

# Mean Semi-absolute Deviation Model for Uncertain Portfolio Optimization Problem

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## Abstract

Semi-absolute deviation is a commonly used downside risk measure in the portfolio optimization problem. However, there is no literature on taking semi-absolute deviation as a risk measure in the framework of uncertainty theory. This paper fills the gap by means of defining semi-absolute deviation for uncertain variables and establishes the corresponding mean semi-absolute deviation models in uncertain environment. Finally, a numerical example are presented to illustrate the application of the proposed approach. ©2012 World Academic Press, UK. All rights reserved.

**Keywords:** uncertain variable, uncertain measure, uncertain portfolio optimization, semi-absolute deviation

## 1 Introduction

Portfolio optimization problem concerns with an individual who is trying to allocate his/her capital to a selected number of securities in order to achieve some investment goal. The first mathematical model was proposed by Markowitz [16], in which expected value was used to describe return and variance was used to measure risk. Since then, variance has been widely used as a risk measure, and a large number of models (e.g. [3, 7]) have been investigated. Different from the quadratic Markowitz model, a linear model for portfolio optimization was provided by Konno and Yamazaki [11] in which the absolute deviation was used to measure the risk of the portfolio. In particular, when the returns of securities are multivariate-normally distributed, the model is equivalent to Markowitz's mean-variance model. Based on absolute deviation, numerous models were developed such as [1, 4, 24].

Although variance and absolute deviation are appropriate risk measures in many cases, they make no distinction between gains and losses. In other words, they regard high returns that investors like as equally undesirable as low returns that investors dislike. In order to solve the problem, downside risk measures were proposed in [5, 13, 16, 23, 25] and proved as reasonable measures [22, 27] since they only take the negative deviations from the expected level into account. Thus, downside risk measures would help investors make proper investments when the returns are asymmetrical. One of the commonly used downside risk measures is semivariance originally introduced in [16] and developed by several scholars such as [6, 17]. The other well-known downside risk measure is semi-absolute deviation proposed by Speranza [25]. In contrast to Markowitz's mean-semivariance model, the mean semi-absolute deviation model for portfolio optimization can be easily evaluated since it can be determined using linear programming models [18, 24, 25].

In the above works, security returns are considered as random variables. However, it is difficult to use probability theory to model the problem in some situations such as the emerging market. In order to deal with vagueness and ambiguity of the returns on securities, fuzzy set theory may be used to describe the fuzzy behavior in portfolio optimization problem by means of fuzzy variable. In fact, fuzz portfolio optimization has been widely investigated in different methods: fuzzy goal programming, fuzzy compromise programming, fuzzy decision theory, possibilistic programming [26], interval programming [12] and other fuzzy formulations [28]. The detailed exposition may be found in [29] which gave an overview on the development of fuzzy

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portfolio optimization. Different from the above approaches, fuzzy portfolio optimization was also considered by Huang [8] and Qin et. al [19, 20] based on credibility measure.

With the development of the theory of dealing with subjective uncertainty, Liu [14] found a paradox when fuzzy variables are employed to describe subjective uncertain phenomena. In order to overcome this shortcoming, Liu [14] proposed uncertain measure and further found uncertainty theory. Surrounding the subject, some works have been done [2, 30, 31]. In particular, several researchers have considered the portfolio optimization problem in which security returns are assumed to be uncertain variables. Qin et al. [21] introduced mean-variance model for uncertain portfolio optimization. Huang [9] proposed mean-risk model for the problem by introducing a risk curve. In addition, Huang [10] also established risk index based models for uncertain portfolio adjusting problem.

