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On Generalized Intuitionistic Fuzzy Divergence (Relative Information) and Their Properties

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Abstract

Atanassov (1986) defined the notion of intuitionistic fuzzy sets, which is a generalization of the concept of fuzzy sets, introduced by the Zadeh (1965). In this paper we introduce divergence (relative information) measure, a kind of a discrimination measure, in the setting of intuitionistic fuzzy set theory. This measure is a generalized version of intuitionistic fuzzy divergence proposed by Wei and Ye (2010), having a flexibility parameter. Some properties of this measure and its applications bringing out the crucial role of the parameter in decision making problems under multi-criteria are demonstrated.

Keywords: fuzzy sets, fuzzy divergence, intuitionistic fuzzy sets, intuitionistic fuzzy divergence, multi-criteria decision making

1 Introduction

In mathematics, while studying a set of objects, we like to associate various quantitative measures defined over the set. Two basic such measures are – quantitative measure with each object and the difference or divergence between any two objects. In Information theory Shannon [10] defined entropy with probability distribution in a set of probability distributions. The measure of divergence, first introduced by Kullback and Leibler [3] is a measure of the extent to which the assumed probability distribution deviates from the true one. There can be and exist other measures of divergence on set of probabilities, with varied names like those of discrimination, distance etc. These find immense applications in decision making and other studies. Paralleling the concept of probability theory is the theory of fuzzy sets (FSs) proposed by Zadeh [15] in 1965. Fuzzy divergence introduced by Bhandari and Pal [2] gives a fuzzy information measure for discrimination of a fuzzy set $\tilde{A}$ relative to some other fuzzy set $\tilde{B}$. It has found wide applications in many areas such as pattern recognition, fuzzy clustering, signal and image processing etc.

An intuitionistic fuzzy set proposed by Atanassov [1], a generalization of fuzzy set, is characterized by two functions expressing the degree of membership and the degree of non-membership, respectively. However for being critical in our considerations it is desirable to additionally take into consideration, what is termed as hesitation degree [8, 9]. This brings us to 'intuitionistic fuzzy sets' and 'information theoretic measures' associated with them that are more appropriate in critical decision making [13, 14], medical diagnosis [6, 11], and pattern recognition [4, 5, 11, 12]. In 2010, Wei and Ye [12] proposed an improved version of Vlachos and Sergiadis [11] intuitionistic fuzzy divergence and studied its applications in pattern recognition. They used the mid-value of the membership, non-membership and hesitation values of two sets to propose a measure. This measure proposed by them seems be rather ad-hoc in nature and lacks the flexibility that it should have. In this paper, we use a flexible approach which provides further leverage of choice to the user, and propose a generalized version of Wei and Ye [12] intuitionistic fuzzy divergence. It may be remarked that the strength of a measure lies in its properties. The new measure has elegant properties, proved in the paper, to enhance the employability of this measure. The strength of this generalization has been demonstrated by an example of multi-criteria decision making.

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The paper is organized as follows: In Section 2 some basic definitions related to intuitionistic fuzzy set theory are briefly given. In Section 3 a generalized intuitionistic fuzzy divergence is proposed and its particular cases discussed. In Section 4 some properties of generalized intuitionistic fuzzy divergence are analyzed. In Section 5 finally, a numerical example is presented to illustrate the application of proposed measure to multi-criteria decision-making and our brief conclusions are presented in Section 6.

2 Preliminaries

In this section we present some basic concepts related to intuitionistic fuzzy sets, which will be needed in the following analysis.

Definition 1 *Fuzzy Set* [15]**:** A fuzzy set $\tilde{A}$ in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is defined as

$$\tilde{A} = \left\{ \left\langle x,\ \mu_{\tilde{A}}(x) \right\rangle \middle| x \in X \right\}, \tag{1}$$

where $\mu_{\tilde{A}}(x): X \to [0,1]$ is measure of *belongingness* or *degree of membership* of an element $x \in X$ to $\tilde{A}$.

In this definition, it may be noted that the measure of non-belongingness of $x \in X$ to $\tilde{A}$ turns out to be $1 - \mu_{\tilde{A}}(x)$.

To introduce additionally the vagueness feature of non-belongingness, Atanassov introduced following generalization of fuzzy sets.

Definition 2 *Intuitionistic Fuzzy Set* [1]**:** An intuitionistic fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is defined as

$$A = \left\{ \left\langle x,\ \mu_A(x), \nu_A(x) \right\rangle \middle| x \in X \right\}, \tag{2}$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$.

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote the *degree of membership* and *degree of non-membership* of $x \in X$ to A, respectively.

Further, we call $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), x \in X$, the *degree of hesitancy* of $x \in X$ to A or the *intuitionistic index*.

Obviously, when $\pi_A(x) = 0$, i.e., $\nu_A(x) = 1 - \mu_A(x)\ \ \forall\ x \in X$, then the IFS set A reduces to Zadeh's fuzzy set. Thus, fuzzy sets are the special cases of IFSs.

