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Some Properties of Product Uncertain Measure

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Abstract

Uncertainty theory is a branch of mathematics for modeling human uncertainty. The first fundamental concept in uncertainty theory is uncertain measure, which is defined by normality axiom, duality axiom and subadditivity axiom. The second fundamental concept is the product uncertain measure, which is defined by product axiom. This paper shows that the product uncertain measure is indeed an uncertain measure, which also means that the product axiom is consistent with other axioms in uncertainty theory.

Keywords: uncertainty theory, uncertain measure, product axiom

1 Introduction

Random phenomena have been well studied by probability theory. However, the sample size in practice is often too small or there is even no-sample to estimate a probability distribution. Thus, we have to invite some domain experts to evaluate their belief degree that each event will occur. Since human beings usually overweight unlikely events, the probability theory may be not suitable for modeling belief degree any more. Distinguished from probability measure, the theory of capacities was proposed by Choquet [2] In 1954. Twenty years later, the fuzzy measure theory was proposed by Sugeno [12] and fuzzy set theory was initiated by Zadeh [14]. For a long time, such belief degree was regarded as subjective probability or fuzzy concept. However, both probability theory and fuzzy set theory may lead to counterintuitive results in some cases. Considering some quantities like "around 25 °C", "roughly 60kg", "very tall", and "long way". For example, it is assumed that the weight of a person is "roughly 60kg". If "roughly 60kg" is regarded as a fuzzy concept and is modeled by a fuzzy set, then we may obtain the weight of the person is "exactly 60kg" with belief degree 1 in possibility measure by the property of membership function. However, it is doubtless that the degree of "roughly 60kg" is almost zero. Anyone with common sense will be not so naive to expect that "exactly 60kg" is the true weight of the person. On the other hand, "exactly 60kg" and "not 60kg" have the same belief degree in possibility measure. Thus we have to regard them "equally likely". It seems that nobody can accept this conclusion. This paradox shows that those imprecise quantities like "roughly 60kg" cannot be quantified by possibility measure and then they are not fuzzy concepts.

In order to model such phenomena, uncertainty theory was founded by Liu [6] in 2007, refined by Liu [11] in 2010, and became a branch of mathematics based on the normality axiom, duality axiom, subadditivity axiom, and product axiom. The first fundamental concept in uncertainty theory is uncertain measure, used to indicate the belief degree that an uncertain event may occur. Some scholars studied the properties of an uncertain measure. Gao [3] gave some mathematical properties of continuous uncertain measure. You [13] proved some convergence theorems of uncertain sequences. Another fundamental concept is product uncertain measure, defined by the fourth axiom called *product axiom*, which is proposed by Liu [7] in 2009, thus producing the operational law for uncertain variables. Based on uncertainty theory, some significant and theoretical work of uncertainty theory such as uncertain programming [8], uncertain process [4], uncertain calculus [7], uncertain differential equation [1] [7], uncertain logic [5], uncertain risk analysis [9] have been established. For exploring the recent developments of uncertainty theory, the readers may consult Liu [11].

In this paper, it is proved that product uncertain measure is indeed an uncertain measure, which means the product axiom is consistent with the other axioms in uncertainty theory. The rest of this paper is organized

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as follows. Some basic concepts of uncertainty theory are recalled in Section 2. The main proof is presented in Section 3. At the end, a brief summary is given in Section 4.

2 Preliminaries

In this paper, we recall some basic concepts in uncertainty theory.

Definition 2.1 [6] *Let Γ be a nonempty set. A collection L of subsets of Γ is a σ-algebra. Each element Λ in the σ-algebra L is called an event. In order to present an axiomatic definition of uncertainty, it is necessary to assign to each event Λ a number $\mathcal{M}\{\Lambda\}$ indicating the belief degree that the event Λ will occur. Whenever the function $\mathcal{M}$ from L to $[0,1]$ satisfies the following three axioms, it is called an uncertain measure:*

Axiom 1. *(Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.*

Axiom 2. *(Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.*

Axiom 3. *(Subadditivity Axiom) For every countable sequence of events $\{\Lambda_k\}$, we have*

$$\mathcal{M}\left\{\bigcup_{k=1}^{\infty} \Lambda_k\right\} \le \sum_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

The triple $(\Gamma, \L, \mathcal{M})$ is called an uncertainty space. Product uncertain measure was defined by Liu [7] in 2009, thus producing the fourth axiom of uncertainty theory called *product axiom.*

Axiom 4.[7]*(Product Axiom) Let $(\Gamma_k, L_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Write*

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots, \quad \L = \L_1 \times \L_2 \times \cdots.$$

Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\bigcup_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{1}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \ldots$, respectively.

