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Functions of Uncertain Variables and Uncertain Programming

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Abstract

An approach for uncertainty distribution of function of an uncertain variable is established based on uncertain measure. Then expected value of function of an uncertain variable is derived from the distribution. To the sake of computation, uncertain simulations are introduced for approximating the uncertainty distribution, optimistic value and expected value. The simulations are integrated to genetic algorithm for solving uncertain expected value models and uncertain chance-constrained programming models. The efficiency of the proposed methods is shown by some examples.

Keywords: uncertain variable, uncertainty distribution, expected value, optimistic value, simulation, uncertain programming

1 Introduction

Uncertainty theory founded by Liu in 2007 [4] and refined in 2010 [6] is a branch of mathematics for dealing with human uncertainty. There are three fundamental concepts in uncertainty theory. The first concept called uncertain measure is introduced based on three axioms: normality axiom, duality axiom and subadditivity axiom for presenting the degree that an uncertain event may occur. The second one called uncertain variable is bringed in for showing quantities in uncertainty. The third one called uncertainty distribution (a real function) is put forwarded for describing uncertain variables. In addition, there are three characters including expected value, optimistic value and pessimistic value for an uncertain variable. These characters play essential roles in uncertain programming [5] and uncertain optimal control [1, 2, 3, 9, 10].

It is known from [6] that the uncertainty distribution of $f(\xi)$ may be provided with the uncertainty distribution of the uncertain variable ξ if the measurable function f is monotone. Then the expected value of $f(\xi)$ may be obtained by the uncertainty distribution of ξ in [7]. However, if f is not monotone, then the above results may be not true. This flaw results in great inconvenience for the use of uncertain expected value in uncertain programming and uncertain optimal control. Now, we hope repair this flaw in the paper.

An uncertain variable is a measurable function from an uncertainty space to the set of real numbers. In practice, e.g. uncertain programming and uncertain optimal control, related uncertain variables are provided with their distributions regardless of the uncertainty spaces on which the uncertain variables are defined. Peng and Iwamura [8] proved that for any uncertainty distribution $\Phi(x)$, there is an uncertain variable defined on the set of real numbers such that its distribution is just $\Phi(x)$. We may call this type of uncertain variable to be common. If we regard uncertain variables provided with uncertainty distributions as common ones, then the distributions and expected values of functions of them may be easily obtained.

The organization of the paper is as follows. In Section 2, some concepts and useful theorems are reviewed. In Section 3, an approach is proposed analytically to obtain the distribution of function of an uncertain variable. In Section 4, the expected value of function of an uncertain variable is investigated. In Section 5, uncertain simulations are introduced for the distribution, optimistic value and expected value of function of an uncertain variable. In Section 6, two uncertain programming models including uncertain expected value model and chance-constrained programming model are studied. Section 7 gives conclusions.

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2 Preliminary

In convenience, we give some notations and concepts. Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M}$ defined on the σ-algebra $\mathcal{L}$ is called an uncertain measure if it satisfies that $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ; $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ; $\mathcal{M}\{\cup_{i=1}^{\infty}\Lambda_i\} \le \Sigma_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}$ for $\Lambda_i \in \mathcal{L}$. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space. An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots, n$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure on the product σ-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying $\mathcal{M}\left\{\prod_{k=1}^{n} \Lambda_k\right\} = \min_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\}$. That is, for each event $\Lambda \in \mathcal{L}$, we have

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \min\limits_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} & \text{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda} \min\limits_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} > 0.5, \\ 1 - \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \min\limits_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} & \text{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c} \min\limits_{1\le k\le n} \mathcal{M}_k\{\Lambda_k\} > 0.5, \\ 0.5, & \text{otherwise.} \end{cases} \quad (1)$$

Suppose f is a measurable function on $\Re^n$, and $\xi_1, \xi_2, \ldots, \xi_n$ are uncertain variables on $(\Gamma, \mathcal{L}, \mathcal{M})$. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable defined as $\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma))$ for $\gamma \in \Gamma$. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if $\mathcal{M}\left\{\cap_{i=1}^{m}\{\xi \in B_i\}\right\} = \min_{1\le i\le m} \mathcal{M}\{\xi \in B_i\}$ for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers. The distribution $\Phi : \mathbb{R} \to [0, 1]$ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\gamma \in \Gamma \mid \xi(\gamma) \le x\}$ for $x \in R$. The expected value of ξ is defined by $E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \le r\}dr$ provided that at least one of the two integrals is finite. The variance of ξ is $V[\xi] = E[(\xi - E[\xi])^2]$. For $\alpha \in (0, 1]$, the α-optimistic value to ξ is defined by $\xi_{\sup}(\alpha) = \sup\{r \mid \mathcal{M}\{\xi \ge r\} \ge \alpha\}$, and the α-pessimistic value to ξ is defined by $\xi_{\inf}(\alpha) = \inf\{r \mid \mathcal{M}\{\xi \le r\} \ge \alpha\}$.

