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# Approximate Solutions of Set-Valued Stochastic Differential Equations

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**Abstract**

In this paper, we consider the problem of approximate solutions of set-valued stochastic differential equations. We firstly prove an inequality of set-valued Itô integrals, which is related to classical Itô isometry, and an inequality of set-valued Lebesgue integrals. Both of the inequalities play an important role to discuss set-valued stochastic differential equations. Then we mainly state the Carathodory's approximate method and the Euler-Maruyama's approximate method for set-valued stochastic differential equations. We also investigate the errors between approximate solutions and accurate solutions.


**Keywords:** set-valued random variables, set-valued stochastic processes, set-valued stochastic differential equations, the Carathodory's approximate solutions, the Euler-Maruyama's approximate solutions

## 1 Introduction

It is well-known that the theory of classical stochastic differential equations has been developed deeply with wide applications in many fields (e.g. [3, 7, 13]). A stochastic differential equation can be written as

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), \tag{1.1}$$

or its integral form

$$x(t) = x(0) + \int_0^t f(s, x(s))ds + \int_0^t g(s, x(s))dB(s),$$

where $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are drift and diffusion coefficients respectively, $\{B(t) : 0 \le t \le T\}$ is an $m$-dimensional Brownian motion.

Recently, the discussion of set-valued stochastic differential equations is becoming one of important topics since the measurements of various uncertainties arise not only from the randomness but also from the impreciseness in some situations. For example, a general option pricing model can be written as

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dB(t)$$

with $S(0) = s_0$, $S(t)$ is the stock price at time $t$, $\mu(t)$, $\sigma(t)$ are given predictable processes, called expected return rate and volatility of the stock price respectively. However, both of $\mu(t)$ and $\sigma(t)$ are difficult to obtain since the factors of affecting the market are too complex. But it is relatively easy to estimate their lower and upper bounds. So we have to discuss the set-valued model

$$dS(t) = US(t)dt + V S(t)dB(t),$$

where $\mu(t) \in [a, b] := U$, $\sigma(t) \in [c, d] := V$ (cf. [15]). This leads us consider the following general set-valued stochastic differential equation

$$dX(t) = F(t, X(t))dt + G(t, X(t))dB(t),$$

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or set-valued stochastic integral equation

$$X(t) = X(0) + \int_0^t F(s, X(s))ds + \int_0^t G(s, X(s))dB(s), \tag{1.2}$$

where $X(t), F(t, X(t)), G(t, X(t))$ are set-valued stochastic processes.

In the set-valued stochastic integral equation (1.2), there are two kinds of integrals. The first one is the set-valued Lebesgue integral of a set-valued stochastic process with respect to time t. For the definitions and properties of set-valued Lebesgue integrals of set-valued stochastic processes, readers may refer to [8, 10, 18] and the references therein. The second integral is the set-valued Itô integral of a set-valued process with respect to a Brownian motion. Kisielewicz [4] first introduced the concept of set-valued stochastic integral by using selection method in 1993. It is generalized concept of the classical stochastic Itô integral. Following his work, there are some results on set-valued differential inclusions and their applications in stochastic control and mathematical economics (e.g. [5, 6, 15]).

However, Kisielewicz's definition of set-valued stochastic integrals is not very satisfactory to discuss set-valued stochastic differential equations, because one can not prove that the set-valued stochastic integral $\xi(t, \omega) =: \int_0^t G(s, \omega)dB(s)$ is an adapted set-valued stochastic process even when $\{G(t)\}$ is a predictable set-valued stochastic process. Jung and Kim [2] modified Kisielewicz's definition and solved the above problem when the basic space is $\mathbb{R}$. Moreover, Zhang et. al [17] slightly modified the definition of [2] again when the basic space is a separable M-type 2 Banach space. In this paper, we shall give an inequality of set-valued stochastic integral (cf. Theorem 1), which is related to classical Itô isometry, and prove an inequality of set-valued Lebesgue integrals. Both of the inequalities play an important role to discuss set-valued stochastic differential equations. This is the first aim of this paper.

Based on the definitions and properties of [9, 10, 18] and the assumptions of set-valued drift coefficient and single-valued diffusion coefficient in (1.2), Li and Li [9], Li et al. [10] discussed set-valued Itô differential (or integral) equations in the Euclidean space $\mathbb{R}^d$ and Zhang et al. [18] investigated set-valued stochastic differential equations in the M-type 2 Banach space. By using the definitions of set-valued Lebesgue integrals in [18] and set-valued Itô stochastic integrals in [17], we discussed the existence and uniqueness of solutions of set-valued stochastic integral equations with set-valued drift and diffusion coefficients in [16].

We would like to metion that Malinowski and Michta [12] and Michta [14] did some work on set-valued stochastic differential equations whose drift and diffusion coefficients are set-valued mappings, by using Kisielewicz's definition of set-valued Itô integral. But they could not prove that the solution of their set-valued stochastic differential equation is adapted, which is a very important point as we have pointed out before.