For any other event $\Lambda \in \L$, we have

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}, & \textit{if} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}, & \textit{if} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, & \textit{otherwise.} \end{cases} \tag{2}$$

In Section 3, we will show that the product uncertain measure $\mathcal{M}$ defined on $\L$ is indeed an uncertain measure, which means the product uncertain measure $\mathcal{M}$ satisfies the normality axiom, duality axiom and subadditivity axiom.

3 Some Properties of Product Uncertain Measure

In this section, we show that the product uncertain measure is an uncertain measure. Firstly, the product uncertain measure $\mathcal{M}$ is proved to be well-defined by Equation (2), which means $\mathcal{M}$ is a single-valued function on $\L$. That is, for any given event $\Lambda \in \L$, there is one and only one number assigned to Λ by Equation (2).

Lemma 3.1 *For any event $\Lambda \in L$, at most one of the following inequation holds,*

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \qquad \textit{and} \qquad \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5.$$

Proof: For any $\Lambda \in \L$, $A_1 \times A_2 \times \cdots \subseteq \Lambda$, and $B_1 \times B_2 \times \cdots \subseteq \Lambda^c$, we have

$$(A_1 \times A_2 \times \cdots) \cap (B_1 \times B_2 \times \cdots) = \emptyset,$$

which means that there is an index j satisfying $A_j \cap B_j = \emptyset$. If not, we have $A_i \cap B_i \neq \emptyset$ for any index i, which leads to $(A_1 \times A_2 \times \cdots) \cap (B_1 \times B_2 \times \cdots) \neq \emptyset$. Hence we have

$$\min_{1 \le k < \infty} \mathcal{M}_k\{A_k\} + \min_{1 \le k < \infty} \mathcal{M}_k\{B_k\} \le \mathcal{M}_j\{A_j\} + \mathcal{M}_j\{B_j\} \le 1.$$

Because the inequation holds for all $(A_1 \times A_2 \times \cdots) \subseteq \Lambda$, and $B_1 \times B_2 \times \cdots \subseteq \Lambda^c$, we have

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} + \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \le 1, \tag{3}$$

which means at most one of the following inequation holds,

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \qquad \text{and} \qquad \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5.$$

The proposition is proved. The proposition shows that Equation (2) is reasonable, which means the set function defined by (2) is a single-valued function.

Secondly, we are going to verify that $\mathcal{M}$ is an uncertain measure, which means that $\mathcal{M}$ satisfies normality axiom, duality axiom and subadditivity axiom in uncertainty theory.

Theorem 3.1 *(Product Uncertain Measure Theorem) Let $(\Gamma_k, L_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Then the product uncertain measure $\mathcal{M}$ defined by Equation (2) is an uncertain measure.*

Proof: In order to prove that the product uncertain measure $\mathcal{M}$ is indeed an uncertain measure, we should verify that the product uncertain measure satisfies the normality axiom, duality axiom and subadditivity axiom.

Step 1: For the universal set $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots$, we have $\mathcal{M}\{\Gamma\} = \min_{1 \le k < \infty} \mathcal{M}_k\{\Gamma_k\} = 1$. Hence the product uncertain measure $\mathcal{M}$ satisfies the normality axiom.

Step 2: For any event $\Lambda \in \mathbb{L}$, we are to show that $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5.$$

Then by Lemma 3.1, we have

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} < 0.5.$$

It follows from Equation (2) that

$$\mathcal{M}\{\Lambda\} = \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\},$$

$$\mathcal{M}\{\Lambda^c\} = 1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq (\Lambda^c)^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} = 1 - \mathcal{M}\{\Lambda\}.$$

The duality is verified. Case 2: Assume

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5.$$

By a similar process as Case 1, we get

$$\mathcal{M}\{\Lambda^c\} = \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\},$$

$$\mathcal{M}\{\Lambda\} = 1 - \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} = 1 - \mathcal{M}\{\Lambda^c\}.$$

The duality is verified. Case 3: Assume

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \le 0.5 \qquad \text{and} \qquad \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subseteq \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \le 0.5.$$

It follows from Equation (2) that $\mathcal{M}\{\Lambda\} = \mathcal{M}\{\Lambda^c\} = 0.5$, which verifies the duality axiom.

