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Aumann-Shapley Values on a Class of Cooperative Fuzzy Games

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Abstract

This paper defines a class of cooperative fuzzy games with infinite players, which is monotonically nondecreasing and continuous with respect to each player's grade of membership. Then, the Aumann-Shapley values for the games with fuzzy coalitions are proposed with the form of Choquet integral. The properties of the Aumann-Shapley values are also shown. Finally, the relationship between the fuzzy Aumann-Shapley value and the fuzzy Shapley function is given.

Keywords: fuzzy cooperative game, Aumann-Shapley value, Shapley function, Choquet integral

1 Introduction

In applying n-person game theory to economic analysis, it is recognized that small games (i.e., games with finite players) are difficult to represent free-market situations. Then, one need to investigate the games with a large number of players (i.e., games with infinite players) that any single player has a negligible effect on the payoffs to the other players [12]. This is a so-called non-atomic game. The Aumann-Shapley value is a well-known solution concept in non-atomic game, which is an important generalization of the Shapley value to the game with infinite players [3].

The Aumann-Shapley value has been investigated by a number of researchers. Most of them treated the games with crisp coalitions. However, there are some situations where some agents do not fully participate in a coalition, but to a certain degree. Such a coalition including some players participating partially is so-called fuzzy coalition introduced by Aubin [1, 2]. It can be represented by a fuzzy subset of the set of players. The membership degree shows to what extent a player transfers his representability and is called a rate of participation. In this paper, we mainly discuss the Aumann-Shapley values for the games with fuzzy coalitions.

Now, the researches on the games with fuzzy coalitions mainly focus on the fuzzy games with finite players, i.e., n-person cooperative games. In 1980, Butnariu [4] firstly defined a fuzzy Shapley function, which maps a fuzzy game to a function deriving the Shapley value from a fuzzy coalition. He gave the expression of the Shapley function on a limited class of fuzzy games − games with proportional values. But most of the games with proportional values are neither monotone non-decreasing nor continuous with regard to the rates of players' participation. Hence, Tsurumi et al. [13] introduced a new class of fuzzy games with Choquet integral form, which was continuous with regard to the rates of players' participation, and studied the Shapley function defined on this class of games. Recently, Butnariu and Kroupa [6] extended the fuzzy games with proportional values to the fuzzy games with weighted function, and the corresponding Shapley function also is given. Following Butnariu and Tsurumi's approaches, there are also many researches on the solutions of the fuzzy games proposed by Butnariu [4] and Tsurumi et al. [13]. Additionally, the Shapley function was generalized in [9] and the simplified expression of the Shapley function was compared with two definitions established by Butnariu [4] and Tsurumi et al. [13]. In [14], the fuzzy cores of the fuzzy games defined by Butnariu and Tsurumi were studied and the nonempty condition of the fuzzy core was discussed. In [10], the specific expression of the Shapley function for the fuzzy cooperative games with multilinear extension form was given, and its existence and uniqueness were discussed.

However, there are few researches on the fuzzy cooperative games with infinite players. In 1993, Butnariu et al. [5] gave the Aummann-Shapley values of non-atomic games with fuzzy coalitions and discussed the

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existence of the Aummann-Shapley values on the different spaces of fuzzy non-atomic games based on the measures of the triangular norm. In this paper, following Tsurumi's approach, we continue to study the Aummann-Shapley values of non-atomic games with fuzzy coalitions. First, we introduce a particular class of non-atomic games with fuzzy coalitions on Choquet integral where the games are both monotone nondecreasing and continuous with regard to the rates of players' participation. Then, we define the Aumann-Shapley values of the class of non-atomic games with fuzzy coalitions and study the properties of the Aumann-Shapley value. We show that the defined Aumann-Shapley values satisfy four axioms and are continuous with regard to the rates of players' participation. Finally, we discuss the relationship between the defined Aumann-Shapley value with the Shapley function proposed by Tsurumi [6]. We show that the defined Aumann-Shapley value is exactly the Shapley function.