Theorem 1 [8] *A function $\Phi : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.*

Given an increasing function $\Phi(x)$, Peng and Iwamura [8] introduced an uncertainty space $(\Re, \mathcal{B}, \mathcal{M})$ as follows. Let $\mathcal{B}$ be the Borel algebra over $\Re$. Let $\mathcal{C}$ be the collection of all intervals of the form $(-\infty, a]$, $(b, +\infty)$, $\emptyset$ and $\Re$. The uncertain measure $\mathcal{M}$ is provided in such a way: first,

$$\mathcal{M}\{(-\infty, a]\} = \Phi(a), \quad \mathcal{M}\{(b, +\infty)\} = 1 - \Phi(b), \quad \mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Re\} = 1.$$

Second, for any $B \in \mathcal{B}$, there exists a sequence $\{A_i\}$ in $\mathcal{C}$ such that

$$B \subset \bigcup_{i=1}^{\infty} A_i.$$

Thus

$$\mathcal{M}\{B\} = \begin{cases} \inf\limits_{B\subset\cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{B\subset\cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 1 - \inf\limits_{B^c\subset\cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{B^c\subset\cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases} \quad (2)$$

The uncertain variable defined by $\xi(\gamma) = \gamma$ from the uncertainty space $(\Re, \mathcal{B}, \mathcal{M})$ to $\Re$ has the uncertainty distribution Φ.

Note that for monotone increasing function $\Phi(x)$ except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$, there may be multiple uncertain variables whose uncertainty distributions are just $\Phi(x)$. However, for any one ξ among them, the uncertain measure of the event $\{\xi \in B\}$ for Borel set B may be not analytically expressed by $\Phi(x)$. For any two ξ and η among them, the uncertain measure of $\{\xi \in B\}$ may differ from that of $\{\eta \in B\}$. These facts result in inconvenience of use in practice. Which one among them should we choose for reasonable and convenient use? Let us consider the uncertain variable ξ defined by $\xi(\gamma) = \gamma$ on the uncertainty space $(\Re, \mathcal{B}, \mathcal{M})$ with the uncertainty distribution $\Phi(x)$, where the uncertain measure $\mathcal{M}$ is defined by (2), and another uncertain variable ξ_1 on the uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$. For each $A \in \mathcal{C}$, we have $\mathcal{M}\{\xi \in A\} = \mathcal{M}_1\{\xi_1 \in A\}$. For any Borel set $B \subset \Re$, if $B \subset \cup_{i=1}^{\infty} A_i$ with $\sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5$, then

$$\mathcal{M}_1\{\xi_1 \in B\} \leq \mathcal{M}_1 \left\{ \bigcup_{i=1}^{\infty} \{\xi_1 \in A_i\} \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}_1\{\xi_1 \in A_i\} = \sum_{i=1}^{\infty} \mathcal{M}\{\xi \in A_i\} < 0.5;$$

if $B^c \subset \cup_{i=1}^{\infty} A_i$ with $\sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5$, then

$$\mathcal{M}_1\{\xi_1 \in B\} = 1 - \mathcal{M}_1\{\xi_1 \in B^c\} \geq 1 - \mathcal{M}_1 \left\{ \bigcup_{i=1}^{\infty} \{\xi_1 \in A_i\} \right\} \geq 1 - \sum_{i=1}^{\infty} \mathcal{M}_1\{\xi_1 \in A_i\} = 1 - \sum_{i=1}^{\infty} \mathcal{M}\{\xi \in A_i\} > 0.5.$$

Thus

$$\mathcal{M}_1\{\xi_1 \in B\} \leq \mathcal{M}\{\xi \in B\} = \inf_{B \subset \cup A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5, \quad \text{if } \inf_{B \subset \cup A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5,$$

$$\mathcal{M}_1\{\xi_1 \in B\} \geq \mathcal{M}\{\xi \in B\} = 1 - \inf_{B^c \subset \cup A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\} > 0.5, \quad \text{if } \inf_{B^c \subset \cup A_i} \sum_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5.$$

In other cases, $\mathcal{M}\{\xi \in B\} = 0.5$. Therefore, the uncertain measure of $\{\xi \in B\}$ is closer to 0.5 than that of $\{\xi_1 \in B\}$. Based on the maximum uncertainty principle, we adopt uncertain variable ξ defined on $(\Re, \mathcal{B}, \mathcal{M})$ for use in our discussion if only the uncertainty distribution is provided.

Definition 1 *An uncertain variable ξ with distribution $\Phi(x)$ is common if it is from the uncertainty space $(\Re, \mathcal{B}, \mathcal{M})$ to $\Re$ defined by $\xi(\gamma) = \gamma$, where $\mathcal{B}$ is the Borel algebra over $\Re$ and $\mathcal{M}$ is defined by (2).*

Let $\Phi(x)$ be continuous. For uncertain measure $\mathcal{M}$ defined by (2), we know that $\mathcal{M}\{(-\infty, a)\} = \Phi(a)$ and $\mathcal{M}\{[b, +\infty)\} = 1 - \Phi(b)$.