For the analogous reasons mentioned in stochastic environment, variance and absolute deviation become unreasonable in the case that uncertain returns are asymmetrical. Therefore, we have to consider downside risk measures for uncertain returns. As a popular downside risk measure, semi-absolute deviation is a direct and comparatively simple approach. However, semi-absolute deviation has not been studied in the framework of uncertainty theory. Thus, this paper will define semi-absolute deviation for uncertain variables and establish new optimization models for uncertain portfolio optimization problem.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts in uncertainty theory. In Section 3, we define semi-absolute deviation for uncertain variables and discuss the mathematical properties. In Section 4, we present three mean semi-absolute deviation models for uncertain portfolio optimization by optimizing different objectives. Section 5 presents a numerical example to illustrate advantages of the proposed approach. Further, some conclusions are given in Section 6.

## 2 Preliminaries

In 2007, Liu [14] proposed the concept of uncertain measure and founded uncertainty theory. In this part, we recall some basic definitions and properties about uncertain measure and uncertain variables, which will be used in the whole paper.

Let  $\Gamma$  be a nonempty set, and let  $\mathcal{L}$  be a  $\sigma$ -algebra over  $\Gamma$ . Each element  $\Lambda \in \mathcal{L}$  is called an event. It is necessary to assign to each event  $\Lambda$  a number  $\mathcal{M}\{\Lambda\}$  which indicates the chance that  $\Lambda$  will occur. Liu [14] proposed the following four axioms to ensure that the number  $\mathcal{M}\{\Lambda\}$  satisfying certain mathematical properties,

**Axiom 1.** (*Normality*)  $\mathcal{M}\{\Gamma\} = 1$ .

**Axiom 2.** (*Monotonicity*)  $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$  whenever  $\Lambda_1 \subset \Lambda_2$ .

**Axiom 3.** (*Self-Duality*)  $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$  for any event  $\Lambda$ .

**Axiom 4.** (*Countable Subadditivity*) For every countable sequence of events  $\{\Lambda_i\}$ , we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \quad (1)$$

The triplet  $(\Gamma, \mathcal{L}, \mathcal{M})$  is called an uncertainty space. An uncertain variable  $\xi$  is defined by Liu [14] as a measurable function from an uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  to the set of real numbers, i.e., for any Borel set  $B$  of real numbers, the set  $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$  is an event. An uncertain variable  $\xi$  can be characterized by an uncertainty distribution which is a function  $\Phi : \mathfrak{R} \rightarrow [0, 1]$  defined by Liu [14] as  $\Phi(t) = \mathcal{M}\{\xi \leq t\}$ . For example, by a linear uncertain variable, we mean that the variable has the following linear uncertainty distribution

$$\Phi(r) = \begin{cases} 0, & \text{if } r \leq a, \\ (r - a)/(b - a), & \text{if } a \leq r \leq b, \\ 1, & \text{if } r \geq b. \end{cases}$$

The linear uncertain variable is denoted by  $\mathcal{L}(a, b)$  where  $a$  and  $b$  are real numbers with  $a < b$ . By a zigzag

uncertain variable, we mean that the variable has the following zigzag uncertainty distribution

$$\Phi(r) = \begin{cases} 0, & \text{if } r \leq a, \\ (r - a)/2(b - a), & \text{if } a \leq r \leq b, \\ (x + c - 2b)/2(c - b), & \text{if } b \leq r \leq c, \\ 1, & \text{if } r \geq c. \end{cases}$$

The zigzag uncertain variable is denoted by  $\mathcal{Z}(a, b, c)$  where  $a, b, c$  are real numbers with  $a < b < c$ . By a normal uncertain variable, we mean that the variable has the following normal uncertainty distribution

$$\Phi(r) = \left( 1 + \exp \left( \frac{\pi(e - r)}{\sqrt{3}\sigma} \right) \right)^{-1}, \quad r \in \mathfrak{R}.$$

The normal uncertain variable is denoted by  $\mathcal{N}(e, \sigma)$  where  $e$  and  $\sigma$  are real numbers with  $\sigma > 0$ .

