

# An Analytic Method for Solving Uncertain Differential Equations

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## Abstract

Uncertain differential equations are a type of differential equations driven by canonical process, and are quite different from stochastic differential equations that are driven by Brownian motion. A solution of an uncertain differential equation is an uncertain process. This paper presents an analytic method to solve a particular class of nonlinear uncertain differential equations and gives some examples to illustrate the proposed analytic method.

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**Keywords:** uncertain differential equation, uncertain process, uncertainty theory

## 1 Introduction

Some information and knowledge are usually represented by human language like “about 100km”, “approximately 80kg”, “fast”, and “heavy”. A lot of surveys showed that these imprecise quantities behave neither like randomness nor like fuzziness [16]. In order to model these imprecise quantities, an uncertainty theory was founded by Liu [8] in 2007 and refined by Liu [12] in 2010. In addition, Liu [11], Gao [3], You [24], Liu and Ha [17], Peng and Iwamura [20], and Liu [14] made significant contributions to the uncertainty theory. Nowadays uncertainty theory has become a branch of mathematics for modeling human uncertainty.

Uncertain statistics is a methodology for collecting and interpreting expert’s experimental data by uncertainty theory. The study of uncertain statistics was started by Liu [12] in 2010 in which a questionnaire survey for collecting expert’s experimental data was designed and the empirical uncertainty distribution (i.e., the linear interpolation method) was proposed. In addition, the principle of least squares [12], the method of moments [21], the B-Spline method [2], and the Delphi method [22] were suggested to determine the uncertainty distributions from expert’s experimental data.

Uncertain programming was first initialized by Liu [10] in 2009 for dealing with optimization problems with uncertain parameters. After that, uncertain programming was applied to machine scheduling problem, vehicle routing problem, and project scheduling problem.

Uncertain logic was designed by Liu [15] in 2011 as a mathematical logic for dealing with uncertain knowledge via uncertain set theory, and provides a flexible means for extracting linguistic summary from a collection of raw data. Uncertain inference is a process of deriving consequences from uncertain knowledge or evidence via uncertain set theory. The first inference rule was proposed by Liu [13] in 2010. Then Gao, Gao and Ralescu [4] extended the inference rule to the case with multiple antecedents and with multiple if-then rules. Uncertain inference was applied to inference control via an inverted pendulum system.

Uncertain calculus, proposed by Liu [9] in 2008 and developed by Liu [11] in 2009, is a branch of mathematics that deals with differentiation and integration of functions of uncertain processes. Based on uncertain calculus, uncertain differential equations were defined by Liu [9] in 2008 as a type of differential equations driven by canonical process. After that, an existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [1]. Uncertain differential equations were also applied to finance by Liu [11] and Peng and Yao [19], and control by Zhu [25].

This paper will provide an analytic method to solve a particular class of nonlinear uncertain differential equations. In Section 2, some basic results on uncertain calculus are recalled. The uncertain differential equations of a special form are proposed in Section 3 and solved by an analytic method in Sections 4 and 5.

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## 2 Uncertain Calculus

In 1827 the botanist Robert Brown observed the irregular movement of pollen suspended in liquid. This movement is now known as *Brownian motion*. A rigorous mathematical definition of Brownian motion was given by Wiener [23] in 1923. After that, Ito [5] extended the classical calculus to Brownian motion and then Ito's calculus was invented in 1944. Based on Ito's calculus, the concept of stochastic differential equation was proposed by Ito [6]. For detailed explosions of stochastic calculus, the readers may consult Karatzas and Shreve [7] and Øksendal [18]. Different from Ito's calculus, Liu [9, 11] developed an uncertain calculus based on uncertainty theory. This section will introduce some basic concepts and theorems of uncertain calculus.

**Definition 1** [11] *Let  $X_t$  be an uncertain process and  $C_t$  a canonical process. For any partition of closed interval  $[a, b]$  with  $a = t_1 < t_2 < \dots < t_{k+1} = b$ , the mesh is written as*

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|. \tag{1}$$

*Then the Liu integral of  $X_t$  with respect to  $C_t$  is*

$$\int_a^b X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}) \tag{2}$$

*provided that the limit exists almost surely and is finite.*

**Definition 2** [11] *Let  $C_t$  be a canonical process and let  $X_t$  be an uncertain process. Assume there exist two uncertain processes  $\mu_t$  and  $\sigma_t$  such that*

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dC_s \tag{3}$$

*for any  $t \geq 0$ . Then we say  $X_t$  has a Liu differential*

$$dX_t = \mu_t dt + \sigma_t dC_t. \tag{4}$$

**Theorem 1** [11] (*Fundamental Theorem of Uncertain Calculus*) *Let  $C_t$  be a canonical process, and let  $h(t, c)$  be a continuously differentiable function. Then the uncertain process  $X_t = h(t, C_t)$  has a Liu integral*

$$dX_t = \frac{\partial h}{\partial t}(t, C_t)dt + \frac{\partial h}{\partial c}(t, C_t)dC_t. \tag{5}$$