Definition 3 *Set Operations on IFSs* [1]**:** Let *IFS(X)* denote the family of all IFSs in the universe X, and let $A, B \in IFS(X)$ be two IFSs, given by

$$A = \left\{ \left\langle x, \mu_A(x), \nu_A(x) \right\rangle | x \in X \right\},\ \ B = \left\{ \left\langle x, \mu_B(x), \nu_B(x) \right\rangle | x \in X \right\}.$$

Then following set operations are defined on *IFS(X)*:

(i) $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)\ \ \forall\ x \in X$;

(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(iii) $A^C = \left\{ \left\langle x,\ \nu_A(x), \mu_A(x) \right\rangle \middle| x \in X \right\}$;

(iv) $A \bigcap B = \left\{ \left\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \right\rangle | x \in X \right\}$;

(v) $A \bigcup B = \left\{ \left\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \right\rangle | x \in X \right\}$

where $\vee, \wedge$ stand respectively for max. and min. operators.

Definition 4 *Intuitionistic Fuzzy Divergence (or Relative Information)*: Given $A \in IFS(X)$, from the definition of intuitionistic fuzzy set, we have:

$$\mu_A(x_i) + \nu_A(x_i) + \pi_A(x_i) = 1,\ \ 0 \le \mu_A(x_i), \nu_A(x_i), \pi_A(x_i) \le 1 \qquad \forall\ x_i \in X.$$

This suggests that $(\mu_A(x_i), \nu_A(x_i), \pi_A(x_i))$ may be regarded as probability distributions. Using this concept, Wei and Ye [12] proposed the following version of intuitionistic fuzzy divergence (relative information) given by

$$D(A|B)=\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\frac{\mu_A(x_i)+\mu_B(x_i)}{2}}+\nu_A(x_i)\log\frac{\nu_A(x_i)}{\frac{\nu_A(x_i)+\nu_B(x_i)}{2}}+\pi_A(x_i)\log\frac{\pi_A(x_i)}{\frac{\pi_A(x_i)+\pi_B(x_i)}{2}}\right]. \quad (3)$$

As a critique of (3), it may be noted that 'intuitionistic fuzzy divergence' is an important concept and a tool for many applications in decision making under vague phenomena. In equation (3) the choice of factor 1/2 with each $\mu_A(x_i)$, $\mu_B(x_i)$ and others, is rather ad-hoc in nature, unrealistic and inflexible. A question naturally arises: Can we choose a flexible way of combining $\mu_A(x_i)$ and $\mu_B(x_i)$, etc.? This is attempted in this paper.

In the next section, we propose a flexible and generalized intuitionistic fuzzy divergence measure and discuss how other divergence measures studied by others arise as its particular cases.

3 A Generalized Measure of Intuitionistic Fuzzy Divergence

Definition 5 *Generalized Intuitionistic Fuzzy Divergence*: Let A and B be two intuitionistic fuzzy sets defined in $X=\{x_1,x_2,...,x_n\}$ having the membership values $\mu_A(x_i),\ i=1,2,...,n$ and $\mu_B(x_i),\ i=1,2,...,n$; non-membership values $\nu_A(x_i),\ i=1,2,...,n$ and $\nu_B(x_i),\ i=1,2,...,n$ respectively.

We define $D_\lambda(A|B)$, the measure of generalized intuitionistic fuzzy divergence between IFSs A and B, as

$$D_\lambda(A|B)=\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}\mu_A(x_i)\log\frac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}\\ +\nu_A(x_i)\log\frac{\nu_A(x_i)}{\lambda\nu_A(x_i)+(1-\lambda)\nu_B(x_i)}\\ +\pi_A(x_i)\log\frac{\pi_A(x_i)}{\lambda\pi_A(x_i)+(1-\lambda)\pi_B(x_i)}\end{bmatrix},\qquad \text{where } 0\le\lambda\le 1. \quad (4)$$

It may be noted that $D_\lambda(A|B)$ is not symmetric, as is the case with Kullback-Leibler [3] measure. To imbue the measure with symmetry, which is logically better suited for any kind of difference a symmetric generalized measure of intuitionistic fuzzy divergence can now be defined as follows:

Definition 6: Given two sets $A,B\in IFS(X)$, we define the symmetric generalized intuitionistic fuzzy divergence (relative information) between IFSs A and B, as

$$D_\lambda(A;B)=D_\lambda(A|B)+D_\lambda(B|A). \quad (5)$$

Note: It can be easily verify that $D_\lambda(A;B)$ and $D_\lambda(A|B)$ satisfy the following properties:

1. $D_\lambda(A|B)$, $D_\lambda(A;B)\ge 0$;
2. When $\lambda\ne 1$, $D_\lambda(A;B)=0$ and $D_\lambda(A|B)=0$, if and only if $A=B$;
3. When $\lambda=1$, $D_\lambda(A;B)$ and $D_\lambda(A|B)$, always gives zero.

Some previously studied particular cases can be immediately noted below.

Particular cases:

i When $\lambda=1/2$, measure (4) reduces to measure (3).

ii When $\lambda=1/2$ and $\pi_A(x)=\pi_B(x)=0$, measure (4) gives the measure of fuzzy divergence proposed by Shang and Jiang in [7].

iii When $\lambda=0$ and $\pi_A(x)=\pi_B(x)=0$, measure (4) reduces to fuzzy divergence defined by Bhandari and Pal in [2].