Definition 2 *An uncertain vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ is common if every uncertain variable ξ_i is common for $i = 1, 2, \ldots, n$.*

3 Distribution of Function of Uncertain Variable

Let us discuss the distribution of $f(\xi)$ for a common uncertain variable ξ or a common uncertain vector. Assume $\mathcal{C}$ is the collection of all intervals of the form $(-\infty, a]$, $(b, +\infty)$, $\emptyset$ and $\Re$. Each element A_i emerging in the sequel is in $\mathcal{C}$.

Theorem 2 *(i) Let ξ be a common uncertain variable with the continuous distribution $\Phi(x)$ and $f(x)$ a Borel function. Then the distribution of the uncertain variable $f(\xi)$ is*

$$\Psi(x) = \mathcal{M}\{f(\xi) \leq x\}$$
$$= \begin{cases} \inf\limits_{\{f(\xi) \leq x\} \subset \cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{\{f(\xi) \leq x\} \subset \cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 1 - \inf\limits_{\{f(\xi) > x\} \subset \cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{\{f(\xi) > x\} \subset \cup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases} \quad (3)$$

(ii) Let $f:\Re^n \to \Re$ be a Borel function, and $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)$ be a common uncertain vector. Then the distribution of the uncertain variable $f(\xi)$ is

$$\Psi(x)=\mathcal{M}\{f(\xi_1,\xi_2,\ldots,\xi_n)\le x\}=\mathcal{M}\{(\xi_1,\xi_2,\ldots,\xi_n)\in f^{-1}(-\infty,x)\}$$
$$=\begin{cases}\sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min\limits_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} & \textit{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\min\limits_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\}>0.5,\\ 1-\sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min\limits_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\} & \textit{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\min\limits_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\}>0.5,\\ 0.5, & \textit{otherwise}\end{cases} \tag{4}$$

where $\Lambda=f^{-1}(-\infty,x)$, and each $\mathcal{M}_k\{\Lambda_k\}$ is derived from (2).

Proof: The conclusions follow directly from (2) and (1), respectively.

Theorem 3 *Let ξ be a common uncertain variable with the continuous distribution $\Phi(x)$. For real numbers b and c, denote*

$$x_1=\frac{-b-\sqrt{b^2-4(c-x)}}{2},\quad x_2=\frac{-b+\sqrt{b^2-4(c-x)}}{2}$$

for $x\ge c-b^2/4$. Then the distribution of the uncertain variable $\xi^2+b\xi+c$ is

$$\Psi(x)=\begin{cases}0, & \textit{if } x<c-\dfrac{b^2}{4}\\ \Phi(x_2)\wedge(1-\Phi(x_1)), & \textit{if } \Phi(x_2)\wedge(1-\Phi(x_1))<0.5\\ \Phi(x_2)-\Phi(x_1), & \textit{if } \Phi(x_2)-\Phi(x_1)>0.5\\ 0.5, & \textit{otherwise.}\end{cases} \tag{5}$$

Proof: For $x<c-b^2/4$, we have

$$\Psi(x)=\mathcal{M}\{\xi^2+b\xi+c\le x\}=\mathcal{M}\{\emptyset\}=0.$$

Let $x\ge c-b^2/4$ in the sequel. Then

$$\Psi(x)=\mathcal{M}\{\xi^2+b\xi+c\le x\}=\mathcal{M}\{x_1\le\xi\le x_2\}=\mathcal{M}\{[x_1,x_2]\}.$$

The conclusion will be proved by (2). Since $[x_1,x_2]\subset(-\infty,x_2]$ and $[x_1,x_2]\subset[x_1,+\infty)$, and $\mathcal{M}\{(-\infty,x_2]\}=\Phi(x_2)$ and $\mathcal{M}\{[x_1,+\infty)\}=1-\Phi(x_1)$, we have $\Psi(x)=\Phi(x_2)\wedge(1-\Phi(x_1))$ if $\Phi(x_2)\wedge(1-\Phi(x_1))<0.5$. Since $[x_1,x_2]^c=(-\infty,x_1)\cup(x_2,+\infty)$, we have

$$\Psi(x)=1-(\Phi(x_1)+1-\Phi(x_2))=\Phi(x_2)-\Phi(x_1)$$

if $\mathcal{M}\{(-\infty,x_1)\}+\mathcal{M}\{(x_2,+\infty)\}=\Phi(x_1)+1-\Phi(x_2)<0.5$, or $\Phi(x_2)-\Phi(x_1)>0.5$. Otherwise $\Psi(x)=0.5$. The proof of the theorem is completed.

