increases from 0.6285 for $\alpha = 0$ to 0.8739 for $\alpha = 1$. This event not only happened for $DMU_1$, but also for the other DMUs on the calculated value of $\theta_o^{a,a}$, $\theta_o^{b,a}$, $\theta_o^{b,b}$ and $\theta_o^{a,b}$.

Table 10: Efficiency of DMUs with various levels of $\alpha$ $(\theta_o^{a,a})$

| $DMU$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 0.9$ | $\alpha = 1$ |
|---|---|---|---|---|---|---|
| 1 | 0.6329 | 0.6418 | 0.7444 | 0.7821 | 0.8360 | 0.8739 |
| 2 | 0.4176 | 0.4224 | 0.5077 | 0.5413 | 0.5879 | 0.6193 |
| 3 | 0.4376 | 0.4433 | 0.5437 | 0.5842 | 0.6417 | 0.6811 |
| 4 | 0.3737 | 0.3781 | 0.5548 | 0.6362 | 0.7560 | 0.8399 |
| 5 | 0.6327 | 0.6412 | 0.7308 | 0.7736 | 0.8363 | 0.8833 |
| 6 | 0.7223 | 0.7338 | 0.8218 | 0.8409 | 0.8593 | 0.8679 |
| 7 | 0.4096 | 0.4483 | 0.9071 | 1.1108 | 1.4462 | 1.6752 |
| 8 | 0.4732 | 0.4818 | 0.5596 | 0.5988 | 0.6521 | 0.6876 |
| 9 | 0.9093 | 0.9283 | 1.1650 | 1.2660 | 1.4090 | 1.5075 |
| 10 | 0.4852 | 0.4910 | 0.5487 | 0.5755 | 0.6146 | 0.6423 |
| 11 | 0.5833 | 0.6606 | 1.6797 | 2.1264 | 2.7699 | 3.2098 |
| 12 | 0.4577 | 0.4626 | 0.5716 | 0.6169 | 0.6822 | 0.7275 |
| 13 | 0.5215 | 0.5369 | 0.7064 | 0.7704 | 0.8623 | 0.9257 |
| 14 | 0.9503 | 0.9616 | 1.0954 | 1.1484 | 1.2186 | 1.2644 |
| 15 | 0.5602 | 0.5674 | 0.6257 | 0.6510 | 0.6870 | 0.7122 |
| 16 | 0.5381 | 0.5458 | 0.6335 | 0.6714 | 0.7229 | 0.7573 |
| 17 | 0.8436 | 0.8645 | 0.9956 | 1.0181 | 1.0393 | 1.0492 |
| 18 | 0.4794 | 0.4854 | 0.5181 | 0.5224 | 0.5412 | 0.5543 |
| 19 | 0.3408 | 0.3873 | 1.8806 | 2.7699 | 4.0578 | 4.9413 |

Table 11: Efficiency of DMUs with various levels of $\alpha$ $(\theta_o^{b,a})$

| $DMU$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 0.9$ | $\alpha = 1$ |
|---|---|---|---|---|---|---|
| 1 | 0.6414 | 0.6501 | 0.7560 | 0.7964 | 0.8313 | 0.8474 |
| 2 | 0.4223 | 0.4307 | 0.5064 | 0.5335 | 0.5628 | 0.5783 |
| 3 | 0.4422 | 0.4510 | 0.5361 | 0.5649 | 0.5959 | 0.6123 |
| 4 | 0.3752 | 0.3803 | 0.5048 | 0.5439 | 0.5864 | 0.6092 |
| 5 | 0.6375 | 0.6464 | 0.7423 | 0.7835 | 0.8213 | 0.8396 |
| 6 | 0.7273 | 0.7393 | 0.8433 | 0.8714 | 0.9013 | 0.9222 |
| 7 | 0.3961 | 0.4396 | 0.7303 | 0.7905 | 0.8563 | 0.8916 |
| 8 | 0.4751 | 0.4845 | 0.5611 | 0.5943 | 0.6301 | 0.6490 |
| 9 | 0.9308 | 0.9492 | 1.1932 | 1.2835 | 1.3694 | 1.4111 |
| 10 | 0.4930 | 0.4982 | 0.5616 | 0.5921 | 0.6181 | 0.6300 |
| 11 | 0.5989 | 0.6787 | 1.1786 | 1.2749 | 1.3820 | 1.4538 |
| 12 | 0.4651 | 0.4717 | 0.5574 | 0.5865 | 0.6179 | 0.6345 |
| 13 | 0.5190 | 0.5288 | 0.6692 | 0.7043 | 0.7422 | 0.7623 |
| 14 | 0.9531 | 0.9661 | 1.1214 | 1.1637 | 1.2145 | 1.2439 |
| 15 | 0.5639 | 0.5716 | 0.6406 | 0.6710 | 0.6989 | 0.7124 |
| 16 | 0.5468 | 0.5565 | 0.6445 | 0.6725 | 0.7092 | 0.7286 |
| 17 | 0.8573 | 0.8765 | 1.0186 | 1.0544 | 1.0959 | 1.1227 |
| 18 | 0.4826 | 0.4890 | 0.5333 | 0.5519 | 0.5725 | 0.5819 |
| 19 | 0.3473 | 0.3977 | 1.2324 | 1.6037 | 2.0153 | 2.2380 |