2 Cooperative Fuzzy Games and the Aumann-Shapley Values

2.1 Preliminaries

In this paper, we consider cooperative fuzzy games with infinite players, i.e., a continuum of players. Let X be the set of players which is a infinite set. A fuzzy coalition is a fuzzy subset of X, which is identified with a measurable function from X to [0,1]. Then for a fuzzy coalition A and player x, $A(x)$ indicates the membership degree of x in A, i.e., the rate of player x's participation in the fuzzy coalition A. For a fuzzy coalition A, the level set is denoted by $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$ for any $\alpha \in [0,1]$. The class of all fuzzy subsets of a fuzzy set $A \subseteq X$ is denoted by $\mathcal{F}(A)$. Particularly, $\mathcal{P}(B)$ denotes the set of all crisp subsets of a crisp set $B \subseteq X$.

A fuzzy game is a function $v : \mathcal{F}(X) \to \mathbb{R}$ with $v(\emptyset) = 0$. $G(X)$ denotes the set of all fuzzy games. A function $v : \mathcal{P}(B) \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a crisp game. We denote the set of all bounded monotone crisp games by $G_0(X)$.

In this paper, union and intersection of two fuzzy sets are defined as follows:

$(A \cup B)(x) = \max\{A(x), B(x)\},\ \forall x \in X;$

$(A \cap B)(x) = \min\{A(x), B(x)\},\ \forall x \in X.$

The complement of a fuzzy set can be defined as follows:

$A^c(x) = 1 - A(x),\ \forall x \in X.$

Then a subfamily $\mathcal{T}$ of $\mathcal{F}(X)$ containing $\emptyset$ and being closed under countable intersections and complement is called a fuzzy $\sigma-$algrebra.

Next, superadditivity and monotonicity in the fuzzy games are defined as follows:

Definition 2.1 *A game $v \in G(X)$ is said to be monotone if $v(A) \geq v(B),\ \forall A, B \in \mathcal{F}(X),\ s.t. B \subseteq A$.*

Definition 2.2 *A game $v \in G(X)$ is said to be superadditive if $v(A \cup B) \geq v(A) + v(B),\ \forall A, B \in \mathcal{F}(X),\ s.t.\ A \cap B = \emptyset$.*

Obviously, if v is superadditive, then v is monotone nondecreasing with respect to set inclusion.

Definition 2.3 *A game $v \in G(X)$ is said to be convex if $v(A \cup B) + v(A \cap B) \geq v(A) + v(B),\ \forall A, B \in \mathcal{F}(X)$.*

Definition 2.4 *A game $v \in G(X)$ is said to be of bounded variation if for each finite family of fuzzy coalitions $\mathcal{C} : \emptyset = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = X$, we have*

$$\sup_{\mathcal{C}}\{\sum_{(A_i, A_{i+1}) \in \mathcal{C}} |v(A_i) - v(A_{i+1})|\} < \infty.$$

Obviously, if the game v is monotone, we can get $\sup_{\mathcal{C}}\{\sum_{(A_i, A_{i+1}) \in \mathcal{C}} |v(A_i) - v(A_{i+1})|\} = v(X)$ for each finite family of fuzzy coalitions $\mathcal{C} : \emptyset = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = X$. Therefore, the game v is of bounded variation. We denote the set of all crisp games which are of bounded variation by BV and the set of all fuzzy games which are of bounded variation by FBV.

We define the value of a fuzzy game as an extension of a usual value of a crisp game as follows.

Definition 2.5 *A function $\varphi[v] : \mathcal{F}(X) \to \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ is said to be a value of a fuzzy game v if it is a finitely additive fuzzy set function, i.e., $\varphi[v](A \cup B) = \varphi[v](A) + \varphi[v](B),\ \forall A, B \in \mathcal{F}(X),\ s.t. A \cap B = \emptyset$.*

We denote the set of all finitely additive crisp set functions by BA and the set of all finitely additive fuzzy set functions by FBA.

2.2 The Aumann-Shapley Values on the Class of Crisp Games

In this subsection, we introduce the Aumann-Shapley values on $G_0(X)$.

Let π be a permutation of X(i.e., a one-one, measurable mapping of X onto itself). Then, for any game v, we define the game $\pi_* v$ by $\pi_* v(A) = v(\pi(A))$ for all $A \in \mathcal{P}(X)$. Let $\mathcal{B}$ be a linear subspace of BV satisfying $\pi_* \mathcal{B} \subseteq \mathcal{B}$ for all permutations π of X. Now, we define the Aumann-Shapley value as follows.