4 Expected Value of Function of Uncertain Variable

If the expected value of uncertain variable ξ with uncertainty distribution $\Phi(x)$ exists, then $E[\xi]=\int_0^{+\infty}(1-\Phi(x))dx-\int_{-\infty}^0\Phi(x)dx$; or $E[\xi]=\int_0^1\Phi^{-1}(\alpha)d\alpha$ provided that $\Phi^{-1}(\alpha)$ exists and unique for each $\alpha\in(0,1)$. Thus, if we obtain the uncertainty distribution $\Psi(x)$ of $f(\xi)$, the expected value of $f(\xi)$ is easily derived from

$$E[f(\xi)]=\int_0^{+\infty}(1-\Psi(x))dx-\int_{-\infty}^0\Psi(x)dx. \tag{6}$$

For a monotone function $f(x)$, Liu and Ha [7] gave a formula to compute the expected value of $f(\xi)$ with the uncertainty distribution $\Phi(x)$ of ξ. However, we may generally not present a formula to compute the expected value of $f(\xi)$ with $\Phi(x)$ for a nonmonotone function $f(x)$ because the uncertainty distribution $\Psi(x)$ of $f(\xi)$ may not be analytically expressed by $\Phi(x)$.

Now if we consider a common uncertain variable ξ, the uncertainty distribution $\Psi(x)$ of $f(\xi)$ may be presented by (3), and then the expected value of $f(\xi)$ can be obtained by (6). Next, we will give some examples to show how to compute the expected value of $f(\xi)$ for a common uncertain variable ξ no matter whether $f(x)$ is monotone.

Example 1: Let ξ be a common linear uncertain variable $\mathcal{L}(a,b)$ with the distribution (also see Fig.1)

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/(b-a), & \text{if } a \le x \le b \\ 1, & \text{if } x \ge b. \end{cases}$$

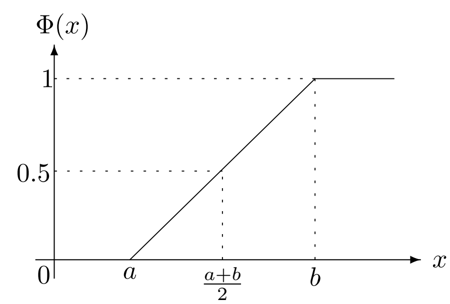

Figure 1: Linear uncertainty distribution

The expected value of ξ is $e = (a+b)/2$. Now we consider the variance of ξ: $V[\xi] = E[(\xi - e)^2]$. Let the uncertainty distribution of $(\xi-e)^2$ be $\Psi(x)$. Let $x \ge 0$, and $x_1 = e - \sqrt{x}$, $x_2 = e + \sqrt{x}$. If $\sqrt{x} \ge (b-a)/2$, then $x_2 \ge b$ and $x_1 \le a$. Thus $\Psi(x) = \Phi(x_2) - \Phi(x_1) = 1$. If $\sqrt{x} \le (b-a)/2$, then $e \le x_2 \le b$ and $a \le x_1 \le e$. Thus $\Phi(x_2) \wedge (1 - \Phi(x_1)) > 0.5$. When $\Phi(x_2) - \Phi(x_1) = 2\sqrt{x}/(b-a) > 0.5$, that is, $\sqrt{x} > (b-a)/4$, $\Psi(x) = \Phi(x_2) - \Phi(x_1) = 2\sqrt{x}/(b-a)$. Hence the uncertainty distribution of $(\xi - e)^2$ (also see Fig.2) is

$$\Psi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.5, & \text{if } 0 \le x \le (b-a)^2/16 \\ 2\sqrt{x}/(b-a), & \text{if } (b-a)^2/16 \le x \le (b-a)^2/4 \\ 1, & \text{if } x \ge (b-a)^2/4 \end{cases}$$

by (5).

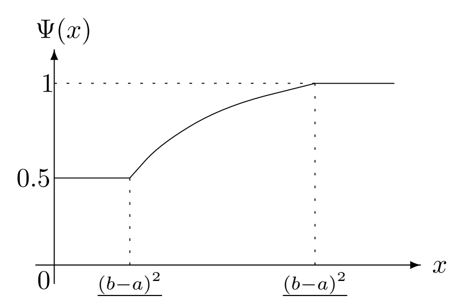

Figure 2: Uncertainty distribution of $(\xi - e)^2$

The variance of ξ is

$$\begin{aligned} V[\xi] = E[(\xi - e)^2] &= \int_0^{+\infty} (1 - \Psi(x))\mathrm{d}x \\ &= \int_0^{(b-a)^2/16} 0.5\mathrm{d}x + \int_{(b-a)^2/16}^{(b-a)^2/4} \left(1 - \frac{2\sqrt{x}}{b-a}\right) \mathrm{d}x \\ &= \frac{7}{96}(b-a)^2. \end{aligned}$$

Example 2: Let ξ be a common linear uncertain variable $\mathcal{L}(-1, 1)$ with the distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le -1 \\ (x+1)/2, & \text{if } -1 \le x \le 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

We will consider the expected value $E[\xi^2 + b\xi]$ for real number b. Let the uncertainty distribution of uncertain variable $\eta = \xi^2 + b\xi$ be $\Psi(x)$. For $x \ge -b^2/4$, denote

$$x_1 = \frac{-b - \sqrt{b^2 + 4x}}{2}, \quad x_2 = \frac{-b + \sqrt{b^2 + 4x}}{2}.$$

(I) If $b = 0$, then $E[\xi^2] = V[\xi] = 7/24$ by Example 1.