The uncertain variables  $\xi_1, \xi_2, \dots, \xi_m$  are said by Liu [14] to be independent if

$$\mathcal{M} \left\{ \bigcap_{i=1}^m \{\xi_i \in B_i\} \right\} = \min_{1 \leq i \leq m} \mathcal{M} \{\xi_i \in B_i\} \tag{2}$$

for any Borel sets  $B_1, B_2, \dots, B_m$  of real numbers. In order to rank uncertain variables, the expected value of  $\xi$  was proposed by Liu [14] as

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\} dr - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\} dr \tag{3}$$

provided that at least one of the two integrals is finite. For example, the linear uncertain variable  $\xi \sim \mathcal{L}(a, b)$  has an expected value  $E[\xi] = (a + b)/4$ ; the zigzag uncertain variable  $\xi \sim \mathcal{Z}(a, b, c)$  has an expected value  $E[\xi] = (a + 2b + c)/4$ ; the normal uncertain variable  $\xi \sim \mathcal{N}(e, \sigma)$  has an expected value  $e$ , i.e.,  $E[\xi] = e$ .

Further, if uncertain variables  $\xi$  and  $\eta$  are independent, then

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{4}$$

for any  $a, b \in \mathfrak{R}$ . In particular, we have  $E[a\xi + b] = aE[\xi] + b$ .

### 3 Semi-absolute Deviation of Uncertain Variables

In this section, we give the definition of the semi-absolute deviation of uncertain variables, and prove some mathematical properties for it. Let  $\xi$  be an uncertain variable and  $e$  a real number. For simplicity, we write  $(\xi - e)^- = \min(\xi - e, 0)$  and  $(\xi - e)^+ = \max(\xi - e, 0)$ .

**Definition 1** Let  $\xi$  be an uncertain variable with finite expected value  $e$ . Then the semi-absolute deviation of  $\xi$  is defined as

$$Sa[\xi] = E [ |(\xi - e)^-| ]. \tag{5}$$

**Remark 1:** Definition 5 tells us that the semi-absolute deviation is the expected value of  $|(\xi - e)^-|$ . Since  $|(\xi - e)^-|$  is a nonnegative uncertain variable, we also have

$$\begin{aligned} Sa[\xi] &= \int_0^{+\infty} \mathcal{M} \{ |(\xi - e)^-| \geq r \} dr \\ &= \int_0^{+\infty} \mathcal{M} \{ e - \xi \geq r \} dr \\ &= \int_{-\infty}^e \mathcal{M} \{ \xi \leq r \} dr \\ &= \int_{-\infty}^e \Phi(r) dr \end{aligned} \tag{6}$$

where  $\Phi(\cdot)$  is the uncertainty distribution of  $\xi$ . This formula will facilitate the calculation of semi-absolute deviation in many cases.

**Example 1:** Let  $\xi \sim \mathcal{L}(a, b)$  be a linear uncertain variable. Then its semi-absolute deviation is

$$Sa[\xi] = \int_{-\infty}^{E[\xi]} \Phi(r) dr = \int_a^{(a+b)/2} \frac{r-a}{b-a} dr = \frac{1}{b-a} \int_a^{(a+b)/2} (r-a) dr = \frac{(b-a)^2}{8}.$$

**Example 2:** Let  $\xi \sim \mathcal{Z}(a, b, c)$  be a zigzag uncertain variable. The semi-absolute deviation of  $\xi$  is

$$Sa[\xi] = \int_{-\infty}^{(a+2b+c)/4} \Phi(r) dr = \begin{cases} \frac{(3(b-a) + (c-b))^2}{64(b-a)}, & \text{if } b-a \geq c-b, \\ \frac{((b-a) + 3(c-b))^2}{64(c-b)}, & \text{if } b-a \leq c-b, \end{cases}$$

which has alternative expression

$$Sa[\xi] = \frac{(2c - 2a + |2b - a - c|)^2}{32(c - a + |2b - a - c|)}.$$

**Example 3:** Let  $\xi \sim \mathcal{N}(e, \sigma)$  be a normal uncertain variable. Then the semi-absolute deviation of  $\xi$  is

$$Sa[\xi] = \int_{-\infty}^e \Phi(r) dr = \int_{-\infty}^e \left(1 + \exp\left(\frac{\pi(e-r)}{\sqrt{3}\sigma}\right)\right)^{-1} dr = \frac{\sqrt{3}\sigma \ln 2}{\pi}.$$

