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Value-at-Risk Criteria for Uncertain Portfolio Optimization Problem with Minimum Regret*

Yanju Chen†, Ying Liu
College of Mathematics and Computer Science, Hebei University, Baoding 071002, Hebei, China

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Abstract

We present a robust method to study fuzzy portfolio selection problem, in which the uncertain parameters are characterized by reduced fuzzy returns with parametric possibility distributions. The reduced fuzzy returns can be obtained by applying mean reduction methods to type-2 fuzzy return rates. If the investors expect to invest a portfolio with minimum regret, then we can formulate a fuzzy portfolio selection model by minimizing the value-at-risk (VaR) of the regret degree. In the case when the return rates are mutually independent type-2 triangular fuzzy variables, we can obtain the analytical representation of the VaR objective and turn the proposed fuzzy portfolio selection model into a parametric programming one so that conventional optimization methods can be used to solve it. Finally, we perform several numerical experiments to illustrate the proposed new modeling idea and the efficiency of solution method.
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Keywords: reduced fuzzy variable, parametric possibility distribution, portfolio selection, value-at-risk, regret degree, parametric programming

1 Introduction

Portfolio selection is to select a combination of securities which is the best to meet the investors' aim. Markowitz [12] applied probability theory to portfolio selection problem and proposed the famous mean-variance model which is the first mathematical formulation of the portfolio selection problem and is the foundation of modern portfolio analysis. He considered return rates of individual bonds as random variables. In the mean-variance model, the expected value and variance of the return rate were taken as the investment return and risk, respectively. In order to improve the mean-variance model and discuss more issues in portfolio field, some meaningful portfolio selection models have been proposed in the literature [13, 17, 19, 23]. In all these models, the return rates of bonds have been treated as random variables and the portfolio selection problems have been formulated as stochastic programming models.

It is well known that an investor will decide the investment proportion to each investment type according to the return rate of each investment type. However, the investor often cannot know the return rate well in the decision-making stage. In order to estimate the return rate, like the return rates of new stocks which are listed in the stock market, the investor will take experts' knowledge into account. In such cases, a fuzzy variable may reflect the experts' knowledge, and treat the return rate of the investment type as a fuzzy variable. On the basis of fuzzy theory [6, 8, 9, 25], some researchers applied various optimization methods to portfolio selection problem. In this respect, the interested reader may refer to the literature [1, 2, 3, 7, 18, 20]. For hybrid uncertain portfolio selection, Liu et al. [11] gave a new chance-variance optimization criterion.

Since the membership function is usually difficult to be determined in practice, Zadeh [24] introduced the concept of a type-2 fuzzy set as an extension of an ordinary fuzzy set. A type-2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership grades themselves are fuzzy sets in $[0, 1]$. Since then, type-2 fuzzy sets have been applied successfully to type-2 fuzzy logic systems [14], pattern recognition [15], and decision making [22]. Fuzzy possibility theory [10] is a variable-based approach to dealing with type-2

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†Corresponding author. Email: yanjuchen@hbu.edu.cn (Y. Chen).

fuzziness. On the basis of this theory, Chen and Wang [4] gave a scalar representative value to characterize the properties of type-2 fuzzy variables, and discussed some basic properties of the representative value operator, and Chen and Zhang [5] discussed the arithmetic of type-2 fuzzy variables. The literature [16] and [21] gave the reduction methods for the type-2 fuzzy variable. In the current development, we study the portfolio selection from a new viewpoint, and assume that the return rates are characterized by type-2 fuzzy variables with known secondary possibility distributions. We first employ a method to reduce the type-2 fuzzy return rates, then formulate a generalized credibility fuzzy portfolio selection model, in which reduced fuzzy returns have parametric possibility distributions. We adopt VaR criteria in the objective when the investors expect to invest with minimum regret. Under mild assumption, the established fuzzy portfolio selection model can be transformed into its equivalent parametric programming problem so that the conventional optimization method can apply.