Definition 2.6 [3] *An operator $\varphi' : \mathcal{B} \to BA$ is said to be an Aumann-Shapley value on $\mathcal{B}$ if it satisfies the following four axioms.*

Axiom C_1: For any game $v \in \mathcal{B}$ and the players set X, $\varphi'[v](X) = v(X)$.

Axiom C_2: For any game $v \in \mathcal{B}$ and permutation π, $\varphi'[\pi_* v] = \pi_* \varphi'[v]$, where $\pi_* v(A) = v(\pi A)$ for any crisp coalition $A \in \mathcal{P}(X)$.

Axiom C_3: For any game $v, w \in \mathcal{B}$ and scalars α, β, $\varphi'[\alpha v + \beta w] = \alpha \varphi'[v] + \beta \varphi'[w]$.

Axiom C_4: If v is monotone, then $\varphi'[v]$ is also monotone (i.e., $\varphi'[v](A) \geq \varphi'[v](B)$, whenever $B \subseteq A$ for any $A, B \in \mathcal{P}(X)$).

Note that the first three axioms are very similar to Shapley value's three axioms, though Axiom C_1 is actually weaker than the Shapley value's. On the other hand, the monotonicity axiom (Axiom C_4) is different from Shapley value's axiom. But it's reasonable. We can see that any coalition always makes nonnegative contributions to any set that it may join if v is monotone. Then it should take a nonnegative value which is all that Axiom C_4 requires.

2.3 The Aumann-Shapley Values on the Class of Fuzzy Games

In this subsection, we dedicate to giving the definition of the Aumann-Shapley value on the fuzzy games.

Let π be a permutation of X. Then, for any fuzzy game v, we define the game $\pi_* v$ by $\pi_* v(A) = v(\pi(A))$ for all $A \in \mathcal{T}$. Let Q be a linear subspace of FBV satisfying $\pi_* Q \subseteq Q$ for all permutations π of X. Now, we define the Aumann-Shapley value as follows.

Definition 2.7 *An operator $\varphi : Q \to FBA$ is said to be an Aumann-Shapley value on Q if it satisfies the following four axioms.*

Axiom F_1: For any game $v \in Q$ and the players set X, $\varphi[v](X) = v(X)$.

Axiom F_2: For any game $v \in Q$ and automorphism π, $\varphi[\pi_* v] = \pi_* \varphi[v]$, where $\pi_* v(A) = v(\pi A)$ for any fuzzy coalition $A \in \mathcal{T}$.

Axiom F_3: For any game $v, w \in Q$ and scalars α, β, $\varphi[\alpha v + \beta w] = \alpha \varphi[v] + \beta \varphi[w]$.

Axiom F_4: If v is monotone, then $\varphi[v]$ is also monotone.

3 The Fuzzy Games with Choquet Integral Form $G_C(X)$

In this section, we shall define a new class of fuzzy games with some good properties, shown in the following.

Definition 3.1 *Let X be a set of players (who participate in interactive activities by forming fuzzy coalition A), and A a fuzzy subset of X. Then a game $v \in G(X)$ is said to be a fuzzy game 'with Choquet integral form' if and only if the following equation holds:*

$$v(A) = \int_0^1 v(A_\alpha) d\alpha. \tag{1}$$

It's apparent that (1) is a choquet integral [7, 8] of the function A with respect to v, i.e., $v(A) = (C) \int_X A dv$. There is a one-to-one correspondence between a crisp game and a fuzzy game 'with Choquet integral form'. We call the crisp game corresponding to a fuzzy game 'with Choquet integral form' the associated crisp game, and the fuzzy game 'with Choquet integral form' corresponding to a crisp game the associated fuzzy game. For simplicity, we denote the crisp game associated with $G_C(X)$ by $G_0(X)$ if there is no fear of confusion.

From the definition of $G_C(X)$, we obtain the following relations. It's clear that any $v \in G_C(X)$ is monotone nondecreasing with respect to A since the associated game is monotone nondecreasing, shown in the next lemma.