The importance of the new measure lies in its elegant properties, which we study in next section, of $D_\lambda(A;B)$, the symmetric generalized intuitionistic fuzzy divergence.

For proofs of the properties, we will consider separation of X into two parts X_1 and X_2, such that

$$X_1 = \{x_i \mid x_i \in X,\ A(x_i) \subseteq B(x_i)\}, \tag{6}$$

$$X_2 = \{x_i \mid x_i \in X,\ A(x_i) \supseteq B(x_i)\}. \tag{7}$$

And note that for all $x_i \in X_1$,

$$\mu_A(x_i) \le \mu_B(x_i) \text{ and } \nu_A(x_i) \ge \nu_B(x_i).$$

As also $\forall\ x_i \in X_2$,

$$\mu_A(x_i) \ge \mu_B(x_i) \text{ and } \nu_A(x_i) \le \nu_B(x_i).$$

In the next section we will denote $\mu_A(x_i)$ by μ_A^i and $\nu_A(x_i)$ by ν_A^i.

4 Properties of Symmetric Generalized Intuitionistic Fuzzy Divergence (Relative Information)

Measure $D_\lambda(A;B)$, the symmetric generalized intuitionistic fuzzy divergence defined in (5), has the following properties:

Theorem 1: For $A, B \in IFS(X)$, $D_\lambda(A \cup B; A \cap B) = D_\lambda(A;B)$.

Proof: To prove the result, we shall start with expressions for each of two terms on the left hand side of following relation

$$D_\lambda(A \cup B; A \cap B) = D_\lambda(A \cup B \mid A \cap B) + D_\lambda(A \cap B \mid A \cup B).$$

So using definition in (4), we first have

$$D_\lambda(A \cup B \mid A \cap B) = \frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_{A\cup B}^i \log\dfrac{\mu_{A\cup B}^i}{\left(\lambda\mu_{A\cup B}^i + (1-\lambda)\mu_{A\cap B}^i\right)} + \nu_{A\cap B}^i \log\dfrac{\nu_{A\cap B}^i}{\left(\lambda\nu_{A\cap B}^i + (1-\lambda)\nu_{A\cup B}^i\right)} \\ +\left(1-\mu_{A\cup B}^i - \nu_{A\cap B}^i\right)\log\dfrac{\left(1-\mu_{A\cup B}^i - \nu_{A\cap B}^i\right)}{\left(\lambda\left(1-\mu_{A\cup B}^i - \nu_{A\cap B}^i\right) + (1-\lambda)\left(1-\mu_{A\cap B}^i - \nu_{A\cup B}^i\right)\right)}\end{array}\right]$$

$$= \frac{1}{n}\left[\begin{array}{l}\displaystyle\sum_{x_i \in X_1}\left\{\begin{array}{l}\mu_B^i \log\dfrac{\mu_B^i}{\left(\lambda\mu_B^i + (1-\lambda)\mu_A^i\right)} + \nu_B^i \log\dfrac{\nu_B^i}{\left(\lambda\nu_B^i + (1-\lambda)\nu_A^i\right)} \\ +\left(1-\mu_B^i - \nu_B^i\right)\log\dfrac{\left(1-\mu_B^i - \nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i - \nu_B^i\right) + (1-\lambda)\left(1-\mu_A^i - \nu_A^i\right)\right)}\end{array}\right\} \\ \displaystyle +\sum_{x_i \in X_2}\left\{\begin{array}{l}\mu_A^i \log\dfrac{\mu_A^i}{\left(\lambda\mu_A^i + (1-\lambda)\mu_B^i\right)} + \nu_A^i \log\dfrac{\nu_A^i}{\left(\lambda\nu_A^i + (1-\lambda)\nu_B^i\right)} \\ +\left(1-\mu_A^i - \nu_A^i\right)\log\dfrac{\left(1-\mu_A^i - \nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i - \nu_A^i\right) + (1-\lambda)\left(1-\mu_B^i - \nu_B^i\right)\right)}\end{array}\right\}\end{array}\right] \tag{8}$$

Next, again from definition in (4), we have

$$D_\lambda(A \cap B \mid A \cup B) = \frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_{A\cap B}^i \log\dfrac{\mu_{A\cap B}^i}{\left(\lambda\mu_{A\cap B}^i + (1-\lambda)\mu_{A\cup B}^i\right)} + \nu_{A\cup B}^i \log\dfrac{\nu_{A\cup B}^i}{\left(\lambda\nu_{A\cup B}^i + (1-\lambda)\nu_{A\cap B}^i\right)} \\ +\left(1-\mu_{A\cap B}^i - \nu_{A\cup B}^i\right)\log\dfrac{\left(1-\mu_{A\cap B}^i - \nu_{A\cup B}^i\right)}{\left(\lambda\left(1-\mu_{A\cap B}^i - \nu_{A\cup B}^i\right) + (1-\lambda)\left(1-\mu_{A\cup B}^i - \nu_{A\cap B}^i\right)\right)}\end{array}\right]$$

$$=\frac{1}{n}\left[\begin{array}{l}\sum_{x_i\in X_1}\left\{\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_B^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_B^i\right)}\\+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}\end{array}\right\}\\+\sum_{x_i\in X_2}\left\{\begin{array}{l}\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_A^i\right)}+\nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_A^i\right)}\\+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right\}\end{array}\right] \quad (9)$$

So that finally:

$$D_\lambda\left(A\cup B;A\cap B\right)=D_\lambda\left(A\cup B\mid A\cap B\right)+D_\lambda\left(A\cap B\mid A\cup B\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_B^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_B^i\right)}\\+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}\end{array}\right]$$

$$+\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_A^i\right)}+\nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_A^i\right)}\\+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right]$$

$$=D_\lambda\left(A\mid B\right)+D_\lambda\left(B\mid A\right)=D_\lambda\left(A;B\right)$$

This proves the theorem.

