Journal of Uncertain Systems
Vol.6, No.4, pp.250-255, 2012
Online at: www.jus.org.uk

Uncertain Integral with respect to Multiple Canonical Processes

Baoding Liu*, Kai Yao
Uncertainty Theory Laboratory, Department of Mathematical Sciences
Tsinghua University, Beijing 100084, China

Received 1 June 2012; Revised 2 September 2012

Abstract

Uncertain calculus is a branch of mathematics that deals with the integral of functions of uncertain process. This paper will extend uncertain integral from single canonical process to multiple ones. Some mathematical properties of uncertain integral with respect to multiple canonical processes are proved, including the fundamental theorem of uncertain calculus.

Keywords: uncertain calculus, uncertain process, uncertainty theory

1 Introduction

In our daily life, due to economical or technical difficulties, very often we are lack of observed data about the unknown state of nature. Then we have to invite some domain experts to evaluate their belief degree that each event will occur. Because human being tends to overweight unlikely events [4], the belief degree usually has a much larger range than the real frequency. In 2012, Liu [9] declared that probability theory fails to model the belief degree under this situation via a counterexample about the strength of a bridge.

In order to deal with the belief degree, an uncertainty theory was founded by Liu [5] in 2007, and refined by Liu [8] in 2010 based on normality, duality, subadditivity and production axioms. Sometimes, the uncertain phenomena evolve with time. For modeling such phenomena, a concept of uncertain process was proposed by Liu [6] as a sequence of uncertain variables indexed by time. After that, Liu [7] designed a canonical process which is an independent and stationary increment uncertain process with normal uncertain variables as the increments. Meanwhile, Liu [7] founded uncertain calculus to deal with the integral and differential of a function of uncertain process with respect to canonical process. Canonical process is a type of continuous uncertain process, and can only model continuous uncertain systems. In order to model the sudden jumps in an uncertain system, Liu [6] proposed uncertain renewal process. Inspired by Liu [7], Yao [13] founded uncertain calculus with respect to uncertain renewal process. After that, Chen [2] generalized the work by Liu [7] and Yao [13], and proposed uncertain calculus with respect to finite variation process.

Based on uncertain calculus, an uncertain differential equation was proposed by Liu [6] in 2008 as a type of differential equation driven by canonical process. Following that, Chen and Liu [1] gave an existence and uniqueness theorem of the solution of an uncertain differential equation. Then Yao *et al.* [14] gave a sufficient condition for an uncertain differential equation being stable. In 2010, Chen and Liu [1] gave an analytic solution for linear uncertain differential equation. Then Liu [11] and Yao [16] presented a spectrum of analytic methods to solve some special classes of nonlinear uncertain differential equations. Besides, Yao and Chen [15] designed a numerical method for solving general uncertain differential equations. Uncertain differential equation found many applications such as uncertain stock model [7, 12], uncertain currency model [10], uncertain interest rate model [3], and uncertain optimal control [17].

In this paper, we will present uncertain calculus with multiple canonical processes. The rest of this paper is structured as follows. The next section is intended to introduce some concepts of uncertainty theory and uncertain calculus with respect to single canonical process. In Section 3, an uncertain integral with respect to multiple canonical processes is proposed. In Section 4, the fundamental theorem of multifactor uncertain integral is proposed. Finally, some remarks are made in Section 5.

---

*Corresponding author. Email: liu@tsinghua.edu.cn (B. Liu).

2 Preliminary

Uncertainty theory is a branch of axiomatic mathematics to deal with human uncertainty. As a fundamental concept, uncertain measure is a set function satisfying normality, duality, subadditivity and production axioms.

Definition 1 [5] *Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0,1]$ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1: *(Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.*
Axiom 2: *(Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.*
Axiom 3: *(Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$ we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Product uncertain measure was defined by Liu [7] in 2009, thus producing the the fourth axiom of uncertainty theory.
Axiom 4: *(Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots,$ respectively.

An uncertain variable is essential a measurable function on an uncertainty space. The formal definition of uncertain variable is given as follows.

Definition 2 [5] *An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set $\Re$ of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 3 [5] *The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any real number x.

The uncertainty distribution Φ is said to be regular if its inverse function Φ^{-1} exists and is unique for each $\alpha \in (0,1)$. In this case, the inverse function Φ^{-1} is called the inverse uncertainty distribution, which plays an important role in the operation of uncertain variables.