Lemma 3.1 *Let $v \in G_C(X)$. Then the following inequality holds: $v(A) \leq v(B),\ \forall A, B \in \mathcal{F}(X),\ s.t.\ A \subseteq B$. In other words, any $v \in G_C(X)$ is monotone nondecreasing with respect to each player's grade of membership.*

Proof. Note that $A \subseteq B$ if and only if $A_\alpha \subseteq B_\alpha$ for any $\alpha \in [0,1]$. Hence, it's apparent that $v(A_\alpha) \leq v(B_\alpha)$ for any $v \in G_0(X)$ since $v \in G_0(X)$ is monotone. From the definition of $v \in G_C(X)$, we obtain $v(A) = \int_0^1 v(A_\alpha)d\alpha \leq \int_0^1 v(B_\alpha)d\alpha = v(B)$. That is $v \in G_C(X)$ is monotone nondecreasing.

It's also apparent that $v \in G_C(X)$ is superadditive if the associated game in $G_0(X)$ is superadditive, shown in the following lemma.

Lemma 3.2 *Let $v \in G_C(X)$ and the associated game in $G_0(X)$ be superadditive. Then the following inequality holds: $v(A \cup B) \geq v(A) + v(B),\ \forall A, B \in \mathcal{F}(X),\ s.t.\ A \cap B = \emptyset$.*

Proof. For any $A, B \in \mathcal{F}(X)$ and $A \cap B = \emptyset$, we have $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$ for any $\alpha \in [0,1]$.

Hence,

$$\begin{aligned} v(A \cup B) &= \int_0^1 v[(A \cup B)_\alpha]d\alpha \\ &= \int_0^1 v(A_\alpha \cup B_\alpha)d\alpha \\ &\geq \int_0^1 [v(A_\alpha) + v(B_\alpha)]d\alpha \\ &= \int_0^1 v(A_\alpha)d\alpha + \int_0^1 v(B_\alpha)d\alpha \\ &= v(A) + v(B). \end{aligned}$$

That is, $v \in G_C(X)$ is superadditive.

It's also clear that $v \in G_C(X)$ is convex if the associated game in $G_0(X)$ is convex, shown in the next lemma.

Lemma 3.3 *Let $v \in G_C(X)$ and the associated game $v \in G_0(X)$ is convex. Then the following holds: $v(A \cup B) + v(A \cap B) \geq v(A) + v(B),\ \forall A, B \in \mathcal{F}(X)$.*

Proof. Note that $\forall A, B \in \mathcal{F}(X)$, $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$ and $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$ for any $\alpha \in [0,1]$. Hence,

$$\begin{aligned} v(A \cup B) + v(A \cap B) &= \int_0^1 v[(A \cup B)_\alpha]d\alpha + \int_0^1 v[(A \cap B)_\alpha]d\alpha \\ &= \int_0^1 [v(A \cup B)_\alpha + v(A \cap B)_\alpha]d\alpha \\ &= \int_0^1 [v(A_\alpha \cup B_\alpha) + v(A_\alpha \cap B_\alpha)]d\alpha \\ &\geq \int_0^1 [v(A_\alpha) + v(B_\alpha)]d\alpha \\ &= \int_0^1 v(A_\alpha)d\alpha + \int_0^1 v(B_\alpha)]d\alpha \\ &= v(A) + v(B). \end{aligned}$$

That is $v \in G_C(X)$ is convex if the associated game $v \in G_0(X)$ is convex.

Theorem 3.1 *Define the distance d in $\mathcal{F}(X)$ by $d(A,B) = \sup_{x \in X} |A(x) - B(x)|$, for any $A, B \in \mathcal{F}(X)$. Then $v \in G_C(X)$ is continuous with respect to the each player's grade of membership.*

Proof. First, we prove that d is the distance of fuzzy sets A and B. From the definition of d, it's apparent that $d \geq 0$, $d(A,A)=0$, and $d(A,B)=d(B,A)$ for any $A,B \in \mathcal{F}(X)$. For any $A,B,C \in \mathcal{F}(X)$, we have

$$\begin{aligned} d(A,C) &= \sup_{x\in X}|A(x)-C(x)| = \sup_{x\in X}|A(x)-B(x)+B(x)-C(x)| \\ &\leq \sup_{x\in X}[|A(x)-B(x)|+|B(x)-C(x)|] \leq \sup_{x\in X}|A(x)-B(x)| + \sup_{x\in X}|B(x)-C(x)| \\ &= d(A,B)+d(B,C). \end{aligned}$$

That is, d is the distance of fuzzy sets A and B.