(II) If $b \ge 2$, then

$$\Psi(x) = \begin{cases} 0, & \text{if } x < 1 - b \\ \Phi(x_2), & \text{if } x \ge 1 - b \end{cases}$$

by (5). Note that $x = x_2^2 + bx_2$. Thus

$$\begin{aligned} E[\xi^2 + b\xi] &= \int_0^{+\infty} (1 - \Psi(x))\mathrm{d}x - \int_{-\infty}^0 \Psi(x)\mathrm{d}x \\ &= \int_0^{1+b} (1 - \Phi(x_2))\mathrm{d}x - \int_{1-b}^0 \Phi(x_2)\mathrm{d}x \\ &= \int_0^1 \left(1 - \frac{y+1}{2}\right)(2y + b)\mathrm{d}y - \int_{-1}^0 \frac{y+1}{2}(2y+b)\mathrm{d}y \\ &= \frac{1}{3}. \end{aligned}$$

(III) If $1 \le b < 2$, then

$$\Psi(x) = \begin{cases} 0, & \text{if } x < -b^2/4 \\ \Phi(x_2), & \text{if } x \ge -b^2/4. \end{cases}$$

Thus

$$\begin{aligned} E[\xi^2 + b\xi] &= \int_0^{+\infty} (1 - \Psi(x))\mathrm{d}x - \int_{-\infty}^0 \Psi(x)\mathrm{d}x \\ &= \int_0^{1+b} (1 - \Phi(x_2))\mathrm{d}x - \int_{-b^2/4}^0 \Phi(x_2)\mathrm{d}x \\ &= \int_0^1 \left(1 - \frac{y+1}{2}\right)(2y + b)\mathrm{d}y - \int_{-b/2}^0 \frac{y+1}{2}(2y+b)\mathrm{d}y \\ &= \frac{1}{48}(b^3 - 6b^2 + 12b + 8). \end{aligned}$$

(IV) If $0 < b < 1$, then

$$\Psi(x) = \begin{cases} 0, & \text{if } x < -b^2/4 \\ \Phi(x_2), & \text{if } -b^2/4 \le x < 0 \\ 0.5, & \text{if } 0 \le x \le (1-b^2)/4 \\ \Phi(x_2) - \Phi(x_1), & \text{if } x > (1-b^2)/4. \end{cases}$$

Thus

$$\begin{aligned}E[\xi^2+b\xi] &= \int_0^{+\infty}(1-\Psi(x))\mathrm{d}x - \int_{-\infty}^0 \Psi(x)\mathrm{d}x \\ &= \int_0^{(1-b^2)/4}\frac{1}{2}\mathrm{d}x + \int_{(1-b^2)/4}^{1-b}(1-\Phi(x_2)+\Phi(x_1))\mathrm{d}x + \int_{1-b}^{1+b}(1-\Phi(x_2))\mathrm{d}x \\ &\quad - \int_{-b^2/4}^0 \Phi(x_2)\mathrm{d}x \\ &= \frac{1-b^2}{8} + \int_{(1-b)/2}^{1-b}\left(1-\frac{y+1}{2}+\frac{-y-b+1}{2}\right)(2y+b)\mathrm{d}y \\ &\quad + \int_{1-b}^{1}\left(1-\frac{y+1}{2}\right)(2y+b)\mathrm{d}y - \int_{-b/2}^{0}\frac{y+1}{2}(2y+b)\mathrm{d}y \\ &= \frac{1}{48}(b^3+12b^2-12b+14).\end{aligned}$$

(V) If $b \le -2$, then

$$\Psi(x) = \begin{cases} 0, & \text{if } x < 1+b \\ 1-\Phi(x_1), & \text{if } x \ge 1+b. \end{cases}$$

Also we have $E[\xi^2+b\xi] = 1/3$.

(VI) If $-2 < b \le -1$, then

$$\Psi(x) = \begin{cases} 0, & \text{if } x < -b^2/4 \\ 1-\Phi(x_1), & \text{if } x \ge -b^2/4. \end{cases}$$

Thus

$$E[\xi^2+b\xi] = \frac{1}{48}(-b^3-6b^2-12b+8).$$

(VII) If $-1 < b < 0$, then

$$\Psi(x) = \begin{cases} 0, & \text{if } x < -b^2/4 \\ 1-\Phi(x_1), & \text{if } -b^2/4 \le x < 0 \\ 0.5, & \text{if } 0 \le x \le (1-b^2)/4 \\ \Phi(x_2)-\Phi(x_1), & \text{if } x > (1-b^2)/4. \end{cases}$$