Step 3: In order to prove the countable subadditivity of $\mathcal{M}$, we firstly prove the monotonicity of $\mathcal{M}$. That is, for any events Λ and Δ in $\mathcal{L}$ with $\Lambda \subseteq \Delta$, we have $\mathcal{M}\{\Lambda\} \leq \mathcal{M}\{\Delta\}$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subseteq\Lambda}\min_{1\leq k<\infty}\mathcal{M}_k\{\Lambda_k\} > 0.5.$$

From $\Lambda \subseteq \Delta$ and the definition of the operation "sup", we have

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subseteq\Delta}\min_{1\leq k<\infty}\mathcal{M}_k\{\Delta_k\} \geq \sup_{\Lambda_1\times\Lambda_2\times\cdots\subseteq\Lambda}\min_{1\leq k<\infty}\mathcal{M}_k\{\Lambda_k\} > 0.5.$$

It follows from Equation (2) that

$$\mathcal{M}\{\Lambda\} = \sup_{\Lambda_1\times\Lambda_2\times\cdots\subseteq\Lambda}\min_{1\leq k<\infty}\mathcal{M}_k\{\Lambda_k\} \leq \sup_{\Delta_1\times\Delta_2\times\cdots\subseteq\Delta}\min_{1\leq k<\infty}\mathcal{M}_k\{\Delta_k\} = \mathcal{M}\{\Delta\}.$$

Case 2: Assume

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subseteq\Delta^c}\min_{1\leq k<\infty}\mathcal{M}_k\{\Delta_k\} > 0.5.$$

Then

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subseteq\Lambda^c}\min_{1\leq k<\infty}\mathcal{M}_k\{\Lambda_k\} \geq \sup_{\Delta_1\times\Delta_2\times\cdots\subseteq\Delta^c}\min_{1\leq k<\infty}\mathcal{M}_k\{\Delta_k\} > 0.5.$$

Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\} &= 1 - \sup_{\Lambda_1\times\Lambda_2\times\cdots\subseteq\Lambda^c}\min_{1\leq k<\infty}\mathcal{M}_k\{\Lambda_k\}\\ &\leq 1 - \sup_{\Delta_1\times\Delta_2\times\cdots\subseteq\Delta^c}\min_{1\leq k<\infty}\mathcal{M}_k\{\Delta_k\} = \mathcal{M}\{\Delta\}.\end{aligned}$$

Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subseteq\Lambda}\min_{1\leq k<\infty}\mathcal{M}_k\{\Lambda_k\} \leq 0.5 \quad \text{and} \quad \sup_{\Delta_1\times\Delta_2\times\cdots\subseteq\Delta^c}\min_{1\leq k<\infty}\mathcal{M}_k\{\Delta_k\} \leq 0.5.$$

Then by Equation (2), we have

$$\mathcal{M}\{\Lambda\} \leq 0.5 \leq 1 - \mathcal{M}\{\Delta^c\} = \mathcal{M}\{\Delta\}.$$

Step 4: Finally, we are to show the countable subadditivity of $\mathcal{M}$. That is, for any countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{+\infty}\Lambda_i\right\} \leq \sum_{i=1}^{+\infty}\mathcal{M}\{\Lambda_i\}.$$

The argument breaks down into three cases. Case 1: Assume that for any event $\Lambda_i \in \mathcal{L}, \mathcal{M}\{\Lambda_i\} < 0.5$. For any given $\varepsilon > 0$, we have

$$A_i^1 \times A_i^2 \times \cdots \subseteq \Lambda_i^c$$

such that

$$\mathcal{M}\{A_i^1 \times A_i^2 \times \cdots\} \geq 1 - \mathcal{M}\{\Lambda_i\} - \varepsilon/2^i,$$

that is,

$$1 - \mathcal{M}\{A_i^1 \times A_i^2 \times \cdots\} \leq \mathcal{M}\{\Lambda_i\} + \varepsilon/2^i$$

holds for all $i \in \mathbb{N}$. Note that

$$\bigcap_{i=1}^{+\infty}A_i^1 \times \bigcap_{i=1}^{+\infty}A_i^2 \times \cdots \times \bigcap_{i=1}^{+\infty}A_i^n \times \cdots \subseteq \bigcap_{i=1}^{+\infty}\Lambda_i^c = \left(\bigcup_{i=1}^{+\infty}\Lambda_i\right)^c.$$