Theorem 2: For $A,B\in IFS\left(X\right)$,

(i) $D_\lambda\left(A;A\cup B\right)=D_\lambda\left(B;A\cap B\right)$;

(ii) $D_\lambda\left(A;A\cap B\right)=D_\lambda\left(B;A\cup B\right)$.

Proof: We prove (i) only, (ii) can be proved analogously.

(i) From definition in (4), we have:

$$D_\lambda\left(A\mid A\cup B\right)=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_{A\cup B}^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_{A\cap B}^i\right)}\\+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)\right)}\end{array}\right]$$

$$=\frac{1}{n}\left[\begin{array}{l}\sum_{x_i\in X_1}\left\{\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_B^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_B^i\right)}\\+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}\end{array}\right\}\\+\sum_{x_i\in X_2}\left\{\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_A^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_A^i\right)}\\+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right\}\end{array}\right]$$

$$=\frac{1}{n}\sum_{x_i\in X_1}\left[\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_B^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_B^i\right)}\\ \quad+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}\end{array}\right] \tag{10}$$

and

$$D_\lambda\left(A\cup B\,|\,A\right)=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_{A\cup B}^i\log\frac{\mu_{A\cup B}^i}{\left(\lambda\mu_{A\cup B}^i+(1-\lambda)\mu_A^i\right)}+\nu_{A\cap B}^i\log\frac{\nu_{A\cap B}^i}{\left(\lambda\nu_{A\cap B}^i+(1-\lambda)\nu_A^i\right)}\\ \quad+\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)\log\frac{\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)}{\left(\lambda\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right]$$

$$=\frac{1}{n}\left[\begin{array}{l}\sum_{x_i\in X_1}\left\{\begin{array}{l}\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_A^i\right)}+\nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_A^i\right)}\\ \quad+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right\}\\ +\sum_{x_i\in X_2}\left\{\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_A^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_A^i\right)}\\ \quad+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right\}\end{array}\right]$$

$$=\frac{1}{n}\sum_{x_i\in X_1}\left[\begin{array}{l}\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_A^i\right)}+\nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_A^i\right)}\\ \quad+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right] \tag{11}$$

Similarly, we get

$$D_\lambda\left(B\,|\,A\cap B\right)=\frac{1}{n}\sum_{x_i\in X_1}\left[\begin{array}{l}\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_A^i\right)}+\nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_A^i\right)}\\ \quad+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right] \tag{12}$$

and

$$D_\lambda\left(A\cap B\,|\,B\right)=\frac{1}{n}\sum_{x_i\in X_1}\left[\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_B^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_B^i\right)}\\ \quad+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}\end{array}\right] \tag{13}$$

Now from the definition of $D_\lambda\left(A;A\cup B\right)$ in (5), we have

$D_\lambda\left(A;A\cup B\right)=D_\lambda\left(A\,|\,A\cup B\right)+D_\lambda\left(A\cup B\,|\,A\right)$

$$= \frac{1}{n}\left[\begin{array}{l} \sum_{x_i \in X_1}\left\{\begin{array}{l} \mu_A^i \log \frac{\mu_A^i}{\left(\lambda\mu_A^i + (1-\lambda)\mu_B^i\right)} + \nu_A^i \log \frac{\nu_A^i}{\left(\lambda\nu_A^i + (1-\lambda)\nu_B^i\right)} \\ + \left(1-\mu_A^i-\nu_A^i\right)\log \frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right) + (1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)} \end{array}\right\} \\ + \sum_{x_i \in X_1}\left\{\begin{array}{l} \mu_B^i \log \frac{\mu_B^i}{\left(\lambda\mu_B^i + (1-\lambda)\mu_A^i\right)} + \nu_B^i \log \frac{\nu_B^i}{\left(\lambda\nu_B^i + (1-\lambda)\nu_A^i\right)} \\ + \left(1-\mu_B^i-\nu_B^i\right)\log \frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right) + (1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)} \end{array}\right\} \end{array}\right] \tag{14}$$

Again, using definition in (5), we have:

$$D_\lambda\left(B; A\cap B\right) = D_\lambda\left(B \mid A\cap B\right) + D_\lambda\left(A\cap B \mid B\right)$$

$$= \frac{1}{n}\left[\begin{array}{l} \sum_{x_i \in X_1}\left\{\begin{array}{l} \mu_B^i \log \frac{\mu_B^i}{\left(\lambda\mu_B^i + (1-\lambda)\mu_A^i\right)} + \nu_B^i \log \frac{\nu_B^i}{\left(\lambda\nu_B^i + (1-\lambda)\nu_A^i\right)} \\ + \left(1-\mu_B^i-\nu_B^i\right)\log \frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right) + (1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)} \end{array}\right\} \\ + \sum_{x_i \in X_1}\left\{\begin{array}{l} \mu_A^i \log \frac{\mu_A^i}{\left(\lambda\mu_A^i + (1-\lambda)\mu_B^i\right)} + \nu_A^i \log \frac{\nu_A^i}{\left(\lambda\nu_A^i + (1-\lambda)\nu_B^i\right)} \\ + \left(1-\mu_A^i-\nu_A^i\right)\log \frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right) + (1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)} \end{array}\right\} \end{array}\right] \tag{15}$$

This proves the theorem.