The paper is organized as follows. In Section 2, we give some VaR formulas of three reduced fuzzy variables for type-2 triangular fuzzy variables. In Section 3, we model the portfolio selection problem in which the return rates are characterized by type-2 fuzzy variables with known secondary possibility distributions. We use the mean reduction methods to reduce the type-2 fuzzy return rates. Then based on generalized credibility measure, we give the portfolio selection model in the case that the investors expect a portfolio with minimum regret. When the return rates are described with mutually independent type-2 triangular fuzzy variables, we can turn the proposed model to be equivalent deterministic parametric programming. To illustrate the modeling idea and the efficiency of the proposed domain decomposition method, one numerical example is given in Section 4. Finally, Section 5 summarizes the main work in this paper.

2 VaR Parametric Formulas of Reduced Fuzzy Variables

Since the reduced fuzzy variables of type-2 triangular fuzzy variables obtained by the mean reduction methods have parametric possibility distribution [16], in this section, we derive VaR parametric formulas for these reduced fuzzy variables. For the reduced fuzzy variable by the E reduction method, its optimistic value is given as follows.

Theorem 1 *Let $\tilde{\xi} = (r_1, r_2, r_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable and ξ its reduced fuzzy variable by the E reduction method.*

(i) If $\alpha \in (0, \frac{4+\theta_r-\theta_l}{16}]$, then $\xi_{\sup}(\alpha) = r_3 - \frac{8\alpha(r_3-r_2)}{4+\theta_r-\theta_l}$;

(ii) If $\alpha \in (\frac{4+\theta_r-\theta_l}{16}, 0.5]$, then $\xi_{\sup}(\alpha) = \frac{4(1-2\alpha)r_3+(8\alpha-\theta_r+\theta_l)r_2}{4-\theta_r+\theta_l}$;

(iii) If $\alpha \in (0.5, \frac{12-\theta_r+\theta_l}{16}]$, then $\xi_{\sup}(\alpha) = \frac{4(2\alpha-1)r_1+(8(1-\alpha)-\theta_r+\theta_l)r_2}{4-\theta_r+\theta_l}$;

(iv) If $\alpha \in (\frac{12-\theta_r+\theta_l}{16}, 1]$, then $\xi_{\sup}(\alpha) = r_1 + \frac{8(1-\alpha)(r_2-r_1)}{4+\theta_r-\theta_l}$.

It is easy to know that $-\tilde{\xi} = (-r_3, -r_2, -r_1; \theta_l, \theta_r)$ if $\tilde{\xi} = (r_1, r_2, r_3; \theta_l, \theta_r)$. Therefore, if ξ is the reduced fuzzy variable of $\tilde{\xi}$, then $-\xi$ is the reduced fuzzy variable of $-\tilde{\xi}$. For any $\alpha \in (0, 1]$, it is easy to check $\xi_{\sup}(\alpha) = -(-\xi)_{\inf}(\alpha)$, and $\xi_{\inf}(\alpha) = -(-\xi)_{\sup}(\alpha)$. The following theorem describes the optimistic value formula of $-\xi$ and the pessimistic value of ξ where ξ is the reduced fuzzy variable of a type-2 triangular fuzzy variable by the E reduction method.

Theorem 2 *Let $\tilde{\xi} = (r_1, r_2, r_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable and ξ its reduced fuzzy variable by the E reduction method.*

(i) If $\alpha \in (0, \frac{4+\theta_r-\theta_l}{16}]$, then $(-\xi)_{\sup}(\alpha) = -r_1 - \frac{8\alpha(r_2-r_1)}{4+\theta_r-\theta_l}$ and $\xi_{\inf}(\alpha) = r_1 + \frac{8\alpha(r_2-r_1)}{4+\theta_r-\theta_l}$;

(ii) If $\alpha \in (\frac{4+\theta_r-\theta_l}{16}, 0.5]$, then $(-\xi)_{\sup}(\alpha) = \frac{4(1-2\alpha)(-r_1)+(8\alpha-\theta_r+\theta_l)(-r_2)}{4-\theta_r+\theta_l}$ and $\xi_{\inf}(\alpha) = \frac{4(1-2\alpha)r_1+(8\alpha-\theta_r+\theta_l)r_2}{4-\theta_r+\theta_l}$;