**Remark 1:** (Chain Rule) *Let  $f$  and  $g$  be continuously differentiable functions. Then the uncertain process  $f(g(C_t))$  has a Liu differential*

$$df(g(C_t)) = f'(g(C_t))g'(C_t)dC_t. \tag{6}$$

**Remark 2:** (Change of Variable) *Let  $f$  and  $g$  be continuously differentiable functions. Then for any  $s > 0$ , we have*

$$\int_0^s f'(g(C_t))g'(C_t)dC_t = f(g(C_s)) - f(g(C_0)). \tag{7}$$

**Remark 3:** (Integration by Parts) *Suppose  $X_t$  and  $Y_t$  are differentiable uncertain processes. Then we have*

$$d(X_t Y_t) = Y_t dX_t + X_t dY_t. \tag{8}$$

## 3 Uncertain Differential Equations

**Definition 3** [9] *Suppose  $C_t$  is a canonical process, and  $f$  and  $g$  are some given functions. Then*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{9}$$

*is called an uncertain differential equation. A solution is an uncertain process  $X_t$  that satisfies (9) identically in  $t$ .*

Chen and Liu [1] proved that the uncertain differential equation has a unique solution if the coefficients  $f(x, t)$  and  $g(x, t)$  satisfy the Lipschitz condition

$$|f(x, t) - f(y, t)| + |g(x, t) - g(y, t)| \leq L|x - y|, \quad \forall x, y \in \mathfrak{R}, t \in [a, b]. \quad (10)$$

and linear growth condition

$$|f(x, t)| + |g(x, t)| \leq L(1 + |x|), \quad \forall x \in \mathfrak{R}, t \in [a, b] \quad (11)$$

for some constant  $L$ . This is the so-called existence and uniqueness theorem.

**Example 1:** [1] Let  $u_{1t}, u_{2t}, v_{1t}, v_{2t}$  be integrable uncertain processes. Then the linear uncertain differential equation

$$dX_t = (u_{1t}X_t + u_{2t})dt + (v_{1t}X_t + v_{2t})dC_t \quad (12)$$

has a solution

$$X_t = U_t \left( X_0 + \int_0^t \frac{u_{2s}}{U_s} ds + \int_0^t \frac{v_{2s}}{U_s} dC_s \right) \quad (13)$$

where

$$U_t = \exp \left( \int_0^t u_{1s} ds + \int_0^t v_{1s} dC_s \right). \quad (14)$$

## 4 Analytic Method - I

**Theorem 2** Let  $f$  be a function of two variables and let  $\sigma_t$  be an integrable uncertain process. Then the uncertain differential equation

$$dX_t = f(t, X_t)dt + \sigma_t X_t dC_t \quad (15)$$

has a solution

$$X_t = Y_t^{-1} Z_t \quad (16)$$

where

$$Y_t = \exp \left( - \int_0^t \sigma_s dC_s \right) \quad (17)$$

and  $Z_t$  is the solution of uncertain differential equation

$$dZ_t = Y_t f(t, Y_t^{-1} Z_t) dt \quad (18)$$

with initial value  $Z_0 = X_0$ .

**Proof:** At first, by using the chain rule, the uncertain process  $Y_t$  has an uncertain differential

$$dY_t = - \exp \left( - \int_0^t \sigma_s dC_s \right) \sigma_t dC_t = -Y_t \sigma_t dC_t.$$

It follows from the integration by parts that

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t = -X_t Y_t \sigma_t dC_t + Y_t f(t, X_t) dt + Y_t \sigma_t X_t dC_t.$$

That is,

$$d(X_t Y_t) = Y_t f(t, X_t) dt.$$

Defining  $Z_t = X_t Y_t$ , we obtain  $X_t = Y_t^{-1} Z_t$  and  $dZ_t = Y_t f(t, Y_t^{-1} Z_t) dt$ . Furthermore, since  $Y_0 = 1$ , the initial value  $Z_0$  is just  $X_0$ . The theorem is thus verified.

**Remark 4:** If  $\sigma_t$  becomes a constant  $\sigma$ , then  $Y_t = \exp(-\sigma C_t)$ , and the uncertain differential equation

$$dX_t = f(t, X_t) dt + \sigma X_t dC_t \quad (19)$$

has a solution

$$X_t = \exp(\sigma C_t) Z_t \quad (20)$$

where  $Z_t$  is the solution of uncertain differential equation

$$dZ_t = \exp(-\sigma C_t)f(t, \exp(\sigma C_t)Z_t)dt \tag{21}$$

with initial value  $Z_0 = X_0$ .