Corollary 1: For $A, B \in IFS\left(X\right)$, $D_\lambda\left(A; A\cup B\right) + D_\lambda\left(A; A\cap B\right) = D_\lambda\left(A; B\right)$.

Proof: It follows straight forwardly from Theorem 2.

Corollary 2: For $A, B \in IFS\left(X\right)$, $D_\lambda\left(B; A\cup B\right) + D_\lambda\left(B; A\cap B\right) = D_\lambda\left(A; B\right)$.

Proof: It also follows straight forwardly from Theorems 2.

Theorem 3: For $A, B, C \in IFS\left(X\right)$,

(i) $D_\lambda\left(A\cup B; C\right) \le D_\lambda\left(A; C\right) + D_\lambda\left(B; C\right)$;

(ii) $D_\lambda\left(A\cap B; C\right) \le D_\lambda\left(A; C\right) + D_\lambda\left(B; C\right)$.

Proof: In the following, we prove only (i), (ii) can be proved analogously.

(i) Let us consider the expression

$$D_\lambda\left(A; C\right) + D_\lambda\left(B; C\right) - D_\lambda\left(A\cup B; C\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l} \mu_A^i \log \frac{\mu_A^i}{\left(\lambda\mu_A^i + (1-\lambda)\mu_C^i\right)} + \nu_A^i \log \frac{\nu_A^i}{\left(\lambda\nu_A^i + (1-\lambda)\nu_C^i\right)} \\ \qquad + \left(1-\mu_A^i-\nu_A^i\right)\log \frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right) + (1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)} \\ + \mu_C^i \log \frac{\mu_C^i}{\left(\lambda\mu_C^i + (1-\lambda)\mu_A^i\right)} + \nu_C^i \log \frac{\nu_C^i}{\left(\lambda\nu_C^i + (1-\lambda)\nu_A^i\right)} \\ \qquad + \left(1-\mu_C^i-\nu_C^i\right)\log \frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right) + (1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)} \end{array}\right]$$

$$
+\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}
\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_C^i\right)}+\nu_A^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_C^i\right)}\\
\qquad+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\\
+\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_B^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_B^i\right)}\\
\qquad+\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}
\end{array}\right]
$$

$$
-\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}
\mu_{A\cup B}^i\log\frac{\mu_{A\cup B}^i}{\left(\lambda\mu_{A\cup B}^i+(1-\lambda)\mu_C^i\right)}+\nu_{A\cap B}^i\log\frac{\nu_{A\cap B}^i}{\left(\lambda\nu_{A\cap B}^i+(1-\lambda)\nu_C^i\right)}\\
\qquad+\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)\log\frac{\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)}{\left(\lambda\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\\
+\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_{A\cup B}^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_{A\cap B}^i\right)}\\
\qquad+\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)\right)}
\end{array}\right]
$$

$$
=\frac{1}{n}\left[\begin{array}{l}
\sum_{x_i\in X_1}\left\{\begin{array}{l}
\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_C^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_C^i\right)}\\
\qquad+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\\
+\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_A^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_A^i\right)}\\
\qquad+\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}
\end{array}\right\}\\
+\sum_{x_i\in X_2}\left\{\begin{array}{l}
\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_C^i\right)}+\nu_A^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_C^i\right)}\\
\qquad+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\\
+\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_B^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_B^i\right)}\\
\qquad+\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}
\end{array}\right\}
\end{array}\right]
$$

$$\geq 0$$

This proves the theorem.

Theorem 4: For $A,B,C\in IFS(X)$, $D_\lambda(A\bigcup B;C)+D_\lambda(A\bigcap B;C)=D_\lambda(A;C)+D_\lambda(B;C)$.

Proof: Using definition in (4), we first have:

$$
D_\lambda(A\bigcup B\,|\,C)=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}
\mu_{A\cup B}^i\log\frac{\mu_{A\cup B}^i}{\left(\lambda\mu_{A\cup B}^i+(1-\lambda)\mu_C^i\right)}+\nu_{A\cap B}^i\log\frac{\nu_{A\cap B}^i}{\left(\lambda\nu_{A\cap B}^i+(1-\lambda)\nu_C^i\right)}\\
\qquad+\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)\log\frac{\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)}{\left(\lambda\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}
\end{array}\right]
$$

$$=\frac{1}{n}\left[\begin{array}{l}\sum_{x_i\in X_1}\left\{\begin{array}{l}\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_C^i\right)}+\nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_C^i\right)}\\ \qquad+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\end{array}\right\}\\ +\sum_{x_i\in X_2}\left\{\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_C^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_C^i\right)}\\ \qquad+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\end{array}\right\}\end{array}\right] \tag{16}$$