(iii) If $\alpha \in (0.5, \frac{12-\theta_r+\theta_l}{16}]$, then $(-\xi)_{\sup}(\alpha) = \frac{4(2\alpha-1)(-r_3)+(8(1-\alpha)-\theta_r+\theta_l)(-r_2)}{4-\theta_r+\theta_l}$ and $\xi_{\inf}(\alpha) = \frac{4(2\alpha-1)r_3+(8(1-\alpha)-\theta_r+\theta_l)r_2}{4-\theta_r+\theta_l}$;

(iv) If $\alpha \in (\frac{12-\theta_r+\theta_l}{16}, 1]$, then $(-\xi)_{\sup}(\alpha) = -r_3 + \frac{8(1-\alpha)(r_3-r_2)}{4+\theta_r-\theta_l}$ and $\xi_{\inf}(\alpha) = r_3 - \frac{8(1-\alpha)(r_3-r_2)}{4+\theta_r-\theta_l}$.

Similarly, for the reduced fuzzy variable ξ of a type-2 triangular fuzzy variable by the E^* reduction method, its optimistic value and pessimistic value are given in the following theorems.

Theorem 3 *Let $\tilde{\xi} = (r_1, r_2, r_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable and ξ its reduced fuzzy variable by the E^* reduction method.*

(i) If $\alpha \in (0, \frac{2+\theta_r}{8}]$, then $\xi_{\sup}(\alpha) = r_3 - \frac{4\alpha(r_3-r_2)}{2+\theta_r}$;

(ii) If $\alpha \in (\frac{2+\theta_r}{8}, 0.5]$, then $\xi_{\sup}(\alpha) = \frac{(4\alpha-\theta_r)r_2+(2-4\alpha)r_3}{2-\theta_r}$;

(iii) If $\alpha \in (0.5, \frac{6-\theta_r}{8}]$, then $\xi_{\sup}(\alpha) = \frac{(4\alpha-2)r_1+(4(1-\alpha)-\theta_r)r_2}{2-\theta_r}$;

(iv) If $\alpha \in (\frac{6-\theta_r}{8}, 1]$, then $\xi_{\sup}(\alpha) = r_1 + \frac{4(1-\alpha)(r_2-r_1)}{2+\theta_r}$.

Theorem 4 *Let $\tilde{\xi} = (r_1, r_2, r_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable and ξ its reduced fuzzy variable by the E^* reduction method.*

(i) If $\alpha \in (0, \frac{2+\theta_r}{8}]$, then $(-\xi)_{\sup}(\alpha) = -r_1 - \frac{4\alpha(r_2-r_1)}{2+\theta_r}$ and $\xi_{\inf}(\alpha) = r_1 + \frac{4\alpha(r_2-r_1)}{2+\theta_r}$;

(ii) If $\alpha \in (\frac{2+\theta_r}{8}, 0.5]$, then $(-\xi)_{\sup}(\alpha) = \frac{(4\alpha-\theta_r)(-r_2)+(2-4\alpha)(-r_1)}{2-\theta_r}$ and $\xi_{\inf}(\alpha) = \frac{(4\alpha-\theta_r)r_2+(2-4\alpha)r_1}{2-\theta_r}$;

(iii) If $\alpha \in (0.5, \frac{6-\theta_r}{8}]$, then $(-\xi)_{\sup}(\alpha) = \frac{(4\alpha-2)(-r_3)+(4(1-\alpha)-\theta_r)(-r_2)}{2-\theta_r}$ and $\xi_{\inf}(\alpha) = \frac{(4\alpha-2)r_3+(4(1-\alpha)-\theta_r)r_2}{2-\theta_r}$;

(iv) If $\alpha \in (\frac{6-\theta_r}{8}, 1]$, then $(-\xi)_{\sup}(\alpha) = -r_3 + \frac{4(1-\alpha)(r_3-r_2)}{2+\theta_r}$ and $\xi_{\inf}(\alpha) = r_3 - \frac{4(1-\alpha)(r_3-r_2)}{2+\theta_r}$.

Before ending the section we provide the results about the VaR formulas of the reduced fuzzy variable ξ of a type-2 triangular fuzzy variable by the E_* reduction method.