Let $A \in \mathcal{F}(X)$, δ be sufficiently small, and $B \in \mathcal{F}(X)$ satisfying $d(A,B) < \delta$. We shall prove that $v(B) \to v(A)$ if $\delta \to 0$. Note that $|A(x)-B(x)| < \delta$ for any $x \in X$ since $d(A,B) = \sup_{x\in X}|A(x)-B(x)| < \delta$. That is, $A(x)-\delta < B(x) < A(x)+\delta$. Hence, for any $\alpha \in [0,1]$, we have $\{x|A(x)-\delta \geq \alpha\} \subseteq \{x|B(x) \geq \alpha\} \subseteq \{x|A(x)+\delta \geq \alpha\}$. The associated game $v \in G_0(X)$ is monotone nondecreasing with respect to set inclusion. Hence, $v\{x|A(x)-\delta \geq \alpha\} \leq v\{x|B(x)\geq\alpha\} \leq v\{x|A(x)+\delta \geq \alpha\}$ holds for any $\alpha \in [0,1]$. That is $v\{x|A(x) \geq \alpha+\delta\} \leq v(B_\alpha) \leq v\{x|A(x) \geq \alpha-\delta\}$. Thus we get $v(B_\alpha) \to v(A_\alpha)$ if $\delta \to 0$.

Note that for any $v \in G_C(X)$, there exists $\xi \in [0,1]$ such that the following holds:

$|v(A)-v(B)| = |\int_0^1 v(A_\alpha)d\alpha - \int_0^1 v(B_\alpha)d\alpha| = |\int_0^1 [v(A_\alpha)-v(B_\alpha)]d\alpha| = |v(A_\xi)-v(B_\xi)| \to 0$ if $\delta \to 0$.

Hence, if $\delta \to 0$, then $v(B) \to v(A)$. The proof is completed.

Lemma 3.4 *Let $v \in G_C(X)$, and $A,B \in \mathcal{F}(X)$ such that $A \subseteq B$. Then $v(A) = v(B)$ if and only if $v(A_\alpha) = v(B_\alpha)$ for any $\alpha \in [0,1]$.*

Proof. It's apparent that $v(A) = v(B)$ if $v(A_\alpha) = v(B_\alpha)$. Assume that $v(A) = v(B)$ but there exists $\alpha \in [0,1]$ such that $v(A_\alpha) \neq v(B_\alpha)$. Then we have

$0 = v(B) - v(A) = \int_0^1 v(B_\alpha)d\alpha - \int_0^1 v(A_\alpha)d\alpha = \int_0^1 [v(B_\alpha)-v(A_\alpha)]d\alpha.$

The associated game $v \in G_0(X)$ is nondecreasing with respect to set inclusion, and $A \subseteq B$ if and only if $A_\alpha \subseteq B_\alpha$ for any $\alpha \in [0,1]$. Hence, $v(A_\alpha) \leq v(B_\alpha)$ and there exists $\alpha \in [0,1]$ such that $v(B_\alpha) > v(A_\alpha)$. That is $v(A_\alpha) \equiv v(B_\alpha)$ can not hold for $\alpha \in [0,1]$. It follows that $\int_0^1 [v(B_\alpha)-v(A_\alpha)]d\alpha > 0$, which is contradicted with $\int_0^1 [v(B_\alpha)-v(A_\alpha)]d\alpha = 0$. Hence, we have $v(A_\alpha) = v(B_\alpha)$ for any $\alpha \in [0,1]$ if $v(A) = v(B)$. The proof is completed.