Ever for classical stochastic differential equations, however, there are only a few types having accurate solutions. So it is necessary to consider numerical solutions of stochastic differential equations. Thus it is important to seek various approximate methods in many models for practical uses (e.g. [13] and its references). For the same reason, in order to develop applications of set-valued stochastic differential equations, we shall investigate the approximate solutions of set-valued stochastic differential equations, which is the second aim of this paper.

We organize our paper as follows: in Section 2, we shall prove an inequality of set-valued stochastic Itô integrals and an inequality of set-valued Lebesgue integrals. Besides, we shall prove some other properties of set-valued Itô integrals and set-valued Lebesgue integrals, and recall set-valued stochastic integral equations. In Section 3, we shall introduce the Caratheodory's approximate solutions and investigate how to seek the size of iterative step and iterative times by the error (distance between approximate solution and accurate solution in some sense). Besides, we shall provide with the Euler-Maruyama's approximate solutions and have a discussion similarly.

## 2 Set-Valued Stochastic Differential (or Integral) Equations

Throughout this paper, assume that $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^+ = [0, \infty)$, $I = [0, T]$, $\mathbb{R}^d$ is the $d$-dimensional Euclidean space with usual norm $\|\cdot\|$, $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a separable Banach space, $\mathcal{B}(\mathbf{E})$ is the Borel field of the space $\mathbf{E}$, $\lambda$ is the Lebesgue measure on $(I, \mathcal{B}(I))$.

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, $\mathcal{A}$ be separable respect to $P$, the $\sigma$-field filtration $\{\mathcal{A}_t : t \in I\}$ satisfy the usual conditions (i.e. complete, non-decreasing and right continuous), $\{B(t) : t \in I\}$ be a Brownian motion, $L^p(\Omega, \mathcal{A}, P; \mathbf{E})$ be the set of all $\mathbf{E}$-valued random variables with finite $p$-order moments ($p \geq 1$).

Assume that $\mathbf{K}(\mathcal{X})$ is the family of all nonempty closed subsets of $\mathcal{X}$, and $\mathbf{K}_{b(c)}(\mathcal{X})$ is the family of all nonempty bounded closed (convex) subsets of $\mathcal{X}$. For any $x \in \mathcal{X}$ and $A \in \mathbf{K}(\mathcal{X})$, the metric between $x$ and $A$ is defined by

$$d(x, A) = \inf_{y \in A} \|x - y\|_{\mathcal{X}}.$$

For any $A, B \in \mathbf{K}(\mathcal{X})$, Hausdorff metric between $A$ and $B$ is defined as

$$H_{\mathcal{X}}(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

In particular, denote $\|A\|_K = H_{\mathcal{X}}(A, \{0\})$. Then $\mathbf{K}_b(\mathcal{X})$ and $\mathbf{K}_{bc}(\mathcal{X})$ are complete spaces with respect to $H_{\mathcal{X}}$ (Theorem 1.1.2 of [11]).

From now on, we focus on the discussion in $\mathbb{R}^d$, which is easy to extend to a separable Banach space or an M-type 2 Banach space as authors have done in [1, 11, 17, 18] respectively.

A set-valued mapping $F : \Omega \to \mathbf{K}(\mathbb{R}^d)$ is called a set-valued random variable if

$$\{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}, \quad \text{for any open set } O \text{ of } \mathbb{R}^d.$$

For the equivalent definitions of a set-valued random variable, readers may refer to Theorems 1.2.3 and 1.2.7 of [11].

An $\mathbb{R}^d$-valued function $f : \Omega \to \mathbb{R}^d$ is called an almost everywhere selection of $F$ if $f(\omega) \in F(\omega)$ for almost everywhere $\omega \in \Omega$, and $F$ is said to be $L^p$-integrably bounded, if $\|F\|_K \in L^p(\Omega, \mathcal{A}, P; \mathbb{R}^+)$.

Let $\mathcal{M}$ be a set of $\mathcal{A}$-measurable mappings $f : \Omega \to \mathbb{R}^d$, $\mathcal{M}$ is called decomposable, if for every $f_1, f_2 \in \mathcal{M}$ and every $A \in \mathcal{A}$ such that $f_1 I_A + f_2 I_{\Omega \setminus A} \in \mathcal{M}$. We have that the $p$-order integral selection set of an $L^p$-integrably bounded set-valued random variable $F$

$$S_F^p = \{f \in L^p(\Omega, \mathcal{A}, P; \mathbb{R}^d) : f(\omega) \in F(\omega) \ a.s. \ (P)\}$$

is nonempty, decomposable with respect to $\mathcal{A}$ and closed in $L^p(\Omega, \mathcal{A}, P; \mathbb{R}^d)$. Conversely, if a given nonempty subset $\Gamma \subset L^p(\Omega, \mathcal{A}, P; \mathbb{R}^d)$ is decomposable with respect to $\mathcal{A}$ and closed, can we find a set-valued random variable $F$ such that its selection set $S_F^p = \Gamma$? The answer is positive given by Hiai and Umegaki in [1]. Readers also may refer to Theorem 1.3.9 of [11].