Theorem 5 *Let $\tilde{\xi} = (r_1, r_2, r_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable and ξ its reduced fuzzy variable by the E_* reduction method.*

(i) If $\alpha \in (0, \frac{2-\theta_l}{8}]$, then $\xi_{\sup}(\alpha) = r_3 - \frac{4\alpha(r_3-r_2)}{2-\theta_l}$;

(ii) If $\alpha \in (\frac{2-\theta_l}{8}, 0.5]$, then $\xi_{\sup}(\alpha) = \frac{(2-4\alpha)r_3+(4\alpha+\theta_l)r_2}{2+\theta_l}$;

(iii) If $\alpha \in (0.5, \frac{6+\theta_l}{8}]$, then $\xi_{\sup}(\alpha) = \frac{(4\alpha-2)r_1+(4(1-\alpha)+\theta_l)r_2}{2+\theta_l}$;

(iv) If $\alpha \in (\frac{6+\theta_l}{8}, 1]$, then $\xi_{\sup}(\alpha) = r_1 + \frac{4(1-\alpha)(r_2-r_1)}{2-\theta_l}$.

Theorem 6 *Let $\tilde{\xi} = (r_1, r_2, r_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable and ξ its reduced fuzzy variable by the E_* reduction method.*

(i) If $\alpha \in (0, \frac{2-\theta_l}{8}]$, then $(-\xi)_{\sup}(\alpha) = -r_1 - \frac{4\alpha(r_2-r_1)}{2-\theta_l}$ and $\xi_{\inf}(\alpha) = r_1 + \frac{4\alpha(r_2-r_1)}{2-\theta_l}$;

(ii) If $\alpha \in (\frac{2-\theta_l}{8}, 0.5]$, then $(-\xi)_{\sup}(\alpha) = \frac{(2-4\alpha)(-r_1)+(4\alpha+\theta_l)(-r_2)}{2+\theta_l}$ and $\xi_{\inf}(\alpha) = \frac{(2-4\alpha)r_1+(4\alpha+\theta_l)r_2}{2+\theta_l}$;

(iii) If $\alpha \in (0.5, \frac{6+\theta_l}{8}]$, then $(-\xi)_{\sup}(\alpha) = \frac{(4\alpha-2)(-r_3)+(4(1-\alpha)+\theta_l)(-r_2)}{2+\theta_l}$ and $\xi_{\inf}(\alpha) = \frac{(4\alpha-2)r_3+(4(1-\alpha)+\theta_l)r_2}{2+\theta_l}$;

(iv) If $\alpha \in (\frac{6+\theta_l}{8}, 1]$, then $(-\xi)_{\sup}(\alpha) = -r_3 + \frac{4(1-\alpha)(r_3-r_2)}{2-\theta_l}$ and $\xi_{\inf}(\alpha) = r_3 - \frac{4(1-\alpha)(r_3-r_2)}{2-\theta_l}$.

3 Portfolio Selection Problem with VaR of the Regret Degree

Suppose that an investor want to allocate his wealth of one unit among n investment types which are risky. It is well known that the investor will decide the investment proportion to the i-th investment type according to the return rate of the i-th investment type. But the investor cannot know the return rates well in the decision-making stage. Sometimes, we can only obtain limited information about the return rates such as the distributions. In this section, we assume that we can only obtain the type-2 distributions of the return rates, i.e., the return rates are characterized by type-2 fuzzy variables. In this case, we assume that all available funds are invested and that short sales are not permitted, then we can describe the portfolio selection problem as

$$\begin{cases} \text{maximize} & \sum_{i=1}^{n} \tilde{\xi}_i x_i \\ \text{subject to:} & \sum_{i=1}^{n} x_i = 1, \\ & x_i \geq 0, i = 1, 2, \ldots, n \end{cases} \tag{1}$$

where $\tilde{\xi}_i$ $(i = 1, 2, \ldots, n)$ represents the type-2 fuzzy return rate of the i-th investment type, and x_i is the decision variable which shows the investment proportion to the i-th investment type, $x_i \geq 0$, $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} x_i = 1$ is a portfolio, $\sum_{i=1}^{n} \tilde{\xi}_i x_i$ represents the corresponding return rate of the portfolio $x_i \geq 0$, $i = 1, 2, \ldots, n$.