Thus

$$\begin{aligned}E[\xi^2+b\xi] &= \int_0^{+\infty}(1-\Psi(x))\mathrm{d}x - \int_{-\infty}^0 \Psi(x)\mathrm{d}x \\ &= \int_0^{(1-b^2)/4}\frac{1}{2}\mathrm{d}x + \int_{(1-b^2)/4}^{1+b}(1-\Phi(x_2)+\Phi(x_1))\mathrm{d}x + \int_{1+b}^{1-b}\Phi(x_1)\mathrm{d}x \\ &\quad - \int_{-b^2/4}^0 (1-\Phi(x_1))\mathrm{d}x \\ &= \frac{1-b^2}{8} + \int_{(1-b)/2}^{1}\left(1-\frac{y+1}{2}+\frac{-y-b+1}{2}\right)(2y+b)\mathrm{d}y \\ &\quad + \int_{(-1-b)/2}^{-1}\frac{y+1}{2}(2y+b)\mathrm{d}y - \int_{-b/2}^{0}\left(1-\frac{y+1}{2}\right)(2y+b)\mathrm{d}y \\ &= \frac{1}{48}(-b^3+12b^2+12b+14)\end{aligned}$$

Example 3: Let ξ be a common normal uncertain variable $\mathcal{N}(e,\sigma)$ with the distribution

$$\Phi(x) = \left(1+\exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1},$$

whose expected value is e. Then the uncertain distribution of $(\xi - e)^2$ is

$$\Psi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.5, & \text{if } 0 \le x \le \dfrac{3\sigma^2(\ln 3)^2}{\pi^2} \\ \dfrac{1-\exp\left(-\frac{\pi\sqrt{x}}{\sqrt{3}\sigma}\right)}{1+\exp\left(-\frac{\pi\sqrt{x}}{\sqrt{3}\sigma}\right)}, & \text{if } x > \dfrac{3\sigma^2(\ln 3)^2}{\pi^2} \end{cases}$$

by (5). Hence the variance of ξ is

$$\begin{aligned} V[\xi] = E[(\xi-e)^2] &= \int_0^{+\infty}(1-\Psi(x))\mathrm{d}x \\ &= \int_0^{\frac{3\sigma^2(\ln 3)^2}{\pi^2}} \frac{1}{2}\mathrm{d}x + \int_{\frac{3\sigma^2(\ln 3)^2}{\pi^2}}^{+\infty} \frac{2\exp\left(-\frac{\pi\sqrt{x}}{\sqrt{3}\sigma}\right)}{1+\exp\left(-\frac{\pi\sqrt{x}}{\sqrt{3}\sigma}\right)}\mathrm{d}x \\ &= \frac{\sigma^2}{\pi^2}\left(\frac{3}{2}(\ln 3)^2 - 12\int_0^{1/3}\frac{\ln z}{1+\ln z}\mathrm{d}z\right) \\ &\approx 0.9432\sigma^2. \end{aligned}$$

5 Uncertain Simulation

It follows from Theorem 3 and the examples in the above section that the uncertainty distribution $\Psi(x)$ of $f(\boldsymbol{\xi})$ may be analytically expressed by (5) for a quadratic function $f(x)$. But $\Psi(x)$ may be hardly analytically expressed for other kinds of functions. Now we will introduce simulation approaches for uncertainty distribution $\Psi(x)$, optimistic value $f_{\sup}$ and expected value $E[f(\boldsymbol{\xi})]$ of $f(\boldsymbol{\xi})$ based on (3) and (4).

(a) Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ be a common uncertain vector where ξ_i is a common uncertain variable with continuous uncertainty distribution $\Phi_i(x)$ for $i = 1, 2, \ldots, n$, and $f : \Re^n \to \Re$ be a Borel function. We will simulate the following uncertain measure:

$$L = \mathcal{M}\{f(\boldsymbol{\xi}) \le 0\}.$$

Algorithm: 1 (Uncertain simulation for L)

Step 1. Set $m_1(i) = 0$ and $m_2(i) = 0$, $i = 1, 2, \ldots, n$.

Step 2. Randomly generate $\boldsymbol{u}_k = (\gamma_k^{(1)}, \gamma_k^{(2)}, \ldots, \gamma_k^{(n)})$ with $0 < \Phi_i(\gamma_k^{(i)}) < 1$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, N$.

Step 3. Rank $\gamma_k^{(i)}$ from small to large as $\gamma_1^{(i)} \le \gamma_2^{(i)} \le \cdots \le \gamma_N^{(i)}$, $i = 1, 2, \ldots, n$.

Step 4. From $k = 1$ to $k = N$, if $f(\boldsymbol{u}_k) \le 0$, $m_1(i) = m_1(i) + 1$, denote $x_{m_1(i)}^{(i)} = \gamma_k^{(i)}$; otherwise, $m_2(i) = m_2(i) + 1$, denote $y_{m_2(i)}^{(i)} = \gamma_k^{(i)}$, $i = 1, 2, \ldots, n$.

Step 5. Set $a^{(i)} = \Phi(x_{m_1(i)}^{(i)}) \wedge (1 - \Phi(x_1^{(i)})) \wedge (\Phi(x_1^{(i)}) + 1 - \Phi(x_2^{(i)})) \wedge \cdots \wedge (\Phi(x_{m_1(i)-1}^{(i)}) + 1 - \Phi(x_{m_1(i)}^{(i)}))$; $b^{(i)} = \Phi(y_{m_2(i)}^{(i)}) \wedge (1 - \Phi(y_1^{(i)})) \wedge (\Phi(y_1^{(i)}) + 1 - \Phi(y_2^{(i)})) \wedge \cdots \wedge (\Phi(y_{m_2(i)-1}^{(i)}) + 1 - \Phi(y_{m_2(i)}^{(i)}))$, $i = 1, 2, \ldots, n$.