**Proposition 1** *If $\Gamma$ is a nonempty closed subset of $L^p(\Omega, \mathcal{A}, P; \mathbb{R}^d)$, then there exists a set-valued random variable $F$ such that $\Gamma = S_F^p$ if and only if $\Gamma$ is decomposable with respect to $\mathcal{A}$. Moreover, $\Gamma$ is bounded in $L^p(\Omega, \mathcal{A}, P; \mathbb{R}^d)$ if and only if $F$ is $L^p$-integrably bounded.*

Let $\mathcal{A}_0$ be a sub-algebra of $\mathcal{A}$. Then the $p$-order integral selection set of $F$ with respect to $\mathcal{A}_0$ is denoted as

$$S_F^p(\mathcal{A}_0) = \{f \in L^p(\Omega, \mathcal{A}_0, P; \mathbb{R}^d) : f(\omega) \in F(\omega) \ a.s.(P)\},$$

where $L^p(\Omega, \mathcal{A}_0, P; \mathbb{R}^d)$ is the set of all $\mathbb{R}^d$-valued $\mathcal{A}_0$-measurable random variables with finite $p$-order moments. Let $\mathcal{N}$ be the $\sigma$-algebra of the progressive events in $I \times \Omega$, i.e.

$$\mathcal{N} = \{A \subset I \times \Omega : A \cap ([0, t] \times \Omega) \in \mathcal{B} \times \mathcal{A}_t, \quad \text{for every} \ \ t \in I\}.$$

A $d$-dimensional stochastic process $f : I \times \Omega \to \mathbb{R}^d$ is called progressive measurable if $f$ is $\mathcal{N}$-measurable. Let $\mathcal{L}^2(\mathbb{R}^d) = L^2(I \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}^d)$ be the set of all $\mathbb{R}^d$-valued progressive processes $f = \{f(t)\}_{t \in I}$ with

$$E[\int_0^T \|f(t)\|^2 dt] < \infty.$$

Next, we shall recall some definitions of set-valued stochastic processes and set-valued stochastic Itô integrals.

The mapping $F : I \times \Omega \to \mathbf{K}(\mathbb{R}^d)$ is called a set-valued stochastic process, if for any $t \in I$, $F(t) : \Omega \to \mathbf{K}(\mathbb{R}^d)$ is a set-valued random variable. Similarly, we have the concepts of set-valued progressive stochastic processes and set-valued progressive $L^p$-integrably bounded stochastic processes, denoted by $\mathcal{L}^p(\mathbf{K}(\mathbb{R}^d))$. If a process $F : I \times \Omega \to \mathbf{K}(\mathbb{R}^d)$ is progressive $L^2$-integrably bounded, then its selection set

$$S_F^2(\mathcal{N}) = \{f \in \mathcal{L}^2(\mathbb{R}^d) : f(t,\omega) \in F(t,\omega), \quad a.s.(\lambda \times P)\}$$

is nonempty and closed. Then we can define

$$\Gamma_t = \{\int_0^t f(s)dB(s) : f \in S_F^2(\mathcal{N})\}, \quad t \in I.$$

We use $L^2$ and $L_t^2$ briefly to denote $L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$ and $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ respectively. Let $\overline{de}\Gamma_t$ be the decomposable closed hull of $\Gamma_t$ with respect to $\mathcal{A}_t$, where the closure is taken in $L_t^2$. That is, for any $g \in \overline{de}\Gamma_t$ and any given $\varepsilon > 0$, there exists a finite $\mathcal{A}_t$-measurable partition $A_1, ..., A_m$ of $\Omega$ and $f_1, ..., f_m \in S_F^2(\mathcal{N})$ such that

$$||g - \sum_{i=1}^m I_{A_i} \int_0^t f_i(s)dB(s)||_{L^2} < \varepsilon.$$

By using Proposition 1, we can obtain the following result (cf. [17]).

**Proposition 2** *If $F : I \times \Omega \to \mathbf{K}(\mathbb{R}^d)$ is a progressive and $L^2$-integrably bounded set-valued process, then for any $t \in I$, there exists an $\mathcal{A}_t$-measurable set-valued random variable $I_0^t(F)$ such that $S^2_{I_0^t(F)}(\mathcal{A}_t) = \overline{de}\Gamma_t$.*

Based on Proposition 2, the following concept of set-valued Itô integral was introduced in [17].

**Definition 1** *For each $t \in I$, the element $I_0^t(F)$ of $\mathbf{K}(\mathbb{R}^d)$ is called Itô integral of the set-valued process $F = \{F_t : t \in I\}$ with respect to a Brownian motion $\{B(t) : t \in I\}$, if $S^2_{I_0^t(F)}(\mathcal{A}_t) = \overline{de}\Gamma_t$, and denoted by $I_0^t(F) = \int_0^t F(s)dB(s)$. For $0 \le u < t \le T$, $\int_u^t F(s)dB(s) = \int_0^t I_{[u,t]}F(s)dB(s)$.*

The following results of this section will play an important role in Section 3. Before showing them, we state the Castaing Representation Theorem of set-valued Itô integrals.