If an investor has invested his money in some investment types according to a portfolio $x_i \geq 0$, $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} x_i = 1$. The investor may feel regretful when the return rates of other investment types become better than those of the invested ones as a result. Since we cannot know the return rates in the decision-making stage, any investment decision may bring regret to the investor. Suppose that an investor is informed about the determined return rates $\bar{\xi}_i$, $i = 1, 2, \ldots, n$ after he (she) has invested his (her) money in some investment types according to a portfolio $x_i \geq 0$, $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} x_i = 1$. The investor will have regret degree $r(\sum_{i=1}^{n} \bar{\xi}_i x_i)$ which can be quantified by many different ways [7].

According to Inuiguchi and Tanino [7], let $F : D_1 \times D_2 \to R(D_1, D_2 \subseteq R)$ be a continuous function such that $F(\cdot, x)$ is strictly increasing and $F(x, \cdot)$ is strictly decreasing. Then $r(\sum_{i=1}^{n} \bar{\xi}_i x_i)$ can be quantified as

$$\max_{\sum_{i=1}^{n} y_i = 1, y_i \geq 0} F(\textstyle\sum_{i=1}^{n} \bar{\xi}_i y_i, \sum_{i=1}^{n} \bar{\xi}_i x_i).$$

We know that $r(\sum_{i=1}^{n} \bar{\xi}_i x_i)$ is continuous, and the higher the return rate of the portfolio $x_i \geq 0$, $i = 1, 2, \ldots, n$ is, the smaller the regret degree is.

The investors are assumed to dislike regret. Thus, the investors would like to minimize the regret degree, i.e., the investors will choose a portfolio with minimum regret degree. Then, if short sales are not permitted and all available funds are invested, we can describe the portfolio selection problem as

$$\begin{cases} \text{minimize} & r(\sum_{i=1}^{n} \tilde{\xi}_i x_i) \\ \text{subject to:} & \sum_{i=1}^{n} x_i = 1, \\ & x_i \geq 0, i = 1, 2, \ldots, n \end{cases} \tag{2}$$

where $\tilde{\xi}_i$ $(i = 1, 2, \ldots, n)$ represents the type-2 fuzzy return rate of the i-th investment type, and x_i is the decision variable which shows the investment proportion to the i-th investment type, $x_i \geq 0$, $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} x_i = 1$ is a portfolio, $\sum_{i=1}^{n} \tilde{\xi}_i x_i$ represents the corresponding return rate of the portfolio $x_i \geq 0$, $i = 1, 2, \ldots, n$, $r(\sum_{i=1}^{n} \tilde{\xi}_i x_i)$ is the corresponding regret degree of the portfolio.

The programming problem (2) is not well-defined due to the type-2 fuzziness of the return rates. In order to obtain a meaningful mathematical model, in this paper, we first use the mean reduction methods to reduce the type-2 fuzzy return rates. Then rank fuzzy variables by pessimistic values, the portfolio selection problem can be built as the following generalized credibility constrained programming with VaR objective

$$\begin{cases} \text{minimize} & \bar{r} \\ \text{subject to:} & \tilde{\text{C}}\text{r}\{r(\sum_{i=1}^{n} \xi_i x_i) \leq \bar{r}\} \geq \alpha, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x_i \geq 0, i = 1, 2, \ldots, n \end{cases} \tag{3}$$

where ξ_i $(i = 1, 2, \ldots, n)$ represents the reduced fuzzy return rate of the type-2 fuzzy return rate $\tilde{\xi}_i$ according to the mean reduction methods, and x_i is the decision variable which shows the investment proportion to the

i-th investment type, $x_i \geq 0$, $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^n x_i = 1$ is a portfolio, $\sum_{i=1}^n \xi_i x_i$ represents the corresponding reduced fuzzy return rate of the portfolio $x_i \geq 0$, $i = 1, 2, \ldots, n$, $r(\sum_{i=1}^n \xi_i x_i)$ is the reduced regret degree, $\tilde{\mathrm{C}}\mathrm{r}$ is the generalized credibility measure, $\alpha \in (0, 1]$ is a prescribed confidence level, and $\bar{r}$ is the level to the regret degree $r(\sum_{i=1}^n \xi_i x_i)$.