Theorem 1 *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution*

$$\Phi^{-1}(r) = f\left(\Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \ldots, \Phi_n^{-1}(1-r)\right).$$

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. Canonical process is one of the most important uncertain processes.

Definition 4 [7] *An uncertain process C_t is said to be a canonical process if*
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \ x \in \Re.$$

Note that ΔC_t and Δt are infinitesimals with the same order. Based on canonical process, Liu [7] defined an uncertain integral, thus founding an uncertain calculus theory.

Definition 5 [7] *Let X_t be an uncertain process and C_t be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the uncertain integral of X_t is defined by

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

For example, a continuous function $f(t)$ is integrable, and

$$\int_0^s f(t)\mathrm{d}C_t \sim \left(0, \int_0^s |f(t)|\mathrm{d}t\right)$$

is a normal uncertain variable at any time s.

Definition 6 [7] *Let C_t be a canonical process and Z_t be an uncertain process. If there exist uncertain processes μ_s and σ_s such that*

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s$$

for any $t \ge 0$, then Z_t is said to have an uncertain differential

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t.$$

In this case, the uncertain process Z_t is called a differentiable uncertain process with drift μ_t and diffusion σ_t.

Liu [7] verified the fundamental theorem of uncertain calculus, i.e., for a canonical process C_t and a continuous differentiable function $h(t, c)$, the uncertain process $Z_t = h(t, C_t)$ is differentiable and has an uncertain differential

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t.$$

Based on the fundamental theorem, Liu [7] proved the chain rule, i.e., for two continuously differentiable functions f and g, the uncertain process $f(g(C_t))$ has an uncertain differential

$$\mathrm{d}f(g(C_t)) = f'(g(C_t))g'(C_t)\mathrm{d}C_t,$$

and the integration by parts theorem, i.e., for two differentiable uncertain processes X_t and Y_t, the uncertain process $X_t Y_t$ has an uncertain differential

$$\mathrm{d}(X_t Y_t) = Y_t \mathrm{d}X_t + X_t \mathrm{d}Y_t.$$

3 Multifactor Uncertain Integral

Definition 7 *Let $X_{1t}, X_{2t}, \ldots, X_{nt}$ be integrable uncertain processes, and $C_{1t}, C_{2t}, \ldots, C_{nt}$ be canonical processes. Then*

$$Z_t = \sum_{i=1}^{n} \int_0^t X_{is} \mathrm{d}C_{is}$$

is called an uncertain integral of $X_{1t}, X_{2t}, \ldots, X_{nt}$ with respect to multiple canonical processes $C_{1t}, C_{2t}, \ldots, C_{nt}$.

Theorem 2 *Assume that Z_t is an uncertain integral of $X_{1t}, X_{2t}, \ldots, X_{nt}$ with respect to multiple canonical processes $C_{1t}, C_{2t}, \ldots, C_{nt}$. Then Z_t is a sample-continuous uncertain process.*

Proof: It follows from the definition of multifactor uncertain integral that

$$|Z_t(\gamma) - Z_r(\gamma)| = \left|\sum_{i=1}^{n}\int_0^t X_{is}(\gamma)\mathrm{d}C_{is}(\gamma) - \sum_{i=1}^{n}\int_0^r X_{is}(\gamma)\mathrm{d}C_{is}(\gamma)\right| = \left|\sum_{i=1}^{n}\int_r^t X_{is}(\gamma)\mathrm{d}C_{is}(\gamma)\right| \to 0$$

for each $\gamma \in \Gamma$ as $r \to t$. Thus Z_t is sample-continuous, and the theorem is proved.