**Theorem 1** Let  $\xi$  be an uncertain variable with a finite expected value. Then for any real numbers  $a$  and  $b$ , we have

$$Sa[a\xi + b] = |a| \cdot Sa[\xi]. \quad (7)$$

**Proof:** It follows from the definition of semi-absolute deviation that

$$Sa[a\xi + b] = E[|(a\xi + b - aE[\xi] - b)^-|] = E[|a(\xi - E[\xi])^-|] = |a| \cdot E[|(\xi - E[\xi])^-|] = |a| \cdot Sa[\xi].$$

The proof is complete.

**Theorem 2** Let  $\xi$  be an uncertain variable with expected value  $e$ , and let  $Sa[\xi]$  be the semi-absolute deviation of  $\xi$ . Then we have  $0 \leq Sa[\xi] \leq E[|\xi - e|]$ .

**Proof:** Since  $|\xi - e|$  and  $|(\xi - e)^-|$  are nonnegative, for any real number  $r$ , we have

$$\{\gamma \mid |\xi(\gamma) - e| \geq r\} \supset \{\gamma \mid |(\xi(\gamma) - e)^-| \geq r\}.$$

It follows from the monotonicity of uncertain measure that

$$\mathcal{M}\{|\xi - e| \geq r\} \geq \mathcal{M}\{|(\xi - e)^-| \geq r\}, \forall r.$$

It immediately follows from the definition of semi-absolute deviation that

$$Sa[\xi] = \int_0^{+\infty} \mathcal{M}\{|(\xi - e)^-| \geq r\} dr \leq \int_0^{+\infty} \mathcal{M}\{|\xi - e| \geq r\} dr = E[|\xi - e|].$$

The proof is complete.

**Theorem 3** Let  $\xi$  be an uncertain variable with finite expected value  $e$ . Then  $Sa[\xi] = 0$  if and only if  $\mathcal{M}\{\xi = e\} = 1$ .

**Proof:** If  $Sa[\xi] = 0$ , then  $E[|(\xi - e)^-|] = 0$ . By the definition of expected value, we have

$$E[|(\xi - e)^-|] = \int_0^{+\infty} \mathcal{M}\{ |(\xi - e)^-| \geq r \} dr,$$

which implies  $\mathcal{M}\{ |(\xi - e)^-| \geq r \} = 0$  for any  $r > 0$ . Further, it is obtained that

$$\mathcal{M}\{ |(\xi - e)^-| = 0 \} = 1, \tag{8}$$

which implies  $\xi - e = (\xi - e)^+ + (\xi - e)^- = (\xi - e)^+$  almost everywhere. As a consequence, we have

$$\int_0^{+\infty} \mathcal{M}\{ (\xi - e)^+ \geq r \} dr = E[(\xi - e)^+] = E[\xi - e] = 0.$$

This indicates that  $\mathcal{M}\{ (\xi - e)^+ \geq r \} = 0$  for any  $r > 0$ . Considering the self-duality of uncertain measure, we have

$$\mathcal{M}\{ (\xi - e)^+ = 0 \} = 1. \tag{9}$$

It follows from equations (8) and (9) that  $\mathcal{M}\{ \xi - e = 0 \} = 1$ , i.e.,  $\mathcal{M}\{ \xi = e \} = 1$ . Conversely, if  $\mathcal{M}\{ \xi = e \} = 1$ , then we have  $\mathcal{M}\{ |\xi - e| = 0 \} = 1$  and  $\mathcal{M}\{ |\xi - e| \geq r \} = 0$  for any real number  $r > 0$ , which implies that  $E[|\xi - e|] = 0$ . Further, it follows from Theorem 2 that  $Sa[\xi] = 0$ . The proof is complete.