**Lemma 1** [17] *For a progressive and $L^2$-integrably bounded set-valued process $F : I \times \Omega \to \mathbf{K}(\mathbb{R}^d)$, there exists a sequence $\{(f_t^i)_{t\in I} : i = 1, 2, \dots\} \subseteq S_F^2(\mathcal{N})$ such that for each $t \in I$,*

$$F(t,\omega) = \mathrm{cl}\{f_t^i(\omega) : i = 1, 2, \dots\} \ a.s.$$

*and*

$$I_0^t(F)(\omega) = \mathrm{cl}\left\{\int_0^t f_s^i(\omega)dB_s(w) : i = 1, 2, \dots\right\} \quad a.s..$$

**Theorem 1** *If two set-valued stochastic processes $F_1, F_2 \in \mathcal{L}^2(\mathbf{K}(\mathbb{R}^d))$, then*
*(i) for every $u, t \in I,\ u < t$, the following inequality holds*

$$H_{L^2}(J_u^t(F_1), J_u^t(F_2)) \le \Big(\int_{[u,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1(s,\omega), F_2(s,\omega))ds \times dP\Big)^{1/2},$$

*where $H_{L^2}$ is Hausdorff metric on $\mathbf{K}(L^2)$;*
*(ii) for every $u, s, t \in I,\ u \le s \le t$, the following equality holds*

$$I_u^t(F_1) = \mathrm{cl}\{I_u^s(F_1) + I_s^t(F_1)\}, \quad a.s.;$$

*(iii) the mapping $I_u^t(F_1) : [u,T] \to \mathbf{K}(\mathbb{R}^d)$ is continuous in $t$ in the sense of $L^2$ for the set-valued case.*

**Proof:** (i) By Lemma 1, for $j = 1, 2$, there exists a sequence $\{(f_t^{ji})_{t\in I} : i = 1, 2, \dots\} \subseteq S_F^2(\mathcal{N})$ such that for each $u, t \in I,\ u < t$, $F_j(t,\omega) = \mathrm{cl}\{(f_t^{ji}(\omega) : i = 1, 2, \dots\}$ $a.s.$ and

$$I_u^t(F_j)(\omega) = \mathrm{cl}\{\int_u^t f_s^{ji}(\omega)dB_s(w) : i = 1, 2, \dots\} \quad a.s..$$

For fixed $f^{1,i}$, we can choose a subsequence $\{f^{2i_k} : k \geq 1\}$ such that

$$d(f^{1i}, f^{2i_k}) \to d(f^{1i}, F_2), a.s.(\lambda \times P), k \to \infty.$$

So we have $d^2(f^{1i}, f^{2i_k}) \to d^2(f^{1i}, F_2), a.s.(\lambda \times P), k \to \infty$. Since

$$d^2(f^{1i}, f^{2i_k}) \leq 2(\|F_1\|^2_{\mathbf{K}} + \|F_2\|^2_{\mathbf{K}}), a.s.(\lambda \times P),$$

and $F_1, F_2$ is $L^2$-integrably bounded, by using dominated convergence theorem, we obtain

$$\int_{[u,t]\times\Omega} d^2(f^{1i}(s), f^{2i_k}(s))ds \times dP \to \int_{[u,t]\times\Omega} d^2(f^{1i}(s), F_2(s))ds \times dP.$$

Thus,

$$\inf_{k\geq 1} \int_{[u,t]\times\Omega} d^2(f^{1i}(s), f^{2i_k}(s))ds \times dP \leq \int_{[u,t]\times\Omega} d^2(f^{1i}(s), F_2(s))ds \times dP.$$

Due to $\int_u^t f_s^{1i}(\omega)dB_s(\omega) \in J_u^t(F_1)$, we have

$$\begin{aligned} & d^2_{L^2}(\textstyle\int_u^t f_s^{1i}(\omega)dB_s(\omega), J_u^t(F_2)) \\ \leq\ & \inf_{k\geq 1} d^2_{L^2}(\textstyle\int_u^t f_s^{1i}(\omega)dB_s(\omega), \int_u^t f_s^{2i_k}(\omega)dB_s(\omega)) \\ =\ & \inf_{k\geq 1} || \textstyle\int_u^t (f_s^{1i}(\omega) - f_s^{2i_k}(\omega))dB_s(\omega)||^2_{L^2} \\ =\ & \inf_{k\geq 1} \textstyle\int_{[u,t]\times\Omega} (f^{1i}(s) - f^{2i_k}(s))^2 ds \times dP \\ \leq\ & \textstyle\int_{[u,t]\times\Omega} d^2(f^{1i}(s), F_2(s))ds \times dP. \end{aligned}$$

So, we get

$$\sup_{i\geq 1} d^2_{L^2}(\int_u^t f_s^{1i}(\omega)dB_s(\omega), J_u^t(F_2)) \leq \int_{[u,t]\times\Omega} \sup_{i\geq 1} d^2(f^{1i}(s), F_2(s))ds \times dP.$$