Theorem 3 *(Linearity of Uncertain Integral) Assume $X_{1t}, X_{2t}, \ldots, X_{nt}$ and $Y_{1t}, Y_{2t}, \ldots, Y_{nt}$ are integrable uncertain processes with respect to $C_{1t}, C_{2t}, \ldots, C_{nt}$ on $[a, b]$. Then for any given real numbers α and β, we have*

$$\sum_{i=1}^{n}\int_a^b (\alpha X_{it} + \beta Y_{it})\,\mathrm{d}C_{it} = \alpha\sum_{i=1}^{n}\int_a^b X_{it}\mathrm{d}C_{it} + \beta\sum_{i=1}^{n}\int_a^b Y_{it}\mathrm{d}C_{it}.$$

Proof: It follows immediately from the linearity of uncertain integral with respect to single canonical process that

$$\sum_{i=1}^{n}\int_a^b (\alpha X_{it} + \beta Y_{it})\,\mathrm{d}C_{it} = \sum_{i=1}^{n}\left(\alpha\int_a^b X_{it}\mathrm{d}C_{it} + \beta\int_a^b Y_{it}\mathrm{d}C_{it}\right) = \alpha\sum_{i=1}^{n}\int_a^b X_{it}\mathrm{d}C_{it} + \beta\sum_{i=1}^{n}\int_a^b Y_{it}\mathrm{d}C_{it}.$$

4 Multifactor Uncertain Differential

Definition 8 *Let $C_{1t}, C_{2t}, \ldots, C_{nt}$ be canonical processes and let Z_t be an uncertain process. If there exist uncertain processes μ_t and $\sigma_{1t}, \sigma_{2t}, \ldots, \sigma_{nt}$ such that*

$$Z_t = Z_0 + \int_0^t \mu_s\mathrm{d}s + \sum_{i=1}^{n}\int_0^t \sigma_{is}\mathrm{d}C_{is} \tag{1}$$

for any $t \geq 0$, then we say Z_t has an uncertain differential

$$\mathrm{d}Z_t = \mu_t\mathrm{d}t + \sum_{i=1}^{n}\sigma_{it}\mathrm{d}C_{it}. \tag{2}$$

In this case, Z_t is called a differentiable uncertain process with drift μ_t and diffusions $\sigma_{1t}, \sigma_{2t}, \ldots, \sigma_{nt}$.

The following theorem gives the uncertain differential of a function of multiple canonical processes.

Theorem 4 *(Fundamental Theorem) Let $C_{1t}, C_{2t}, \ldots, C_{nt}$ be canonical processes. If $h(t, c_1, c_2, \ldots, c_n)$ is a continuously differentiable function, then the uncertain process $Z_t = h(t, C_{1t}, C_{2t}, \ldots, C_{nt})$ is differentiable and has an uncertain differential*

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_{1t}, C_{2t}, \ldots, C_{nt})\mathrm{d}t + \sum_{i=1}^{n}\frac{\partial h}{\partial c_i}(t, C_{1t}, C_{2t}, \ldots, C_{nt})\mathrm{d}C_{it}.$$

Proof: Since the function h is continuously differentiable, by using Taylor series expansion, the infinitesimal increment of Z_t has a first-order approximation

$$\Delta Z_t = \frac{\partial h}{\partial t}(t, C_{1t}, C_{2t}, \ldots, C_{nt})\Delta t + \sum_{i=1}^{n}\frac{\partial h}{\partial c_i}(t, C_{1t}, C_{2t}, \ldots, C_{nt})\Delta C_{it}.$$

Thus the theorem is proved.

Example 1 Let us calculate the uncertain differential of $C_{1t}+C_{2t}$. In this case, we have $h(t,c_1,c_2)=c_1+c_2$. It is clear that

$$\frac{\partial h}{\partial t}(t,c_1,c_2)=0,\quad \frac{\partial h}{\partial c_1}(t,c_1,c_2)=1,\quad \frac{\partial h}{\partial c_2}(t,c_1,c_2)=1.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}(C_{1t}+C_{2t})=\mathrm{d}C_{1t}+\mathrm{d}C_{2t}.$$

Example 2 Let us calculate the uncertain differential of $\exp(C_{1t}+C_{2t})$. In this case, we have $h(t,c_1,c_2)=\exp(c_1+c_2)$. It is clear that

$$\frac{\partial h}{\partial t}(t,c_1,c_2)=0,\quad \frac{\partial h}{\partial c_1}(t,c_1,c_2)=\exp(c_1+c_2),\quad \frac{\partial h}{\partial c_2}(t,c_1,c_2)=\exp(c_1+c_2).$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}\exp(C_{1t}+C_{2t})=\exp(C_{1t}+C_{2t})\mathrm{d}C_{1t}+\exp(C_{1t}+C_{2t})\mathrm{d}C_{2t}.$$