In the following, Theorem 7 implies that the model (3) can be turned into a equivalent crisp form when the regret degree $r(\sum_{i=1}^n \xi_i x_i)$ is quantified as a mathematical expression.

Theorem 7 *Suppose that ξ_i is the reduced fuzzy variable of mutually independent type-2 fuzzy variables $\tilde{\xi}_i$, $i = 1, 2, \ldots, n$ in the model (2) with the mean reduction methods. Let $r(\sum_{i=1}^n \xi_i x_i) = \max_t \{\xi_t - \sum_{i=1}^n \xi_i x_i,\ t = 1, 2, \ldots, n\}$. Then the model (3) is equivalent to the following crisp form*

$$\begin{cases} \text{minimize} & \bar{r} \\ \text{subject to:} & \sum_{i=1, i\neq t}^n (\xi_i)_{sup}(\alpha) x_i + (1 - x_t)(-\xi_t)_{sup}(\alpha) + \bar{r} \geq 0,\ t = 1, 2, \ldots, n, \\ & \sum_{i=1}^n x_i = 1, \\ & x_i \geq 0, i = 1, 2, \ldots, n. \end{cases} \tag{4}$$

If the return rates are characterized by mutually independent type-2 triangular fuzzy variable $\tilde{\xi}_i = (r_{1,i}, r_{2,i}, r_{3,i}; \theta_{l,i}, \theta_{r,i})$, $i = 1, 2, \ldots, n$ and the parameters are known, then for any given α, by VaR parametric formulas of ξ_i in Section 2, we know that the model (4) is a linear programming problem.

In the model (3), we know that the larger α needs the larger $\bar{r}$, vice versa. But we want to maximize α and minimize $\bar{r}$ simultaneously, in order to balance the two objects, we take $\bar{r}/\alpha$ as objective function. Similar to the model (4), we have the following equivalent deterministic programming problem

$$\begin{cases} \text{minimize} & \frac{\bar{r}}{\alpha} \\ \text{subject to:} & \sum_{i=1, i\neq t}^n (\xi_i)_{sup}(\alpha) x_i + (1 - x_t)(-\xi_t)_{sup}(\alpha) + \bar{r} \geq 0,\ t = 1, 2, \ldots, n, \\ & \sum_{i=1}^n x_i = 1, \\ & x_i \geq 0, i = 1, 2, \ldots, n. \end{cases} \tag{5}$$

If the return rates are characterized by mutually independent type-2 triangular fuzzy variable $\tilde{\xi}_i = (r_{1,i}, r_{2,i}, r_{3,i}; \theta_{l,i}, \theta_{r,i})$, $i = 1, 2, \ldots, n$ and the parameters $\theta_{l,i} = \theta_l$ and $\theta_{r,i} = \theta_r$ for any $i = 1, 2, \ldots, n$ are unknown, then the model (5) can be turned into equivalent deterministic programming problems under some conditions. Now we suggest a domain decomposition method to find the optimal solutions to the deterministic programming models. We take E reduction method as an example to describe the idea of domain decomposition method. The similar idea can be applied to other mean reduction methods. We restrict the domain of $\bar{r}/\alpha$ in the following four disjoint feasible subregions. For the sake of presentation, the four feasible subregions are denoted by

$$D_1 = \left\{ \begin{array}{l} \sum_{i=1, i\neq t}^n \left(r_{1,i} + \frac{8(1-\alpha)(r_{2,i} - r_{1,i})}{4 + \theta_r - \theta_l} \right) x_i - r_{3,t}(1 - x_t) + \frac{8(1-\alpha)(r_{3,t} - r_{2,t})(1 - x_t)}{4 + \theta_r - \theta_l} + \bar{r} \geq 0, \\ t = 1, 2, \ldots, n, \frac{12 - \theta_r + \theta_l}{16} < \alpha \leq 1, \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \ldots, n \end{array} \right\}, \tag{6}$$