Step 6. If $a^{(i)} < 0.5$, return $L_1^{(i)} = a^{(i)}$, $L_2^{(i)} = 1 - a^{(i)}$; if $b^{(i)} < 0.5$, return $L_1^{(i)} = 1 - b^{(i)}$, $L_2^{(i)} = b^{(i)}$; otherwise, return $L_1^{(i)} = 0.5$, $L_2^{(i)} = 0.5$, $i = 1, 2, \ldots, n$.

Step 7. If $a = L_1^{(1)} \wedge L_1^{(2)} \wedge \cdots \wedge L_1^{(n)} > 0.5$, then $L = a$; if $b = L_2^{(1)} \wedge L_2^{(2)} \wedge \cdots \wedge L_2^{(n)} > 0.5$, then $L = 1 - b$; otherwise, $L = 0.5$.

(b) Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ be a common uncertain vector where ξ_i is a common uncertain variable with continuous uncertainty distribution $\Phi_i(x)$ for $i = 1, 2, \ldots, n$, and $f : \Re^n \to \Re$ be a Borel function. The following algorithm is used to simulate the optimistic value:

$$f_{\sup} = \sup\{r \mid \mathcal{M}\{f(\boldsymbol{\xi}) \ge r\} \ge \alpha\}$$

where $\alpha \in (0,1)$ is a predetermined confidence level.

Algorithm: 2 (Uncertain simulation for $f_{\sup}$)

Step 1. Randomly generate $\boldsymbol{u}_k = (\gamma_k^{(1)}, \gamma_k^{(2)}, \ldots, \gamma_k^{(n)})$ with $0 < \Phi_i(\gamma_k^{(i)}) < 1$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, m$.
Step 2. Set $a = f(\boldsymbol{u}_1) \wedge f(\boldsymbol{u}_2) \wedge \cdots \wedge f(\boldsymbol{u}_m)$, $b = f(\boldsymbol{u}_1) \vee f(\boldsymbol{u}_2) \vee \cdots \vee f(\boldsymbol{u}_m)$.
Step 3. Set $r = (a+b)/2$.
Step 4. If $\mathcal{M}\{f(\boldsymbol{\xi}) \geq r\} \geq \alpha$, then $a \leftarrow r$.
Step 5. If $\mathcal{M}\{f(\boldsymbol{\xi}) \geq r\} < \alpha$, then $b \leftarrow r$.
Step 6. Repeat the third to fifth steps until $b - a < \epsilon$ for a sufficiently small number ϵ.
Step 7. $f_{\sup} = (a+b)/2$.

(c) Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ be a common uncertain vector where ξ_i is a common uncertain variable with continuous uncertainty distribution $\Phi_i(x)$ for $i = 1, 2, \ldots, n$, and $f : \Re^n \to \Re$ be a Borel function. The expected value $E[f(\boldsymbol{\xi})]$ is approached by the following algorithm:

Algorithm: 3 (Uncertain simulation for E)

Step 1. Set $E = 0$.
Step 2. Randomly generate $\boldsymbol{u}_k = (\gamma_k^{(1)}, \gamma_k^{(2)}, \ldots, \gamma_k^{(n)})$ with $0 < \Phi_i(\gamma_k^{(i)}) < 1$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, m$.
Step 3. Set $a = f(\boldsymbol{u}_1) \wedge f(\boldsymbol{u}_2) \wedge \cdots \wedge f(\boldsymbol{u}_m)$, $b = f(\boldsymbol{u}_1) \vee f(\boldsymbol{u}_2) \vee \cdots \vee f(\boldsymbol{u}_m)$.
Step 4. Randomly generate r from $[a, b]$.
Step 5. If $r \geq 0$, then $E \leftarrow E + \mathcal{M}\{f(\boldsymbol{\xi}) \geq r\}$.
Step 6. If $r < 0$, then $E \leftarrow E + \mathcal{M}\{f(\boldsymbol{\xi}) \leq r\}$.
Step 7. Repeat the fourth to sixth steps for N times.
Step 8. $E[f(\boldsymbol{\xi})] = a \vee 0 + b \wedge 0 + E \cdot (b-a)/N$.

6 Uncertain Programming

Uncertain Expected Value Model

Genetic algorithm (GA) integrated with the simulation method (Algorithm 3) introduced in the previous section may be used to solve the following uncertain expected value model:

$$\begin{cases} \max\limits_{\boldsymbol{x}} E\left[f(\boldsymbol{x}, \boldsymbol{\xi})\right] \\ \text{subject to:} \\ \qquad E[g_j(\boldsymbol{x}, \boldsymbol{\xi})] \leq 0, \ j = 1, 2, \ldots, p \end{cases} \tag{7}$$

where $\boldsymbol{x}$ is a decision vector, $\boldsymbol{\xi}$ is a common uncertain vector, $f(\boldsymbol{x}, \boldsymbol{\xi})$ is the return function, and $g_j(\boldsymbol{x}, \boldsymbol{\xi})$ is the constraint function, $j = 1, 2, \ldots, p$.