Example 3 Let us calculate the uncertain differential of $tC_{1t}C_{2t}$. In this case, we have $h(t,c_1,c_2)=tc_1c_2$. It is clear that

$$\frac{\partial h}{\partial t}(t,c_1,c_2)=c_1c_2,\quad \frac{\partial h}{\partial c_1}(t,c_1,c_2)=tc_2,\quad \frac{\partial h}{\partial c_2}(t,c_1,c_2)=tc_1.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}(tC_{1t}C_{2t})=C_{1t}C_{2t}\mathrm{d}t+tC_{2t}\mathrm{d}C_{1t}+tC_{1t}\mathrm{d}C_{2t}.$$

Example 4 Let us calculate the uncertain differential of $t\sin C_{1t}\sin C_{2t}$. In this case, we have $h(t,c_1,c_2)=t\sin c_1\sin c_2$. It is clear that

$$\frac{\partial h}{\partial t}(t,c_1,c_2)=\sin c_1\sin c_2,\quad \frac{\partial h}{\partial c_1}(t,c_1,c_2)=t\cos c_1\sin c_2,\quad \frac{\partial h}{\partial c_2}(t,c_1,c_2)=t\sin c_1\cos c_2.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}\,t\sin C_{1t}\sin C_{2t}=\sin C_{1t}\sin C_{2t}\mathrm{d}t+t\cos C_{1t}\sin C_{2t}\mathrm{d}C_{1t}+t\sin C_{1t}\cos C_{2t}\mathrm{d}C_{2t}.$$

5 Conclusion

This paper presented the concepts of uncertain integral and uncertain differential of uncertain processes with respect to multiple canonical processes. Besides, it proved the sample-continuity and linearity of uncertain integral, as well as the fundamental theorem of uncertain calculus.

Acknowledgements

This work was supported by National Natural Science Foundation of China Grant No.61273044.

References

[1] Chen, X., and B. Liu, Existence and uniqueness theorem for uncertain differential equations, *Fuzzy Optimization and Decision Making*, vol.9, no.1, pp.69–81, 2010.

[2] Chen, X., Uncertain calculus with finite variation processes, http://orsc.edu.cn/online/110613.pdf.

[3] Chen, X., and J. Gao, Uncertain term structure model of interest rate, *Soft Computing*, to be published.

[4] Kahneman, D., and A. Tversky, Prospect theory: an analysis of decision under risk, *Econometrica*, vol.47, no.2, pp.263–292, 1979.

[5] Liu, B., *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.

[6] Liu, B., Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, vol.2, no.1, pp.3–16, 2008.

[7] Liu, B., Some research problems in uncertainty theory, *Journal of Uncertain Systems*, vol.3, no.1, pp.3–10, 2009.

[8] Liu, B., *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, 2010.

[9] Liu, B., Why is there a need for uncertainty theory?, *Journal of Uncertain Systems*, vol.6, no.1, pp.3–10, 2012.

[10] Liu, Y.H., and X. Chen, Uncertain currency model and currency pricing formula, http://orsc.edu.cn/online/091010.pdf.

[11] Liu, Y.H., An analytic method for solving uncertain differential equations, *Journal of Uncertain Systems*, vol.6, no.4, pp.244–249, 2012.

[12] Peng, J., and K. Yao, A new option pricing model for stocks in uncertainty markets, *International Journal of Operations Research*, vol.7, no.4, pp.213–224, 2010.

[13] Yao, K., Uncertain calculus with renewal process, *Fuzzy Optimization and Decision Making*, vol.11, no.3, pp.285–297, 2012.

[14] Yao, K., J. Gao, and Y. Gao, Some stability theorems of uncertain differential equation, *Fuzzy Optimization and Decision Making*, to be published.

[15] Yao, K., and X. Chen, A numerical method for solving uncertain differential equations, *Journal of Intelligent and Fuzzy Systems*, to be published.

[16] Yao, K., A type of nonlinear uncertain differential equations with analytic solution, http://orsc.edu.cn/online/110928.pdf.

[17] Zhu, Y., Uncertain optimal control with application to a portfolio selection model, *Cybernetics and Systems*, vol.41, no.7, pp.535–547, 2010.