## 4 Mean Semi-absolute Deviation Models

In this section, we establish new portfolio optimization models in uncertain environment by employing the semi-absolute deviation of the portfolio as the measure of risk. Let  $\xi_i (i = 1, 2, \dots, n)$  be the uncertain return of the  $i$ th security, and  $x_i$  be the proportion of total amount of funds invested in security  $i$ . By uncertain arithmetic, the total return of the portfolio is  $\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$  which is also an uncertain variable. This means that the portfolio is risky. If an investor wants to maximize the expected return at the given risk level, then it can express in a single-objective non-linear programming model as follows,

$$\begin{cases} \text{maximize} & E[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \\ \text{subject to:} & Sa[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \leq d, \\ & x_1 + x_2 + \dots + x_n = 1, \\ & x_i \geq 0, \quad i = 1, 2, \dots, n, \end{cases} \tag{10}$$

where  $d$  denotes the maximum risk level the investors can tolerate. The constraints ensure that all the capital will be invested to  $n$  securities and short sales are not allowed. It is worth pointing out that the proposed model mainly deal with the case with asymmetrical return.

When an investor wants to minimize the risk of investment with an acceptable expected return level, the mathematical formulation of the problem is given as follows,

$$\begin{cases} \text{minimize} & Sa[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \\ \text{subject to:} & E[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \geq r, \\ & x_1 + x_2 + \dots + x_n = 1, \\ & x_i \geq 0, \quad i = 1, 2, \dots, n, \end{cases} \tag{11}$$

where  $r$  is the minimum expected return level accepted by the investor.

A risk-averse investor always wants to maximize the return and minimize the risk of the portfolio. However, these two objects are inconsistent. To determine an optimal portfolio with a given degree of risk aversion, we formalize the following optimization model,

$$\begin{cases} \text{maximize} & E[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] - \phi \cdot Sa[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \\ \text{subject to:} & x_1 + x_2 + \dots + x_n = 1, \\ & x_i \geq 0, \quad i = 1, 2, \dots, n, \end{cases} \tag{12}$$

where  $\phi \in [0, +\infty)$  represents the degree of absolute risk aversion. Here, the greater the value of  $\phi$  is, the more risk-averse the investors are. Note that  $\phi = 0$  means that the investor does not consider risk, and  $\phi$  approaching infinity means that the investor will allocate all the money to risk-less securities.

Note that in uncertain environment,  $E[\xi_1x_1 + \xi_2x_2 \cdots + \xi_nx_n] \neq x_1E[\xi_1] + x_2E[\xi_2] + \cdots + x_nE[\xi_n]$  for general uncertain variables  $\xi_1, \xi_2, \dots, \xi_n$ . However, the inequality will become equality when  $\xi_1, \xi_2, \dots, \xi_n$  are independent. In particular, if security returns are all linear uncertain variables, denoting the return of security  $i$  by  $\xi_i = (a_i, b_i)$ . It follows from Theorem 1.21 of [15] that the portfolio return  $\sum_{i=1}^n \xi_i x_i = (\sum_{i=1}^n x_i a_i, \sum_{i=1}^n x_i b_i)$  is also a linear uncertain variable. Based on this, model (11) may be converted into the following quadratic programming,

$$\left\{ \begin{array}{l} \text{minimize} \quad \left( \sum_{i=1}^n x_i (b_i - a_i) \right)^2 \\ \text{subject to:} \quad \sum_{i=1}^n x_i (a_i + b_i) \geq 2r, \\ \quad \quad \quad x_1 + x_2 + \cdots + x_n = 1, \\ \quad \quad \quad x_i \geq 0, \quad i = 1, 2, \dots, n. \end{array} \right. \tag{13}$$