4 The Aumann-Shapley Values on $G_C(X)$

Define a function $\varphi : G_C(X) \to \mathbb{R}_+$ by

$$\varphi[v](A) = \int_0^1 \varphi'[v](A_\alpha)d\alpha, \tag{2}$$

where φ' is the Aumann-Shapley value given in Definition 2.6 and $A \in \mathcal{T}$. Note that (2) is a choquet integral of the function A with regard to φ' and it can be represented by $(C)\int_X Ad\varphi'[v]$. Now we show that φ is an Aumann-Shapley value on $G_C(X)$.

Lemma 4.1 *The function φ defined by (2) is finitely additive fuzzy set function.*

Proof. Let any $A,B \in \mathcal{T}$ satisfy $A \cap B = \emptyset$. Hence we have $A_\alpha \cap B_\alpha = \emptyset$ for any $\alpha \in [0,1]$. Since φ' is finitely additive set function, we can get $\varphi'[v][(A\cup B)_\alpha] = \varphi'[v](A_\alpha \cup B_\alpha) = \varphi'[v](A_\alpha) + \varphi'[v](B_\alpha)$ for any $\alpha \in [0,1]$. It follows that

$$\begin{aligned} \varphi[v](A\cup B) &= \int_0^1 \varphi'[v][(A\cup B)_\alpha]d\alpha = \int_0^1 \varphi'[v](A_\alpha \cup B_\alpha)d\alpha \\ &= \int_0^1 \{\varphi'[v](A_\alpha) + \varphi'[v](B_\alpha)\}d\alpha = \int_0^1 \varphi'[v](A_\alpha)d\alpha + \int_0^1 \varphi'[v](B_\alpha)d\alpha \\ &= \varphi[v](A) + \varphi[v](B). \end{aligned}$$

Theorem 4.1 *The function defined by (2)is an Aumann-Shapley value on $G_C(X)$.*

Proof. We shall prove that the function defined by (2) satisfies Axioms $F_1 - F_4$.

Axiom F_1: Let $v \in G_C(X)$. Since $X(x) = 1$ for any $x \in X$ and $\varphi'[v](X) = v(X)$ holds from Axiom C_1, we obtain

$\varphi[v](X) = \int_0^1 \varphi'[v][(X)_\alpha]d\alpha = \int_0^1 \varphi'[v](X)d\alpha = v(X).$

Axiom F_2: For any game $v \in G_C(X)$, permutation π and any $A \in \mathcal{T}$, from $\pi_* v(A) = v(\pi A)$ and Axiom C_2, we have

$\varphi[\pi_* v](A) = \varphi[v](\pi A) = \int_0^1 \varphi'[v][(\pi A)_\alpha]d\alpha = \int_0^1 \varphi'[\pi_* v](A)_\alpha d\alpha = \int_0^1 \pi_* \varphi'[v](A)_\alpha d\alpha = \pi_* \varphi[v](A).$

This completes the proof of Axiom F_2.

Axiom F_3: For $v, w \in G_C(X)$, scalars α, β and any $A \in \mathcal{T}$, it is clear that $\alpha v + \beta w \in G_C(X)$ from the definition of $G_C(X)$. Since $\varphi'[\alpha v + \beta w] = \alpha\varphi'[v] + \beta\varphi'[w]$ holds from Axiom C_3, we obtain

$$\begin{aligned}\varphi[\alpha v + \beta w](A) &= \int_0^1 \varphi'[\alpha v + \beta w](A_\alpha)d\alpha = \int_0^1 \{\alpha\varphi'[v] + \beta\varphi'[w]\}(A_\alpha)d\alpha \\ &= \int_0^1 \alpha\varphi'[v](A_\alpha)d\alpha + \int_0^1 \beta\varphi'[w](A_\alpha)d\alpha = \alpha\int_0^1 \varphi'[v](A_\alpha)d\alpha + \beta\int_0^1 \varphi'[w](A_\alpha)d\alpha \\ &= \alpha\varphi[v](A) + \beta\varphi[w](A) = \{\alpha\varphi[v] + \beta\varphi[w]\}(A).\end{aligned}$$

Axiom F_4: Let the associated game $v \in G_0(X)$ be monotone and $A, B \in \mathcal{T}$ satisfy $A \subseteq B$. Using Lemma 3.1, we have v is monotone for $v \in G_C(X)$. Since $A \subseteq B$ if and only if $A_\alpha \subseteq B_\alpha$ for any $\alpha \in [0,1]$, we have $v(A_\alpha) \le v(B_\alpha)$ for the associated game v. It follows that $\varphi'[v](A_\alpha) \le \varphi'[v](B_\alpha)$ from Axiom C_4. Hence we obtain $\varphi[v](A) = \int_0^1 \varphi'[v](A_\alpha)d\alpha \le \int_0^1 \varphi'[v](B_\alpha)d\alpha = \varphi[v](B)$. That is, $\varphi[v]$ is monotone.