Similarly, we can show that

$$\sup_{i\geq 1} d^2_{L^2}(\int_u^t f_s^{2i}(\omega)dB_s(\omega), J_u^t(F_1)) \leq \int_{[u,t]\times\Omega} \sup_{i\geq 1} d^2(f^{2i}(s), F_1(s))ds \times dP.$$

Hence, we have

$$H^2_{L^2}(J_u^t(F_1), J_u^t(F_2)) \leq \int_{[u,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1, F_2)ds \times dP.$$

(ii) By Lemma 1, there exists a sequence $\{(f_t^i)_{t\in I} : i = 1, 2, \ldots\} \subseteq S^2_F(\mathcal{N})$ such that for each $u, s, t \in I,\ u \leq s < t$,

$$I_u^t(F)(\omega) = \text{cl}\left\{\int_u^t f_r^i(\omega)dB_r(w) : i = 1, 2, \ldots\right\} \quad a.s.,$$

$$I_u^s(F)(\omega) = \text{cl}\left\{\int_u^s f_r^i(\omega)dB_r(w) : i = 1, 2, \ldots\right\} \quad a.s.,$$

$$I_s^t(F)(\omega) = \text{cl}\left\{\int_s^t f_r^i(\omega)dB_r(w) : i = 1, 2, \ldots\right\} \quad a.s..$$

Thus

$$I_u^t(F) \subseteq \text{cl}\{I_u^s(F) + I_s^t(F)\}, \quad a.s..$$

Conversely, for any $a \in I_u^s(F) + I_s^t(F)$, it follows from that, for any $\varepsilon > 0$, there exists $m(\varepsilon), n(\varepsilon)$ such that

$$\left\|a - (\int_u^s f_r^{m(\varepsilon)}(\omega)dB_r(w) + \int_s^t f_r^{n(\varepsilon)}(\omega)dB_r(w))\right\|_{L^2} < \varepsilon.$$

Or equivalently,

$$\begin{aligned} & \left\|a - \left(\int_u^s f_r^{m(\varepsilon)}(\omega)dB_r(w) + \int_s^t f_r^{n(\varepsilon)}(\omega)dB_r(w)\right)\right\|_{L^2} \\ =\ & \left\|a - \left(\int_u^t \left(I_{[u,s]} f_r^{m(\varepsilon)}(\omega) + I_{[s,t]} f_r^{n(\varepsilon)}(\omega)\right)dB_r(w)\right)\right\|_{L^2} \\ <\ & \varepsilon. \end{aligned}$$

Since $\int_u^t \Big(I_{[u,s]} f_r^{m(\varepsilon)}(\omega) + I_{[s,t]} f_r^{n(\varepsilon)}(\omega)\Big) dB_r(w) \in I_u^t(F)$ and $I_u^t(F)$ is closed, we have $a \in I_u^t(F)$. Hence

$$I_u^t(F) = \text{cl}\{I_u^s(F) + I_s^t(F)\}, \quad a.s..$$

(iii) Using (i) and (ii) we get that, for any $s, t \in I,\ s \leq t$,

$$\begin{aligned} H_{L^2}^2(I_0^t(F), I_0^s(F)) &= H_{L^2}^2(I_s^t(F) + I_0^s(F), I_0^s(F)) \\ &\leq H_{L^2}^2(I_s^t(F), \{0\}) \\ &\leq \int_{[s,t]\times\Omega} \|F\|_K^2 ds \times dP. \end{aligned}$$

Since $F$ is $L^2$-integrably bounded, $H_{L^2}^2(I_0^t(F), I_0^s(F))$ converges to 0 as $t - s$ goes to 0, which yields the continuity of $I_u^t(F)$. The proof is completed. □

Now we recall the definition of set-valued Lebesgue integral given in [18]. Take a set-valued progressive $L^2$-integrably bounded stochastic process $F : I \times \Omega \to \mathbf{K}(\mathbb{R}^d)$, $L_u^t(F) := \int_u^t F(s)ds$ is called set-valued Lebesgue integral of $F$, if

$$S^2_{L_u^t(F)}(\mathcal{A}_t) = \overline{de}\left\{\int_u^t f(s)ds : f \in S_F^2(\mathcal{N})\right\}, \quad u, t \in I,\ u < t.$$

Similar to Theorem 1 and its proof, we have the following results.