Further, we assume that security returns are all zigzag uncertain variables, denoting the return of security  $i$  by  $\xi_i = (a_i, b_i, c_i)$ . It follows from Theorem 1.22 of [15] that the portfolio return  $\sum_{i=1}^n \xi_i x_i = (\sum_{i=1}^n x_i a_i, \sum_{i=1}^n x_i b_i, \sum_{i=1}^n x_i c_i)$  is also a zigzag uncertain variable. Then the model (11) may be converted into the following deterministic form,

$$\left\{ \begin{array}{l} \text{minimize} \quad \frac{\left( \sum_{i=1}^n 2x_i (c_i - a_i) + \left| \sum_{i=1}^n x_i (2b_i - a_i - c_i) \right| \right)^2}{\sum_{i=1}^n x_i (c_i - a_i) + \left| \sum_{i=1}^n x_i (2b_i - a_i - c_i) \right|} \\ \text{subject to:} \quad \sum_{i=1}^n x_i (a_i + 2b_i + c_i) \geq 4r, \\ \quad \quad \quad x_1 + x_2 + \cdots + x_n = 1, \\ \quad \quad \quad x_i \geq 0, \quad i = 1, 2, \dots, n. \end{array} \right. \tag{14}$$

**Remark 2:** When the security returns are all linear or zigzag uncertain variables, both models (10) and (12) can be converted into deterministic programming problems in a similar way.

## 5 A Numerical Example

In this section, we present a numerical example about uncertain portfolio optimization problem to demonstrate the new modeling idea of this paper. For this purpose, we consider the problem that an investor plans to invest his fund among twenty securities. It is worth pointing out that the experiment process may be repeated for more and/or less securities. Further, all the future returns of securities are zigzag uncertain variables denoted by  $\xi_i = \mathcal{Z}(a_i, b_i, c_i)$  for  $i = 1, 2, \dots, 20$ . For the practical investment problem, the key is to accurately estimate the values of parameters  $a_i, b_i$  and  $c_i$  for  $i = 1, 2, \dots, 20$ . The detailed method may be found in the chapter of Uncertain Statistics in [15]. Once these parameters are obtained, we can apply the proposed model to construct an optimal portfolio for the investor according to his/her requirements. For illustrative purpose, the assumed returns of securities are shown in Table 1.

Since all the security returns are zigzag uncertain variables, the proposed mean semi-absolute deviation models can be converted into equivalent deterministic models. Here, we use the model (11) to determine the optimal portfolio. The transformed deterministic model is as follows,

$$\left\{ \begin{array}{l} \text{minimize} \quad \frac{\left( \sum_{i=1}^{20} 2x_i \alpha_i + \left| \sum_{i=1}^{20} x_i \beta_i \right| \right)^2}{\left( \sum_{i=1}^{20} x_i \alpha_i + \left| \sum_{i=1}^{20} x_i \beta_i \right| \right)} \\ \text{subject to:} \quad x_1 \mu_1 + x_2 \mu_2 + \cdots + x_{20} \mu_{20} \geq r, \\ \quad \quad \quad x_1 + x_2 + \cdots + x_{20} = 1, \\ \quad \quad \quad 0 \leq x_i \leq 0.5, \quad i = 1, 2, \dots, 20 \end{array} \right. \tag{15}$$

Table 1: Zigzag uncertain returns of 20 securities

Security No.	Uncertain Return	Security No.	Uncertain Return
1	(-0.12, 0.05, 0.21)	11	(-0.16, -0.01, 0.14)
2	(-0.19, 0.02, 0.22)	12	(-0.13, 0.01, 0.19)
3	(-0.11, -0.01, 0.13)	13	(-0.14, 0.03, 0.21)
4	(-0.17, 0.02, 0.16)	14	(-0.18, 0.02, 0.16)
5	(-0.19, 0.03, 0.19)	15	(-0.14, 0.00, 0.18)
6	(-0.13, 0.01, 0.20)	16	(-0.17, 0.00, 0.33)
7	(-0.18, 0.01, 0.25)	17	(-0.24, 0.02, 0.45)
8	(-0.15, 0.05, 0.18)	18	(-0.14, 0.03, 0.21)
9	(-0.21, -0.01, 0.18)	19	(-0.12, 0.02, 0.14)
10	(-0.09, 0.02, 0.13)	20	(-0.20, 0.01, 0.24)