Theorem 4.2 *Define the distance d in $\mathcal{F}(X)$ by $d(A,B) = \sup_{x\in X}|A(x) - B(x)|$ for any $A, B \in \mathcal{F}(X)$. If $v \in G_C(X)$ is monotone, then $\varphi[v]$ is continuous with respect to each player's grade of membership.*

Proof. The theorem can be proved in the same manner as Theorem 3.1.

Corollary 4.1 *Let the associated game $v = f \circ (\mu_1, \mu_2, \dots, \mu_n)$ where $\mu_i (i = 1, 2, \dots, n)$ are non-atomic measures satisfying $\mu_i(X) = 1$ and $f : \mathcal{R}(\mu_1, \mu_2, \dots, \mu_n) \to \mathbb{R}$ is a continuously differentiable function of n variables satisfying $f(0) = 0$. Then for any $A \in \mathcal{T}$, the Aumann-Shapley value of fuzzy game $v(A) = (C)\int_X Ad[f \circ (\mu_1, \mu_2, \dots, \mu_n)]$ has the following formula*

$$\varphi[v](A) = \sum_{i=1}^{n} \alpha_i \cdot (C)\int_X Ad\mu_i$$

where $\alpha_i = \int_0^1 f_i(t, t, \dots, t)dt$ and f_i is the partial derivative to the ith variable of f.

Proof. For the associated game $v = f \circ (\mu_1, \mu_2, \dots, \mu_n)$, the Aumann-Shapley value $\varphi'[v] = \sum_{i=1}^n \alpha_i \cdot \mu_i$, where $\alpha_i = \int_0^1 f_i(t, t, \dots, t)dt$. Hence using Formula (2), we can get Aumann-Shapley value $\varphi[v](A)$ of the fuzzy game $v(A) = (C)\int_X Ad[f \circ (\mu_1, \mu_2, \dots, \mu_n)]$. That is,

$\varphi[v](A) = (C)\int_X Ad\varphi'[v] = \int_0^1 \varphi'[v](A_\alpha)d\alpha = \sum_{i=1}^n \alpha_i \cdot \int_0^1 \mu_i(A_\alpha)d\alpha = \sum_{i=1}^n \alpha_i \cdot (C)\int_X Ad\mu_i.$

Example 4.1 *Let $\mu_i (i = 1, 2, 3)$ be three distinct crisp measures on X with $\mu_i(X) = 1$. Suppose the associated game $v = \mu_1\mu_2 + \mu_2\mu_3^2$. Then we can get $\varphi'[v] = \frac{1}{2}\mu_1 + \frac{5}{6}\mu_2 + \frac{2}{3}\mu_3$[12]. Thus the value of the fuzzy coalition A can be evaluated by Choquet integral of A with respect to fuzzy game v. That is,*

$$\begin{aligned}v(A) = (C)\int_X Adv &= \int_0^1 v(A_\alpha)d\alpha = \int_0^1 (\mu_1\mu_2 + \mu_2\mu_3^2)(A_\alpha)d\alpha \\ &= \int_0^1 (\mu_1\mu_2)(A_\alpha)d\alpha + \int_0^1 (\mu_2\mu_3^2)(A_\alpha)d\alpha = (C)\int_X Ad(\mu_1\mu_2) + (C)\int_X Ad(\mu_2\mu_3^2).\end{aligned}$$