**Theorem 2** *If two set-valued stochastic processes $F_1, F_2 \in \mathcal{L}^2(\mathbf{K}(\mathbb{R}^d))$, then*
*(i) for every $u, t \in I,\ u < t$, the following inequality holds*

$$H_{L^2}^2(L_u^t(F_1), L_u^t(F_2)) \leq (t-u) \int_{[u,t]\times\Omega} H_{\mathbb{R}^d}^2(F_1(s,\omega), F_2(s,\omega)ds \times dP;$$

*(ii) for every $u, s, t \in I$ with $u \leq s \leq t$, it holds $L_u^t(F_1) = \text{cl}\{L_u^s(F_1) + L_s^t(F_1)\}$, a.s.;*
*(iii) $L_u^t(F_1)$ is continuous in $t$ in the sense of $L^2$ for set-valued case.*

In the following we state the existence and uniqueness of solutions of set-valued stochastic integral equations. Let $F, G : I \times \mathbf{K}(\mathbb{R}^d) \to \mathbf{K}(\mathbb{R}^d)$ be the elements of $\mathcal{L}^2(\mathbf{K}(\mathbb{R}^d))$. We consider the following set-valued stochastic integral equation

$$X(t) = X_0 + L_0^t(F(s, X(s))) + I_0^t(G(s, X(s))),\ \forall\ t \in I. \tag{2.1}$$

**Theorem 3** [16] *Assume that $F(t, X)$ and $G(t, X)$ satisfy the following conditions:*
*(A1) Linear growth condition: for every $X \in \mathbf{K}_b(L^2)$, there exists constant $K > 0$ such that*

$$E(H_{\mathbb{R}^d}^2(F(t,X), \{0\}) + H_{\mathbb{R}^d}^2(G(t,X), \{0\})) \leq K(1 + \|X\|_K^2), \quad \forall\ t \in I$$

*where $\|X\|_K = H_{L^2}(X, \{0\})$.*
*(A2) Lipschitz continuous condition: for every $X, Y \in \mathbf{K}_b(L^2)$, there exists constant $\overline{K} > 0$ such that*

$$E(H_{\mathbb{R}^d}^2(F(t,X), F(t,Y)) + H_{\mathbb{R}^d}^2(G(t,X), G(t,Y))) \leq \overline{K} H_{L^2}^2(X, Y), \quad \forall\ t \in I.$$

*Then for any $X_0 \in \mathbf{K}_b(L_0^2)$, there exists a unique adapted set-valued solution $\{X(t)\}$ of equation (2.1) and $\{X(t)\}$ is continuous in $t$ in the sense of $L^2$ for set-valued case.*

# 3 Approximate Solutions

In the previous section we have established existence and uniqueness of solutions of set-valued stochastic differential equations (2.1). However, in general, the solutions do not have explicit formulas. In practice, we therefore seek the approximate solutions. We call constant $\varepsilon > 0$ is error, if the following inequality holds

$$\sup_{t\in I} H_{L^2}^2(X_n(t), X(t)) < \varepsilon,$$

where $H^2_{L^2}(X_n(t), X(t))$ represents the $L^2$ distance between the approximate solution $X_n(t)$ and accurate solution $X(t)$.

In the proof of the existence and uniqueness of solutions for set-valued stochastic differential equations, we use the Picard iterative in [16]. Hence, we can use it to find approximate solution $X_n(t)$. But it has a disadvantage that for any $t$, for each $n$, one has to compute all $X_0(t), X_1(t), \ldots, X_{n-1}(t)$ in order to compute $X_n(t)$, and it involves a lot of calculations. More efficient ways are the Caratheodory's and the Euler-Maruyama's approximate methods because they do not need to compute all $X_1(t), \ldots, X_{n-1}(t)$ but compute $X_n(t)$ directly by using the results of previous calculations before $t$. Now we will discuss them in the following.

## 3.1 The Caratheodory's Approximate Solutions

Now we introduce the Caratheodory's approximate solutions. For every integer $n \geq 1$, define $X_n(t) = X_0$ for $-1 \leq t \leq 0$, and $\forall\ t \in (0, T]$, let

$$X_n(t) = X_0 + \int_0^t F(s, X_n(s - 1/n))ds + \int_0^t G(s, X_n(s - 1/n))dB(s). \tag{3.1}$$

Note that for $0 \leq t \leq 1/n$, $X_n(t)$ can be computed by

$$X_n(t) = X_0 + \int_0^t F(s, X_0)ds + \int_0^t G(s, X_0)dB(s),$$

then for $1/n < t \leq 2/n$,

$$X_n(t) = X_n(1/n) + \int_{1/n}^t F(s, X_n(s - 1/n))ds + \int_{1/n}^t G(s, X_n(s - 1/n))dB(s)$$

and so on. In other words, $X_n(t)$ can be computed step-by-step on the intervals $[0, 1/n], (1/n, 2/n], \ldots$.

In calculations, for any given $\varepsilon > 0$, how to decide at which step $n$ we may stop the computations? That is how to find the size of iterative steps and iterative times so that the $L^2$-distance between the approximate solution and accurate solution is less than any given $\varepsilon > 0$. Our main result will answer this question. Firstly, we introduce two Lemmas in order to establish the main result.