where  $\alpha_i = c_i - a_i$ ,  $\beta_i = 2b_i - a_i - c_i$  and  $\mu_i = (a_i + 2b_i + c_i)/4$ . We employ the `fmincon` function in MATLAB 7.1 to solve the above model. For a given minimal return level  $r$ , we obtain a series of optimal investment strategies. The computational results are shown in Table 2 in which the first column is the preset minimal return level and the second column is semi-absolute deviation of the optimal portfolio. Further, the efficient frontier of model (15) are shown in Figure 1.

Table 2: Investment proportions to 20 securities with zigzag uncertain return rates (%)

$r$	Semi-Absolute Deviation	( $x_1$ , $x_3$ , $x_8$ , $x_{10}$ , $x_{17}$ , $x_{19}$ )
0.005	0.922	( 0.00, 50.00, 0.00, 50.00, 0.00, 0.00)
0.010	0.922	( 0.00, 50.00, 0.00, 50.00, 0.00, 0.00)
0.015	0.947	( 0.00, 16.67, 0.00, 50.00, 0.00, 33.33)
0.020	0.982	( 7.69, 0.00, 0.00, 50.00, 0.00, 42.31)
0.025	1.025	(23.08, 0.00, 0.00, 50.00, 0.00, 26.92)
0.030	1.068	(38.46, 0.00, 0.00, 50.00, 0.00, 11.54)
0.035	1.145	(50.00, 0.00, 10.00, 40.00, 0.00, 0.00)
0.040	1.324	(50.00, 0.00, 50.00, 0.00, 0.00, 0.00)
0.045	1.560	(50.00, 0.00, 33.33, 0.00, 16.67, 0.00)
0.050	1.803	(50.00, 0.00, 16.67, 0.00, 33.33, 0.00)
0.055	2.051	(50.00, 0.00, 0.00, 0.00, 50.00, 0.00)

## 6 Conclusion

In this paper, the concept of semi-absolute deviation was first defined for uncertain variables, and some properties were proven. By means of taking semi-absolute deviation as a risk measure, three mean semi-absolute deviation models were presented in this work. A numerical example was presented and solved in the situation that all the security returns are zigzag uncertain variables. The computational results implied that the proposed model is feasible and significant for uncertain portfolio optimization problem.

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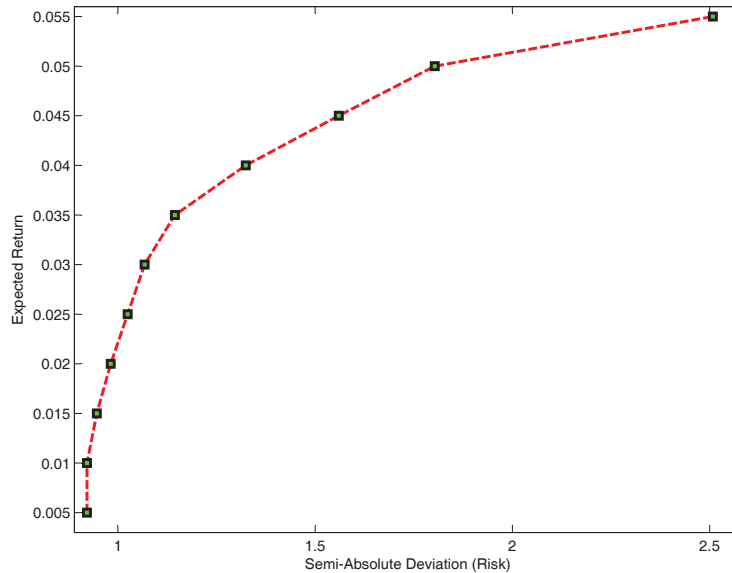


Figure 1: Efficient frontier of mean semi-absolute deviation model (15)

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