Example 4: Let $\xi_1 \sim \mathcal{Z}(-2, 1, 2)$, $\xi_2 \sim \mathcal{Z}(-1, 1, 3)$, $\xi_3 \sim \mathcal{Z}(1, 2, 3)$. Consider the following uncertain expected value model:

$$\begin{cases} \overline{E} \equiv \max\limits_{x_1, x_2, x_3} E\left[\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}\right] \\ \text{subject to:} \\ \qquad x_1^2 + x_2^2 + x_3^2 \leq 6. \end{cases}$$

Algorithm 3 is employed to approximate the objective function

$$(x_1, x_2, x_3) \to E\left[\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}\right],$$

and GA is employed for providing the optimal solution

$$\boldsymbol{x} = (x_1, x_2, x_3) = (-0.6613, -0.9383, -2.1635), \quad \overline{E} = 4.8084.$$

Uncertain Chance-Constrained Programming Model

When we want to maximize the optimistic return, we have the following uncertain maximax CCP model:

$$\begin{cases} \max\limits_{\boldsymbol{x}} \max\limits_{\overline{f}} \overline{f} \\ \text{subject to:} \\ \qquad \mathcal{M}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq \overline{f}\} \geq \beta \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0,\ j = 1, 2, \ldots, p\} \geq \alpha \end{cases} \tag{8}$$

where α and β are the predetermined confidence levels, and $\max \overline{f}$ is the β-optimistic return.

If we want to maximize the pessimistic return, then we have the following minimax CCP model:

$$\begin{cases} \max\limits_{\boldsymbol{x}} \min\limits_{\overline{f}} \overline{f} \\ \text{subject to:} \\ \qquad \mathcal{M}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq \overline{f}\} \geq \beta \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0,\ j = 1, 2, \ldots, p\} \geq \alpha \end{cases} \tag{9}$$

where $\min \overline{f}$ is the β-pessimistic return.

Example 5: Let $\xi_1 \sim \mathcal{L}(-2, 0)$, $\xi_2 \sim \mathcal{L}(-1, 1)$, $\xi_3 \sim \mathcal{L}(1, 3)$. Consider the following CCP model:

$$\begin{cases} \hat{f} \equiv \max\limits_{\boldsymbol{x}} \max\limits_{\overline{f}} \overline{f} \\ \text{subject to:} \\ \qquad \mathcal{M}\{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2 \geq \overline{f}\} \geq 0.95 \\ \qquad \mathcal{M}\{|x_1 - \xi_1| + |x_2 - \xi_2| + |x_3 - \xi_3| \leq 10\} \geq 0.90 \\ \qquad x_1, x_2, x_3 \geq 0. \end{cases}$$

Algorithm 2 is employed to approximate the function

$$(x_1, x_2, x_3) \rightarrow \max\{r \mid \mathcal{M}\{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2 \geq r\} \geq 0.95\},$$

and Algorithm 1 is employed to approximate the function

$$(x_1, x_2, x_3) \rightarrow \mathcal{M}\{|x_1 - \xi_1| + |x_2 - \xi_2| + |x_3 - \xi_3| \leq 10\}.$$

Then GA is employed for providing the optimal solution

$$\boldsymbol{x} = (x_1, x_2, x_3) = (6.3223, 0.0174, 2.0090), \quad \hat{f} = 40.3047.$$

7 Conclusion

Generally speaking, the uncertainty distribution of function $f(\boldsymbol{\xi})$ of an uncertain vector $\boldsymbol{\xi}$ may be not derived from the distribution of $\boldsymbol{\xi}$ except that the function $f(\boldsymbol{x})$ is monotone. In this paper, a scheme is introduced for establishing the uncertainty distribution of function $f(\boldsymbol{\xi})$ directly from the distribution of $\boldsymbol{\xi}$ without the monotonicity of $f(\boldsymbol{x})$ for a common uncertain vector $\boldsymbol{\xi}$. Then the expected value and optimistic value of $f(\boldsymbol{\xi})$ may be obtained also from the distribution of $\boldsymbol{\xi}$. For the sake of numeric computation, uncertain simulation schemes are presented for the uncertainty distribution, the expected value and optimistic value of $f(\boldsymbol{\xi})$. Finally, the uncertain simulation schemes are integrated to genetic algorithm (GA) for solving uncertain expected value models and uncertain chance-constrained programming models. Examples show the efficiency of the methods. It should be pointed out that uncertain simulations are time-consuming processes. Thus, in order to speed up the solution process, neural network (NN) may be employed to approximate uncertain expected value and optimistic value based on the data produced by uncertain simulations. Numerical experiments of uncertain programming models by employing uncertain simulations, NN and GA remain for the interested readers.

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