**Lemma 2** *Assume that linear growth condition (A1) holds, then for all $n \geq 1$, we have*

$$\sup_{t \in I} \|X_n(t)\|_K^2 \leq C_1 := (1 + 3\|X_0\|_K^2)e^{3K(T+1)T}. \tag{3.2}$$

**Proof:** By using Theorems 1 and 2, and the fact that $X_n(t)$ satisfies equation (3.1), we have

$$\begin{aligned}
\|X_n(t)\|_K^2 &\leq 3\|X_0\|_K^2 + 3\Big\| \textstyle\int_0^t F(s, X_n(s - 1/n))ds\Big\|_K^2 \\
&\quad + 3\Big\| \int_0^t G(s, X_n(s - 1/n))dB(s)\Big\|_K^2 \\
&\leq 3\|X_0\|_K^2 + 3t \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(s, X_n(s - 1/n)), \{0\})ds \times dP \\
&\quad + 3\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(s, X_n(s - 1/n)), \{0\})ds \times dP \\
&\leq 3\|X_0\|_K^2 + 3K(t + 1)\int_0^t (1 + \|X_n(s - 1/n)\|_K^2)ds \\
&\leq 3\|X_0\|_K^2 + 3K(T + 1)\int_0^t (1 + \sup_{r \in [0,s]} \|X_n(r)\|_K^2)ds
\end{aligned}$$

for all $t \in I$. Consequently, we have

$$1 + \sup_{r \in [0,t]} \|X_n(r)\|_K^2 \leq 1 + 3\|X_0\|_K^2 + 3K(T + 1)\int_0^t (1 + \sup_{r \in [0,s]} \|X_n(r)\|_K^2)ds.$$

Then the Gronwall inequality (cf. Page 45 of [13]) implies

$$\sup_{r\in[0,t]}\|X_n(r)\|_K^2 \leq (1+3\|X_0\|_K^2)e^{3K(T+1)t}$$

for all $t \in I$. In particular, (3.2) follows when $t = T$. □

**Lemma 3** *Assume that linear growth condition (A1) holds, then for all $n \geq 1$, and $0 \leq s < t \leq T$ with $t - s \leq 1$, we have*

$$H_{L^2}^2(X_n(t), X_n(s)) \leq C_2(t-s),$$

*where $C_2 = 4K(1 + C_1)$ and $C_1$ is defined in Lemma 2.*

**Proof:** By virtue of Theorems 1 and 2 and inequality $H_{\mathcal{X}}(A+B, C+D) \leq H_{\mathcal{X}}(A, C) + H_{\mathcal{X}}(B, D)$, for $A, B, C, D \in \mathbf{K}(\mathcal{X})$ (cf. Lemma1.1.11 of [11]). we have

$$\begin{aligned}
H_{L^2}^2(X_n(t), X_n(s)) &\leq 2H_{L^2}^2\Big(\int_0^t F(r, X_n(r-1/n))dr, \int_0^s F(r, X_n(r-1/n))dr\Big)\\
&\quad +2H_{L^2}^2\Big(\int_0^t G(r, X_n(r-1/n))dB(r), \int_0^s G(r, X_n(r-1/n))dB(r)\Big)\\
&\leq 2H_{L^2}^2\Big(\int_s^t F(r, X_n(r-1/n))dr, \{0\}\Big)\\
&\quad +2H_{L^2}^2\Big(\int_s^t G(r, X_n(r-1/n))dB(r), \{0\}\Big)\\
&\leq 2(t-s)\int_{[s,t]\times\Omega} H_{\mathbb{R}^d}^2(F(s, X_n(s-1/n)), \{0\})dr \times dP\\
&\quad +2\int_{[s,t]\times\Omega} H_{\mathbb{R}^d}^2(G(s, X_n(s-1/n)), \{0\})dr \times dP\\
&\leq 2K(t-s+1)\int_{[s,t]} (1 + \|X_n(r-1/n)\|_K^2)dr\\
&\leq 4K(1+C_1)(t-s),
\end{aligned}$$

where in the last inequality we used the assumption $t - s \leq 1$ and Lemma 2. The proof is completed. □

**Theorem 4** *Assume that the linear growth condition (A1) and the Lipschitz condition (A2) hold, $X(t)$ is the unique solution of equation (2.1). Then, for $n \geq 1$,*

$$\sup_{t\in I} H_{L^2}^2(X_n(t), X(t)) \leq \frac{C_3}{n},$$

*where $C_3 = 4C_2\overline{K}T(T+1)\exp[4\overline{K}T(T+1)]$ and $C_2$ is defined in Lemma 3.*