Next, we can estimate the share of fuzzy coalition A. From Formula (2), we can obtain

$$\begin{aligned}\varphi[v](A) &= \int_0^1 (\frac{1}{2}\mu_1 + \frac{5}{6}\mu_2 + \frac{2}{3}\mu_3)(A_\alpha)d\alpha \\ &= \frac{1}{2}\int_0^1 \mu_1(A_\alpha)d\alpha + \frac{5}{6}\int_0^1 \mu_2(A_\alpha)d\alpha + \frac{2}{3}\int_0^1 \mu_3(A_\alpha)d\alpha \\ &= \frac{1}{2}(C)\int_0^1 Ad\mu_1 + \frac{5}{6}(C)\int_0^1 Ad\mu_2 + \frac{2}{3}(C)\int_0^1 Ad\mu_3.\end{aligned}$$

5 The Relationship Between Aumann-Shapley Value and Shapley Function

In this section, we will discuss the relationship between the Aumann-Shapley value and the Shapley function which both have the form of Choquet integral. First, we give the relationship between the crisp Aumann-Shapley value and the crisp Shapley function.

Under the multilinear extension operator, Owen [11] has shown the following statement.

Proposition 5.1 [11] *If X is a finite set and $\mathcal{A}$ consists of all crisp subsets of X, then the Aumann-Shapley value with respect to v is exactly the Shapley value of v and*

$$\varphi_i'[v] = \varphi'[v](\{i\}) = \sum_{S\in\mathcal{A},\ i\in S} \frac{|S|!(|X|-|S|-1)!}{|X|!}[v(S) - v(S\backslash\{i\})].$$

Note that if X is a finite set and $\mathcal{A}$ consists of all crisp subsets of $A \subseteq X$, then from Proposition 5.1 the Shapley function of v can be

$$\varphi_i'[v](A) = \sum_{S\in\mathcal{A},\ i\in S} \frac{|S|!(|A|-|S|-1)!}{|A|!}[v(S) - v(S\backslash\{i\})]. \tag{3}$$

Let X be a finite set. For any $A \in \mathcal{F}(X)$, let $Q(A) = \{A(i)|A(i) > 0, i \in X\}$ and $q(A)$ be the cardinality of $Q(A)$. We write the elements of $Q(A)$ in the increasing order as $h_1 < h_2 < \cdots < h_{q(A)}$. Note that $(C)\int_X Adv = \sum_{l=1}^{q(A)} v(A_l)\cdot(h_l - h_{l-1})$. Hence it is apparent that the fuzzy game v defined by (1) can be written as $v(A) = \sum_{l=1}^{q(A)} v[(A)_l]\cdot(h_l - h_{l-1})$, which is exactly the fuzzy game defined by Tsurumi et al [13]. It is clear that the game defined by (1) is the extension of the game defined by Tsurumi et al [13] .

Let X be a finite set and A any fuzzy set of X. Then the Aumann-Shapley value $\varphi[v](A)$ of the game $v \in G_C(X)$ has the form of (2) which can be represented by $\sum_{l=1}^{q(A)} \varphi'[v](A_l)\cdot(h_l - h_{l-1})$ and the vector $(\varphi_i[v](A))_{i\in X}$ with coordinates $\varphi_i[v](A) = \sum_{l=1}^{q(A)} \varphi_i'[v](A_l)\cdot(h_l - h_{l-1})$, where $\varphi_i'[v]$ has the form of (3). Hence, the vector $(\varphi_i[v](A))_{i\in X}$ coincides with the Shapley function proposed by Tsurumi et al [13].

6 Conclusion

The Aumann-Shapley value is a well-known solution concept in non-atomic game theory. In the fuzzy game, the Aumann-Shapley value of non-atomic game with fuzzy coalitions, as an important solution concept, has been researched by the game theory researchers. In this paper, a class of particular fuzzy games with infinite players were proposed and the properties of the class of the fuzzy games were studied. Based on the class of the proposed fuzzy games, we gave the fuzzy Aumann-Shapley values with special forms. Furthermore, the relationship between the fuzzy Aumann-Shapley value and the fuzzy Shapley function of the proposed fuzzy cooperative games was discussed. However, the uniqueness of the Aumann-Shapley value even in the proposed class has not been proven yet. This can be one of the future topics of our research. Additionally, the existence of the fuzzy Aumann-Shapley value with the Choquet integral form in the special spaces is also an interesting topic for the further research.

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