and

$$D_\lambda\left(C\,|\,A\bigcup B\right)=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_{A\cup B}^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_{A\cap B}^i\right)}\\ \qquad+\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_{A\cup B}^i-\nu_{A\cap B}^i\right)\right)}\end{array}\right]$$

$$=\frac{1}{n}\left[\begin{array}{l}\sum_{x_i\in X_1}\left\{\begin{array}{l}\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_B^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_B^i\right)}\\ \qquad+\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}\end{array}\right\}\\ +\sum_{x_i\in X_2}\left\{\begin{array}{l}\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_A^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_A^i\right)}\\ \qquad+\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right\}\end{array}\right] \tag{17}$$

Next, again from definition in (4), we have:

$$D_\lambda\left(A\bigcap B\,|\,C\right)=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_{A\cap B}^i\log\frac{\mu_{A\cap B}^i}{\left(\lambda\mu_{A\cap B}^i+(1-\lambda)\mu_C^i\right)}+\nu_{A\cup B}^i\log\frac{\nu_{A\cup B}^i}{\left(\lambda\nu_{A\cup B}^i+(1-\lambda)\nu_C^i\right)}\\ \qquad+\left(1-\mu_{A\cap B}^i-\nu_{A\cup B}^i\right)\log\frac{\left(1-\mu_{A\cap B}^i-\nu_{A\cup B}^i\right)}{\left(\lambda\left(1-\mu_{A\cap B}^i-\nu_{A\cup B}^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\end{array}\right]$$

$$=\frac{1}{n}\left[\begin{array}{l}\sum_{x_i\in X_1}\left\{\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\mu_C^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\nu_C^i\right)}\\ \qquad+\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\end{array}\right\}\\ +\sum_{x_i\in X_2}\left\{\begin{array}{l}\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\mu_C^i\right)}+\nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\nu_C^i\right)}\\ \qquad+\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\mu_C^i-\nu_C^i\right)\right)}\end{array}\right\}\end{array}\right] \tag{18}$$

and

$$D_\lambda(C\mid A\cap B)=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_{A\cap B}^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_{A\cup B}^i\right)}\\ +\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_{A\cap B}^i-\nu_{A\cup B}^i\right)\right)}\end{array}\right]$$

$$=\frac{1}{n}\left[\begin{array}{l}\sum_{x_i\in X_1}\left\{\begin{array}{l}\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_A^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_A^i\right)}\\ +\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right\}\\ +\sum_{x_i\in X_2}\left\{\begin{array}{l}\mu_C^i\log\frac{\mu_C^i}{\left(\lambda\mu_C^i+(1-\lambda)\mu_B^i\right)}+\nu_C^i\log\frac{\nu_C^i}{\left(\lambda\nu_C^i+(1-\lambda)\nu_B^i\right)}\\ +\left(1-\mu_C^i-\nu_C^i\right)\log\frac{\left(1-\mu_C^i-\nu_C^i\right)}{\left(\lambda\left(1-\mu_C^i-\nu_C^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}\end{array}\right\}\end{array}\right] \tag{19}$$

After adding (16), (17), (18), (19), we get the result.

This proves the theorem.

Theorem 5: For $A,B\in IFS(X)$,

(a) $D_\lambda(A;B)=D_\lambda\left(A^C;B^C\right)$;

(b) $D_\lambda\left(A;B^C\right)=D_\lambda\left(A^C;B\right)$;

(c) $D_\lambda(A;B)+D_\lambda\left(A^C;\,B\right)=D_\lambda\left(A^C;B^C\right)+D_\lambda\left(A;B^C\right)$

where A^C and B^C represent respectively the complements of intuitionistic fuzzy sets *A* and *B*.

Proof: (a) It simply follows from the relation of membership and non-membership of an element in a set and its complement.

(b) Let us consider the expression

$$D_\lambda\left(A;B^C\right)-D_\lambda\left(A^C;B\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\nu_B^i\right)}+\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\mu_B^i\right)}\\ +\left(1-\mu_A^i-\nu_A^i\right)\log\frac{\left(1-\mu_A^i-\nu_A^i\right)}{\left(\lambda\left(1-\mu_A^i-\nu_A^i\right)+(1-\lambda)\left(1-\nu_B^i-\mu_B^i\right)\right)}\\ \nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\mu_A^i\right)}+\mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\nu_A^i\right)}\\ +\left(1-\nu_B^i-\mu_B^i\right)\log\frac{\left(1-\nu_B^i-\mu_B(x_i)\right)}{\left(\lambda\left(1-\nu_B^i-\mu_B^i\right)+(1-\lambda)\left(1-\mu_A^i-\nu_A^i\right)\right)}\end{array}\right]$$