**Example 2:** Let  $\alpha$  and  $\sigma$  be real numbers with  $\alpha \neq 1$ . Consider the uncertain differential equation

$$dX_t = X_t^\alpha dt + \sigma X_t dC_t. \tag{22}$$

At first,  $Y_t = \exp(-\sigma C_t)$  and  $Z_t$  satisfies the uncertain differential equation,

$$dZ_t = \exp(-\sigma C_t)(\exp(\sigma C_t)Z_t)^\alpha dt = \exp((\alpha - 1)\sigma C_t)Z_t^\alpha dt.$$

Since  $\alpha \neq 1$ , we have

$$dZ_t^{1-\alpha} = (1 - \alpha) \exp((\alpha - 1)\sigma C_t)dt.$$

It follows from the fundamental theorem of uncertain calculus that

$$Z_t^{1-\alpha} = Z_0^{1-\alpha} + (1 - \alpha) \int_0^t \exp((\alpha - 1)\sigma C_s)ds.$$

Theorem 2 says the uncertain differential equation (22) has a solution  $X_t = \exp(\sigma C_t)Z_t$ , i.e.,

$$X_t = \exp(\sigma C_t) \left( X_0^{1-\alpha} + (1 - \alpha) \int_0^t \exp((\alpha - 1)\sigma C_s)ds \right)^{1/(1-\alpha)}.$$

## 5 Analytic Method - II

**Theorem 3** Let  $g$  be a function of two variables and let  $\alpha_t$  be an integrable uncertain process. Then the uncertain differential equation

$$dX_t = \alpha_t X_t dt + g(t, X_t) dC_t \tag{23}$$

has a solution

$$X_t = Y_t^{-1} Z_t \tag{24}$$

where

$$Y_t = \exp\left(-\int_0^t \alpha_s ds\right) \tag{25}$$

and  $Z_t$  is the solution of uncertain differential equation

$$dZ_t = Y_t g(t, Y_t^{-1} Z_t) dC_t \tag{26}$$

with initial value  $Z_0 = X_0$ .

**Proof:** At first, by using the chain rule, the uncertain process  $Y_t$  has an uncertain differential

$$dY_t = -\exp\left(-\int_0^t \alpha_s ds\right) \alpha_t dt = -Y_t \alpha_t dt.$$

It follows from the integration by parts that

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t = -X_t Y_t \alpha_t dt + Y_t \alpha_t X_t dt + Y_t g(t, X_t) dC_t.$$

That is,

$$d(X_t Y_t) = Y_t g(t, X_t) dC_t.$$

Defining  $Z_t = X_t Y_t$ , we obtain  $X_t = Y_t^{-1} Z_t$  and  $dZ_t = Y_t g(t, Y_t^{-1} Z_t) dC_t$ . Furthermore, since  $Y_0 = 1$ , the initial value  $Z_0$  is just  $X_0$ . The theorem is thus verified.

**Remark 5:** If  $\alpha_t$  becomes a constant  $\alpha$ , then  $Y_t = \exp(-\alpha t)$ , and the uncertain differential equation

$$dX_t = \alpha X_t dt + g(t, X_t) dC_t \quad (27)$$

has a solution

$$X_t = \exp(\alpha t) Z_t \quad (28)$$

where  $Z_t$  is the solution of uncertain differential equation

$$dZ_t = \exp(-\alpha t) g(t, \exp(\alpha t) Z_t) dC_t \quad (29)$$

with initial value  $Z_0 = X_0$ .

**Example 3:** Let  $\alpha$  and  $\beta$  be real numbers with  $\beta \neq 1$ . Consider the uncertain differential equation

$$dX_t = \alpha X_t dt + X_t^\beta dC_t. \quad (30)$$

At first,  $Y_t = \exp(-\alpha t)$  and  $Z_t$  satisfies the uncertain differential equation,

$$dZ_t = \exp(-\alpha t) (\exp(\alpha t) Z_t)^\beta dC_t = \exp((\beta - 1)\alpha t) Z_t^\beta dC_t.$$

Since  $\beta \neq 1$ , we have

$$dZ_t^{1-\beta} = (1 - \beta) \exp((\beta - 1)\alpha t) dC_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$Z_t^{1-\beta} = Z_0^{1-\beta} + (1 - \beta) \int_0^t \exp((\beta - 1)\alpha s) dC_s.$$

Theorem 3 says the uncertain differential equation (30) has a solution  $X_t = \exp(\alpha t) Z_t$ , i.e.,

$$X_t = \exp(\alpha t) \left( X_0^{1-\beta} + (1 - \beta) \int_0^t \exp((\beta - 1)\alpha s) dC_s \right)^{1/(1-\beta)}.$$

## 6 Conclusion

This paper presents an analytic method to solve a particular class of nonlinear uncertain differential equations. Some examples are also presented for illustrating the effectiveness of the proposed method.

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