Table 12: Efficiency of DMUs with various levels of $\alpha$ ($\theta_o^{b,b}$)

| $DMU$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 0.9$ | $\alpha = 1$ |
|---|---|---|---|---|---|---|
| 1 | 10.6641 | 0.6763 | 0.8496 | 0.9010 | 0.9491 | 0.9729 |
| 2 | 0.4350 | 0.4435 | 0.5869 | 0.6222 | 0.6603 | 0.6807 |
| 3 | 0.3237 | 0.3599 | 0.5814 | 0.6304 | 0.6834 | 0.7116 |
| 4 | 0.3939 | 0.3986 | 0.5818 | 0.6179 | 0.6569 | 0.6777 |
| 5 | 0.4319 | 0.4609 | 0.6762 | 0.7332 | 0.7912 | 0.8208 |
| 6 | 0.7158 | 0.7289 | 0.8084 | 0.8325 | 0.8707 | 0.8881 |
| 7 | 0.5135 | 0.5492 | 0.7928 | 0.8422 | 0.8959 | 0.9246 |
| 8 | 0.5283 | 0.5386 | 0.6487 | 0.6881 | 0.7306 | 0.7531 |
| 9 | 0.9207 | 0.9380 | 1.1939 | 1.2755 | 1.3533 | 1.3919 |
| 10 | 0.4885 | 0.4988 | 0.6110 | 0.6501 | 0.6876 | 0.7061 |
| 11 | 0.7583 | 0.8256 | 1.1449 | 1.2477 | 1.3725 | 1.4451 |
| 12 | 0.3316 | 0.3528 | 0.4959 | 0.5406 | 0.5855 | 0.6142 |
| 13 | 0.5745 | 0.5843 | 0.7497 | 0.7930 | 0.8484 | 0.8780 |
| 14 | 0.8720 | 0.9013 | 1.1438 | 1.2117 | 1.2848 | 1.3236 |
| 15 | 0.4696 | 0.4843 | 0.6343 | 0.744 | 0.7141 | 0.7342 |
| 16 | 0.5597 | 0.6180 | 0.9314 | 0.9725 | 1.0159 | 1.0411 |
| 17 | 0.7696 | 0.7925 | 0.9874 | 1.0589 | 1.1237 | 1.1536 |
| 18 | 0.5109 | 0.5716 | 0.8385 | 0.8901 | 0.9453 | 0.9743 |
| 19 | 0.3284 | 0.3679 | 0.6852 | 0.7694 | 0.8523 | 0.8993 |

Table 13: Efficiency of DMUs with various levels of $\alpha$ ($\theta_o^{a,b}$)

| $DMU$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 0.9$ | $\alpha = 1$ |
|---|---|---|---|---|---|---|
| 1 | 0.6585 | 0.6703 | 0.8662 | 0.9392 | 1.0618 | 1.1568 |
| 2 | 0.4280 | 0.4349 | 0.6082 | 0.6657 | 0.7486 | 0.8060 |
| 3 | 0.3306 | 0.3656 | 0.6352 | 0.7235 | 0.8470 | 0.9308 |
| 4 | 0.3920 | 0.4061 | 0.6404 | 0.7192 | 0.8344 | 0.9148 |
| 5 | 0.4323 | 0.4679 | 0.6913 | 0.7600 | 0.8644 | 0.9326 |
| 6 | 0.7109 | 0.7235 | 0.7864 | 0.7944 | 0.8153 | 0.8404 |
| 7 | 0.5193 | 0.5527 | 0.9797 | 1.1508 | 1.4601 | 1.6710 |
| 8 | 0.5265 | 0.5359 | 0.6477 | 0.6946 | 0.7585 | 0.8011 |
| 9 | 0.9086 | 0.9267 | 1.2133 | 1.3254 | 1.5085 | 1.6509 |
| 10 | 0.4854 | 0.4951 | 0.6059 | 0.6448 | 0.6976 | 0.7333 |
| 11 | 0.7301 | 0.8104 | 1.7041 | 2.1386 | 2.7636 | 3.1905 |
| 12 | 0.3376 | 0.3589 | 0.5272 | 0.5910 | 0.6740 | 0.7273 |
| 13 | 0.5715 | 0.5806 | 0.7962 | 0.8933 | 1.0343 | 1.1322 |
| 14 | 0.8692 | 0.8968 | 1.1661 | 1.2601 | 1.3905 | 1.4784 |
| 15 | 0.4667 | 0.4809 | 0.6366 | 0.6804 | 0.7492 | 0.7963 |
| 16 | 0.5507 | 0.6061 | 0.8835 | 0.9356 | 1.0037 | 1.0477 |
| 17 | 0.7506 | 0.7745 | 0.9718 | 1.0012 | 1.0305 | 1.0458 |
| 18 | 0.5083 | 0.5681 | 0.8032 | 0.8480 | 0.9048 | 0.9464 |
| 19 | 0.3305 | 0.3667 | 0.8391 | 1.0496 | 1.4365 | 1.7013 |