**Proof:** By the Lipschitz condition (A2), Theorems 1 and 2, it is not difficult to obtain

$$\begin{aligned}
&H_{L^2}^2(X_n(t), X(t))\\
\leq\ & 2(t+1)\overline{K}\int_0^t H_{L^2}^2(X_n(r-1/n), X(r))dr\\
\leq\ & 4\overline{K}(T+1)\int_0^t (H_{L^2}^2(X_n(r-1/n), X_n(r)) + H_{L^2}^2(X_n(r), X(r)))dr,
\end{aligned}$$

where in last inequality we used

$$H_{\mathcal{X}}(A, B) = H_{\mathcal{X}}(A+C, C+B) \leq H_{\mathcal{X}}(A, C) + H_{\mathcal{X}}(B, C)$$

for any $A, B, C \in \mathbf{K}(\mathcal{X})$. Then using Lemma 3, we have $H_{L^2}^2(X_n(r-1/n), X_n(r)) \leq C_2/n$ if $r \geq 1/n$, otherwise if $0 \leq r < 1/n$, $H_{L^2}^2(X_n(r-1/n), X_n(r)) = H_{L^2}^2(X_n(0), X_n(r)) \leq C_2 r \leq C_2/n$. Hence, we have

$$\begin{aligned}
&\sup\nolimits_{r\in[0,t]} H_{L^2}^2(X_n(r), X(r))\\
\leq\ & (4/n)C_2\overline{K}T(T+1) + 4\overline{K}(T+1)\int_0^t \sup_{r\in[0,s]} H_{L^2}^2(X_n(r), X(r))ds.
\end{aligned}$$

Finally, by applying the Gronwall inequality, we have

$$\sup_{t\in I} H^2_{L^2}(X_n(t), X(t)) \leq \frac{C_3}{n},$$

and the proof is completed. □

Now we can make sure $n$, such that $n$ is an integer larger than $C_3/\varepsilon$. Then we can compute $X_n(t)$ over the intervals $[0,1/n],(1/n,2/n],\ldots$, step by step. Theorem 4 can guarantee that the approximate solution $X_n(t)$ is closed enough to the accurate solution $X(t)$ uniformly, that is

$$\sup_{t\in I} H^2_{L^2}(X_n(t), X(t)) < \varepsilon.$$

## 3.2 The Euler-Maruyama's Approximate Solutions

The Euler-Maruyama's approximate solution is defined as follows: for every integer $n \geq 1$, define $X_n(0) = X_0$, and then for $(k-1)/n < t \leq k/n \wedge T$, $k = 1, 2, \ldots$,

$$X_n(t) = X_n\big(\frac{k-1}{n}\big) + \int_{\frac{k-1}{n}}^{t} F\big(s, X_n\big(\frac{k-1}{n}\big)\big)ds + \int_{\frac{k-1}{n}}^{t} G\big(s, X_n\big(\frac{k-1}{n}\big)\big)dB(s).$$

Then define

$$\hat{X}_n(t) = X_0 I_{\{0\}}(t) + \sum_{k\geq 1} X_n\big(\frac{k-1}{n}\big) I_{((k-1)/n,k/n]}(t)$$

for $t \in I$, then we have

$$X_n(t) = X_n(0) + \int_0^t F(s, \hat{X}_n(s))ds + \int_0^t G(s, \hat{X}_n(s))dB(s),\ t \in I.$$

Similar to the Caratheodory's approximate solutions, we now investigate how to make sure the size of iterative step and iterative times. We have the following results.

**Lemma 4** *Assume that linear growth condition (A1) holds, then for all $n \geq 1$, the Euler-Maruyama's approximate solutions $X_n(t)$ satisfy*

$$\sup_{t\in I} \|X_n(t)\|_K \leq C_4 := (1 + 3\|X_0\|^2_K)e^{3K(T+1)T}.$$

**Lemma 5** *Assume that linear growth condition (A1) holds, for all $n \geq 1$, and $0 \leq s < t \leq T$ with $t - s \leq 1$, the Euler-Maruyama's approximate solutions $X_n(t)$ satisfy*

$$H^2_{L^2}(X_n(t), X_n(s)) \leq C_5(t - s),$$

*where $C_5 = 8K(1 + C_4)$ and $C_4$ is defined in Lemma 4.*

**Theorem 5** *Assume that the linear growth condition (A1) and the Lipschitz condition (A2) hold, and $X(t)$ is the unique set-valued solution of equation (2.1). Then, for $n \geq 1$, the Euler-Maruyama's approximate solutions $X_n(t)$ satisfy*

$$\sup_{t\in I} H^2_{L^2}(X_n(t), X(t)) \leq \frac{C_6}{n},$$

*where $C_6 = 4C_5\overline{K}T(T + 1)\exp[4\overline{K}T(T + 1)]$ and $C_5$ is defined in Lemma 5.*

The proofs of above Lemmas 4 and 5 and Theorem 5 are omitted because of the similarity of the proofs of Lemmas 2 and 3 and Theorem 4.

# 4 Conclusions

It is necessary to consider set-valued differential (or integral) equations because the measurements of various uncertainties arise not only from randomness but also from the impreciseness in many situations. In this paper, we firstly prove an inequality of set-valued Itô integrals and an inequality of set-valued Lebesgue integrals. Both of the inequalities play an important role to discuss set-valued stochastic differential equations. In order to develop applications of set-valued stochastic differential equations, it is necessary to study approximate solutions of set-valued stochastic differential equations. In our paper, we introduced the Caratheodory's approximate solutions and the Euler-Maruyama's approximate solutions of set-valued stochastic differential equations. Besides, we gave how to make sure their the size of iterative step and iterative times.

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