It follows from Equation (1) and the monotonicity of $\mathcal{M}$ that

$$\mathcal{M}\left\{\bigcap_{i=1}^{+\infty}A_i^1 \times \bigcap_{i=1}^{+\infty}A_i^2 \times \cdots\right\} = \min_{1\leq k<\infty}\mathcal{M}_k\left\{\bigcap_{i=1}^{+\infty}A_i^k\right\} \leq \mathcal{M}\left\{\left(\bigcup_{i=1}^{+\infty}\Lambda_i\right)^c\right\}.$$

Thus

$$\begin{aligned}
\mathcal{M}\left\{\bigcup_{i=1}^{+\infty}\Lambda_i\right\} &\le 1-\min_{1\le k<\infty}\mathcal{M}_k\left\{\bigcap_{i=1}^{+\infty}A_i^k\right\}\\
&=\max_{1\le k<\infty}\mathcal{M}_k\left\{\bigcup_{i=1}^{+\infty}(A_i^k)^c\right\}\\
&\le\max_{1\le k<\infty}\sum_{i=1}^{+\infty}\mathcal{M}_k\{(A_i^k)^c\}\\
&\le\sum_{i=1}^{+\infty}\max_{1\le k<\infty}\mathcal{M}_k\{(A_i^k)^c\}\\
&=\sum_{i=1}^{+\infty}\left(1-\min_{1\le k<\infty}\mathcal{M}_k\{A_i^k\}\right)\\
&=\sum_{i=1}^{+\infty}\left(1-\mathcal{M}\left\{A_i^1\times A_i^2\times\cdots\right\}\right)\\
&\le\sum_{i=1}^{+\infty}\left(\mathcal{M}\{\Lambda_i\}+\varepsilon/2^i\right)\\
&\le\sum_{i=1}^{+\infty}\mathcal{M}\{\Lambda_i\}+\varepsilon.
\end{aligned}$$

That is

$$\mathcal{M}\left\{\bigcup_{i=1}^{+\infty}\Lambda_i\right\}\le\sum_{i=1}^{+\infty}\mathcal{M}\{\Lambda_i\}+\varepsilon.$$

Letting $\varepsilon\to 0$, we obtain

$$\mathcal{M}\left\{\bigcup_{i=1}^{+\infty}\Lambda_i\right\}\le\sum_{i=1}^{+\infty}\mathcal{M}\{\Lambda_i\}.$$

Case 2: Assume that there is one and only one event Λ_i, say Λ_1, such that $\mathcal{M}\{\Lambda_1\}\ge 0.5$ and $\mathcal{M}\{\Lambda_i\}<0.5$, for all $i\ge 2$. Letting $\Lambda=\Lambda_1\cup\Lambda_2\cup\cdots$, for the monotonicity of $\mathcal{M}$, $\mathcal{M}\{\Lambda\}\ge\mathcal{M}\{\Lambda_1\}\ge 0.5$. If $\mathcal{M}\{\Lambda\}=0.5$, Then $\mathcal{M}\{\Lambda\}=0.5\le\mathcal{M}\{\Lambda_1\}$, the subadditivity is proved. If $\mathcal{M}\{\Lambda\}>0.5$, then we have $\mathcal{M}\{\Lambda^c\}<0.5$. From Case 1 and

$$\Lambda_1^c\subseteq\Lambda^c\cup\bigcup_{i=2}^{+\infty}\Lambda_i,$$

we get

$$\mathcal{M}\left\{\Lambda_1^c\right\}\le\mathcal{M}\left\{\Lambda^c\right\}+\sum_{i=2}^{+\infty}\mathcal{M}\left\{\Lambda_i\right\}.$$

Thus

$$\mathcal{M}\left\{\Lambda\right\}=1-\mathcal{M}\left\{\Lambda^c\right\}\quad\le 1-\mathcal{M}\left\{\Lambda_1^c\right\}+\sum_{i=2}^{+\infty}\mathcal{M}\left\{\Lambda_i\right\}=\mathcal{M}\{\Lambda_1\}+\sum_{i=2}^{+\infty}\mathcal{M}\left\{\Lambda_i\right\}=\sum_{i=1}^{+\infty}\mathcal{M}\left\{\Lambda_i\right\}.$$

The subadditivity of $\mathcal{M}$ is proved. Case 3: If there are at least two events $\Lambda_i,\Lambda_j,i\ne j,\mathcal{M}\{\Lambda_i\},\mathcal{M}\{\Lambda_j\}\ge 0.5$, then the subadditivity follows from $\sum_{i=1}^{+\infty}\mathcal{M}\{\Lambda_i\}\ge 1$. Now we have verified that $\mathcal{M}$ satisfies the normality axiom, duality axiom and subadditivity axiom, which means $\mathcal{M}$ is an uncertain measure. The theorem is proved.