Based on the data on Tables 10, 11, 12, and 13 and MPI formulation in (14), the productivity change with the fuzzy data can be calculated for all DMUs. The results of assessing the MPI for different $\alpha$'s ($0 \leq \alpha \leq 1$) in the two mentioned periods are provided in Table 14. Therefore, we can survey the productivity growth for all the fuzzy DMUs by solving linear programming problems for each given $\alpha$. All computations were done with LINGO software.

Table 14: MPI with various levels of $\alpha$

| DMU | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 0.9$ | $\alpha = 1$ |
|---|---|---|---|---|---|---|
| 1 | 1.0378 | 1.0422 | 1.1435 | 1.1655 | 1.2041 | 1.2327 |
| 2 | 1.0275 | 1.0297 | 1.1783 | 1.1975 | 1.2223 | 1.2376 |
| 3 | 0.7438 | 0.8112 | 1.1256 | 1.1756 | 1.2304 | 1.2602 |
| 4 | 1.0494 | 1.0610 | 1.1535 | 1.1332 | 1.1119 | 1.1007 |
| 5 | 0.6804 | 0.7213 | 0.9283 | 0.9587 | 0.9978 | 1.0160 |
| 6 | 0.9840 | 0.9858 | 0.9578 | 0.9500 | 0.9574 | 0.9656 |
| 7 | 1.2820 | 1.2410 | 1.0827 | 1.0505 | 1.0277 | 1.0171 |
| 8 | 1.1122 | 1.1119 | 1.1567 | 1.1588 | 1.1612 | 1.1627 |
| 9 | 0.9942 | 0.9932 | 1.0207 | 1.0200 | 1.0286 | 1.0393 |
| 10 | 0.9955 | 1.0047 | 1.0961 | 1.1091 | 1.1237 | 1.1311 |
| 11 | 1.2588 | 1.2215 | 0.9927 | 0.9920 | 0.9954 | 0.9940 |
| 12 | 0.7251 | 0.7617 | 0.9058 | 0.9397 | 0.9702 | 0.9837 |
| 13 | 1.1014 | 1.0932 | 1.1237 | 1.1425 | 1.1709 | 1.1868 |
| 14 | 0.9147 | 0.9328 | 1.0420 | 1.0688 | 1.0987 | 1.1154 |
| 15 | 0.8329 | 0.8473 | 1.0036 | 1.0249 | 1.0555 | 1.0734 |
| 16 | 1.0235 | 1.1104 | 1.4196 | 1.4195 | 1.4101 | 1.4059 |
| 17 | 0.8937 | 0.9000 | 0.9727 | 0.9938 | 1.0083 | 1.0120 |
| 18 | 1.0594 | 1.1697 | 1.5612 | 1.6178 | 1.6613 | 1.6908 |
| 19 | 0.9576 | 0.9358 | 0.4980 | 0.4251 | 0.3869 | 0.3719 |

According to Table 14, DMUs can be divided into four categories: 1) DMUs with progress in productivity for all $\alpha$'s, 2) DMUs with regress in productivity for all $\alpha$'s, 3) DMUs with change in productivity from progress to regress with increasing $\alpha$, 4) DMUs with change in productivity from regress to progress with increasing $\alpha$. This classification can be done in order to analyze the productivity growth in relation to $\alpha$. In this situation, units 1, 2, 4, 7, 8, 13, and 18 are located in the first category, units 6, 12, and 19 belong to the second category, category three only consists of unit 11, and units 3, 5, 9, 10, 14, 15, 16, and 17 construct the 4th category.

# 5 Concluding Remarks

In this work, after studying the credibility theory for facing with fuzzy problems, we applied it to fuzzy DEA and, particularly, CCR models with the fuzzy data. We established that by increasing the confidence level of $\alpha$, the amount of technical efficiency will increase, as well. This conclusion is in line with the concept of the credibility level. Then, the discussed matters on the fuzzy CCR were used for determining the MPI with the fuzzy data. In a particular case that inputs and outputs are trapezoidal fuzzy numbers and the credibility levels are available, the utilized models for calculating the MPI were relaxed to linear programs. All of the discussed matters were accompanied with one empirical example. Measuring the MPI with stochastic data can be done as an interesting work in future.

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