$$-\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{l}\nu_A^i\log\frac{\nu_A^i}{\left(\lambda\nu_A^i+(1-\lambda)\mu_B^i\right)}+\mu_A^i\log\frac{\mu_A^i}{\left(\lambda\mu_A^i+(1-\lambda)\nu_B^i\right)}\\ +\left(1-\nu_A^i-\mu_A^i\right)\log\frac{\left(1-\nu_A^i-\mu_A(x_i)\right)}{\left(\lambda\left(1-\nu_A^i-\mu_A^i\right)+(1-\lambda)\left(1-\mu_B^i-\nu_B^i\right)\right)}\\ \mu_B^i\log\frac{\mu_B^i}{\left(\lambda\mu_B^i+(1-\lambda)\nu_A^i\right)}+\nu_B^i\log\frac{\nu_B^i}{\left(\lambda\nu_B^i+(1-\lambda)\mu_A^i\right)}\\ +\left(1-\mu_B^i-\nu_B^i\right)\log\frac{\left(1-\mu_B^i-\nu_B^i\right)}{\left(\lambda\left(1-\mu_B^i-\nu_B^i\right)+(1-\lambda)\left(1-\nu_A^i-\mu_A^i\right)\right)}\end{array}\right]$$

$$=0$$

This proves the result.

(c) It obviously follows (a) and (b).

5 Application to Multi-Criteria Decision Making Problem

Representation of imperfect phenomena is usually best done through IFSs. In this section, we present a method based on proposed symmetric generalized intuitionistic fuzzy divergence, to solve multi-criteria decision making problems.

Let us consider a decision problem involving a set of options $M=\{M_1,M_2,...,M_m\}$ to be considered under a set of criteria $C=\{C_1,C_2,...,C_n\}$. For decision making, characteristic sets for each option are determined as IFSs assigning appropriate values to $\mu-$ and $\nu-$functions. So let the characteristic-set of the option M_i in terms of the set of criteria C be an IFS:

$$M_i=\left\{\left\langle C_j,\mu_{ij},\nu_{ij}\right\rangle \mid C_j\in C\right\},\qquad i=1,2,...,m \text{ and } j=1,2,...,n,$$

where μ_{ij} indicates the degree with which the option M_i satisfies the criterion C_j and ν_{ij} indicates the degree with which the option M_i does not satisfy the criterion C_j.

Using the measure defined by (5), we introduce the following approach to solve the above multi-criteria intuitionistic fuzzy decision making problem:

Step 1: Find the ideal solution M^*, given by:

$$M^*=\left\{\left\langle\mu_{1*},\nu_{1*}\right\rangle,\left\langle\mu_{2*},\nu_{2*}\right\rangle,...,\left\langle\mu_{n*},\nu_{n*}\right\rangle\right\},$$

where, for each $j=1,2,...,n$,

$$\left\langle\mu_{j*},\nu_{j*}\right\rangle=\left\langle\max_i\mu_{ij},\min_i\nu_{ij}\right\rangle.$$

Step 2: Calculate $D_\lambda\left(M_i;M^*\right)$ given by the following:

$$D_\lambda\left(M_i,M^*\right)=\sum_{j=1}^{n}\left[\begin{array}{l}\mu_{ij}\log\dfrac{\mu_{ij}}{\left(\lambda\mu_{ij}+(1-\lambda)\mu_{j*}\right)}+\nu_{ij}\log\dfrac{\nu_{ij}}{\left(\lambda\nu_{ij}+(1-\lambda)\nu_{j*}\right)}\\ \quad+\left(1-\mu_{ij}-\nu_{ij}\right)\log\dfrac{\left(1-\mu_{ij}-\nu_{ij}\right)}{\left(\lambda\left(1-\mu_{ij}-\nu_{ij}\right)+(1-\lambda)\left(1-\mu_{j*}-\nu_{j*}\right)\right)}\\ +\mu_{j*}\log\dfrac{\mu_{j*}}{\left(\lambda\mu_{j*}+(1-\lambda)\mu_{ij}\right)}+\nu_{j*}\log\dfrac{\nu_{j*}}{\left(\lambda\nu_{j*}+(1-\lambda)\nu_{ij}\right)}\\ \quad+\left(1-\mu_{j*}-\nu_{j*}\right)\log\dfrac{\left(1-\mu_{j*}-\nu_{j*}\right)}{\left(\lambda\left(1-\mu_{j*}-\nu_{j*}\right)+(1-\lambda)\left(1-\mu_{ij}-\nu_{ij}\right)\right)}\end{array}\right]. \tag{20}$$

Step 3: Select the option M_k with smallest $D_\lambda\left(M_k,M^*\right)$.

In order to demonstrate the applicability of the proposed method to multicriteria- decision making, we consider below an investment company decision-making problem.

Example: Suppose that an investment company wants to invest a certain amount of money in the best option out of five options: A car company M_1, a food company M_2, a computer company M_3, an arms company M_4 and a TV company M_5. The investment company needs to take a decision according to the following four criteria: (1) G_1, the risk analysis; (2) G_2, the growth analysis; (3) G_3, the social-political impact analysis; and (4) G_4, the environmental impact analysis. For evaluating the five possible alternatives $M_i\left(i=1,2,...,5\right)$, the decision maker, on the basis of available data, has formed IFSs as the following five characteristic sets:

$$M_1 = \{\langle G_1, 0.5, 0.4\rangle, \langle G_2, 0.6, 0.3\rangle, \langle G_3, 0.3, 0.6\rangle, \langle G_4, 0.2, 0.7\rangle\},$$
$$M_2 = \{\langle G_1, 0.7, 0.3\rangle, \langle G_2, 0.7, 0.2\rangle, \langle G_3, 0.7, 0.2\rangle, \langle G_4, 0.4, 0.5\rangle\},$$
$$M_3 = \{\langle G_1, 0.6, 0.4\rangle, \langle G_2, 0.5, 0.4\rangle, \langle G_3, 0.5, 0.3\rangle, \langle G_4, 0.6, 0.3\rangle\},$$
$$M_4 = \{\langle G_1, 0.8, 0.1\rangle, \langle G_2, 0.6, 0.3\rangle, \langle G_3, 0.3, 0.4\rangle, \langle G_4, 0.2, 0.6\rangle\},$$
$$M_5 = \{\langle G_1, 0.6, 0.2\rangle, \langle G_2, 0.4, 0.3\rangle, \langle G_3, 0.7, 0.1\rangle, \langle G_4, 0.5, 0.3\rangle\}.$$

Step 1. We obtain M^*:

$$M^* = \{\langle G_1, 0.8, 0.1\rangle, \langle G_2, 0.7, 0.2\rangle, \langle G_3, 0.7, 0.1\rangle, \langle G_4, 0.6, 0.3\rangle\}.$$

Step 2. We use formula (20) to measure $D_\lambda(M_i, M^*)$, taking $\lambda = 0.2$, $\lambda = 0.4$, $\lambda = 0.5$, $\lambda = 0.6$ and $\lambda = 0.8$ respectively, we get the following table:

Table 1: Values of $D_\lambda(M_i, M^*)$ for $\lambda = 0.2$, $\lambda = 0.4$, $\lambda = 0.5$, $\lambda = 0.6$ and $\lambda = 0.8$

	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.5$	$\lambda = 0.6$	$\lambda = 0.8$	Ranking
$D_\lambda(M_1, M^*)$	0.5758	0.3217	0.2271	0.0585	0.0417	5th
$D_\lambda(M_2, M^*)$	0.1823	0.1027	0.0737	0.0500	0.0163	2nd
$D_\lambda(M_3, M^*)$	0.2695	0.1514	0.1081	0.0728	0.0227	3rd
$D_\lambda(M_4, M^*)$	0.3445	0.1927	0.1353	0.0884	0.0238	4th
$D_\lambda(M_5, M^*)$	0.1230	0.0639	0.0419	0.0245	0.0030	1st

Table 1 shows that the ranking order of alternatives is same, as long as λ, takes the same value for all alternatives, that is:

$$M_5 \succ M_2 \succ M_3 \succ M_4 \succ M_1.$$

Thus M_5 is the most preferable alternative.

Change of Consideration: In the above consideration, same value of λ for all alternatives was taken. But in realistic situations it can be different for different alternatives. The value of λ may then depend on an un-explicit (like past experience) of the decision maker.

Let us next consider relative information measures $D_\lambda(M_i, M^*)$ for different values of λ: taking $\lambda = 0.7$ for M_1, $\lambda = 0.4$ for M_2, $\lambda = 0.5$ for M_3, $\lambda = 0.1$ for M_4 and $\lambda = 0.2$ for M_5.

Calculating $D_\lambda(M_i, M^*)$ for different values of λ we get the following table:

Table 2: Values of $D_\lambda(M_i, M^*)$ for different values of λ

		Ranking
$D_{\lambda=0.7}(M_1, M^*)$	0.0880	1st
$D_{\lambda=0.4}(M_2, M^*)$	0.1027	2nd
$D_{\lambda=0.5}(M_3, M^*)$	0.2597	4th
$D_{\lambda=0.1}(M_4, M^*)$	0.4440	5th
$D_{\lambda=0.2}(M_5, M^*)$	0.1766	3rd

The order of rankings is now

$$M_1 \succ M_2 \succ M_5 \succ M_3 \succ M_4.$$

Next again taking $\lambda = 0.2$ for M_1, $\lambda = 0.8$ for M_2, $\lambda = 0.9$ for M_3, $\lambda = 0.4$ for M_4 and $\lambda = 0.1$ for M_5, the corresponding table is

Table 3: Values of $D_\lambda\left(M_i, M^*\right)$ for different values of λ

		Ranking
$D_{\lambda=0.2}\left(M_1, M^*\right)$	0.5758	5th
$D_{\lambda=0.8}\left(M_2, M^*\right)$	0.0163	1st
$D_{\lambda=0.9}\left(M_3, M^*\right)$	0.0232	2nd
$D_{\lambda=0.4}\left(M_4, M^*\right)$	0.1927	3rd
$D_{\lambda=0.1}\left(M_5, M^*\right)$	0.2044	4th

The order of rankings has once again changed to

$$M_2 \succ M_3 \succ M_4 \succ M_5 \succ M_1 .$$

We thus find the change in order of the rankings and this brings in the role of parameter λ .

6 Conclusions

In this paper, we proposed a new divergence measure called *generalized intuitionistic fuzzy divergence* in the setting of intuitionistic fuzzy set theory, introducing a parameter λ . This measure generalizes intuitionistic fuzzy relative information measure proposed by Wei and Ye [13]. The parameter introduced provides flexibility criteria for decision making. Further study of this measure will be reported separately.

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