The following theorem shows that the uncertain measure $\mathcal{M}$ defined by Equation (2) also satisfies Equation (1).

Theorem 3.2 *For any event $\Lambda=\Lambda_1\times\Lambda_2\times\cdots$, the uncertain measure $\mathcal{M}$ defined by Equation (2) satisfies that*

$$\mathcal{M}\left\{\bigcup_{k=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}.$$

Proof: Assume $\mathcal{M}\{\Lambda\}$ is defined by Equation (2). The argument breaks down into three cases. Case 1: Assume $\min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} > 0.5$. By Equation (2), we immediately have

$$\mathcal{M}\{\Lambda_1 \times \Lambda_2 \times \cdots\} = \min_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\}.$$

Case 2: Assume $\min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} = 0.5$, we have

$$\sup_{A_1\times A_2\times\cdots\subseteq\Lambda_1\times\Lambda_2\times\cdots} \min_{1\le k<\infty} \mathcal{M}_k\{A_k\} = 0.5.$$

By Equation (3), we have

$$\sup_{A_1\times A_2\times\cdots\subseteq(\Lambda_1\times\Lambda_2\times\cdots)^c} \min_{1\le k<\infty} \mathcal{M}_k\{A_k\} \le 1 - \min_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} \le 0.5.$$

It follows from Equation (2) that $\mathcal{M}\{\Lambda_1\times\Lambda_2\times\cdots\} = 0.5 = \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\}$. Case 3: Assume $\min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} < 0.5$. For any given small number $\varepsilon > 0$, for simplicity, assuming $0.5 > \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} + \varepsilon > \mathcal{M}_1\{\Lambda_1\}$, we have

$$(\Gamma_1\backslash\Lambda_1) \times \Gamma_2 \times \cdots \subseteq (\Lambda_1 \times \Lambda_2 \times \cdots)^c$$

and

$$\sup_{A_1\times A_2\times\cdots\subseteq(\Gamma_1\backslash\Lambda_1)\times\Gamma_2\times\cdots} \min_{1\le k<\infty} \mathcal{M}_k\{A_k\} > 0.5.$$

By Equation (2) and the definition of the operation "sup", we have

$$0.5 < \mathcal{M}\{(\Gamma_1\backslash\Lambda_1) \times \Gamma_2 \times \cdots\} \le \mathcal{M}\{(\Lambda_1 \times \Lambda_2 \times \cdots)^c\}.$$

Thus

$$1 - \mathcal{M}_1\{\Lambda_1\} \le 1 - \mathcal{M}\{\Lambda_1 \times \Lambda_2 \times \cdots\}.$$

That is,

$$\mathcal{M}\{\Lambda_1 \times \Lambda_2 \times \cdots\} \le \mathcal{M}_1\{\Lambda_1\} \le \min_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} + \varepsilon.$$

Letting $\varepsilon \to 0$, we have

$$\mathcal{M}\{\Lambda_1 \times \Lambda_2 \times \cdots\} \le \min_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\}.$$

On the other hand, by Equation (3), we obtain

$$\min_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} \le 1 - \sup_{A_1\times A_2\times\cdots\subseteq(\Lambda_1\times\Lambda_2\times\cdots)^c} \min_{1\le k<\infty} \mathcal{M}_k\{A_k\} = \mathcal{M}\{\Lambda_1 \times \Lambda_2 \times \cdots\}.$$

Thus

$$\mathcal{M}\{\Lambda_1 \times \Lambda_2 \times \cdots\} = \min_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\}$$

holds. The theorem is proved.

By now, we showed that the product uncertain measure is an uncertain measure, which also means that the product axiom is consistent with other axioms. The consistence among normality axiom, duality axiom and subadditivity axiom is shown in [6]. Hence the axiomatic system of uncertainty theory is consistent.

4 Conclusion

In this paper, we showed that uncertain measure $\mathcal{M}$ defined by Liu [7] is an uncertain measure, which also means that the product axiom is consistent with other axioms.

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