$$D_2 = \left\{ \begin{array}{l} \sum_{i=1, i\neq t}^n \frac{(8 - 8\alpha - \theta_r + \theta_l) r_{2,i} + 4(2\alpha - 1) r_{1,i}}{4 - \theta_r + \theta_l} x_i - \frac{(8 - 8\alpha - \theta_r + \theta_l) r_{2,t} + 4(2\alpha - 1) r_{3,t}}{4 - \theta_r + \theta_l} (1 - x_t) + \bar{r} \geq 0, \\ t = 1, 2, \ldots, n, 0.5 < \alpha \leq \frac{12 - \theta_r + \theta_l}{16}, \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \ldots, n \end{array} \right\}, \tag{7}$$

$$D_3 = \left\{ \begin{array}{l} \sum_{i=1, i\neq t}^n \frac{4(1 - 2\alpha) r_{3,i} + (8\alpha - \theta_r + \theta_l) r_{2,i}}{4 - \theta_r + \theta_l} x_i - \frac{4(1 - 2\alpha) r_{1,t} + (8\alpha - \theta_r + \theta_l) r_{2,t}}{4 - \theta_r + \theta_l} (1 - x_t) + \bar{r} \geq 0, \\ t = 1, 2, \ldots, n, \frac{4 + \theta_r - \theta_l}{16} < \alpha \leq 0.5, \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \ldots, n \end{array} \right\}, \tag{8}$$

$$D_4 = \left\{ \begin{array}{l} \sum_{i=1, i\neq t}^n \left(r_{3,i} - \frac{8\alpha(r_{3,i} - r_{2,i})}{4 + \theta_r - \theta_l} \right) x_i - r_{1,t}(1 - x_t) - \frac{8\alpha(r_{2,t} - r_{1,t})(1 - x_t)}{4 + \theta_r - \theta_l} + \bar{r} \geq 0, \\ t = 1, 2, \ldots, n, 0 < \alpha \leq \frac{4 + \theta_r - \theta_l}{16}, \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \ldots, n \end{array} \right\}. \tag{9}$$

If we denote the feasible region of problem (5) as D, then $D = D_1 \cup D_2 \cup D_3 \cup D_4$ and problem (5) can be equivalently written as

$$\begin{cases} \text{minimize} & \frac{\bar{r}}{\alpha} \\ \text{subject to:} & x \in D \end{cases} \tag{10}$$

where $x = (x_1, x_2, \ldots, x_n)$. As a consequence, the corresponding four subproblems are as follows:

$$\begin{cases} \text{minimize} & \frac{\bar{r}}{\alpha} \\ \text{subject to:} & x \in D_i \end{cases} \tag{11}$$

for $i = 1, 2, 3, 4$. Therefore, we can solve problem (10) indirectly by solving their corresponding subproblems (11) via conventional solution methods or general-purpose software. After the four subproblems have been solved, the solution to problem (10) can be found by comparing the solutions to the four subproblems. In the next section, we will demonstrate the effectiveness of the domain decomposition method via numerical experiments, in which the software Lingo is employ to solve the subproblems.

4 A Numerical Example

In this section, we suppose that an investor want to allocate his wealth of one unit among 20 investment types which are risky and short sales are not permitted. We assume that the return rates are characterized by mutually independent type-2 triangular fuzzy variables $\tilde{\xi}_i,\ i = 1, 2, \ldots, 20$, which are shown in Table 1. Let ξ_i $(i = 1, 2, \ldots, 20)$ represents the reduced fuzzy return rate according to the E reduction method, and x_i is the decision variable which shows the investment proportion to the i-th investment type, $\sum_{i=1}^{20} \xi_i x_i$ represents the corresponding reduced fuzzy return rate of the portfolio $x_i \geq 0,\ i = 1, 2, \ldots, 20$. According to model (5), the portfolio selection problem can be built as

$$\begin{cases} \text{minimize} & \bar{r} \\ \text{subject to:} & \sum_{i=1}^{20} (\xi_i)_{sup}(\alpha) x_i + (1 - x_t)(-\xi_t)_{sup}(\alpha) + \bar{r} \geq 0,\ t = 1, 2, \ldots, 20, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x_i \geq 0, i = 1, 2, \ldots, 20 \end{cases} \tag{12}$$

where $\alpha \in (0, 1]$ is a confidence level, and $\bar{r}$ is the level to the regret degree $r(\sum_{i=1}^{20} \xi_i x_i) = \max_t\{\xi_t - \sum_{i=1}^{20} \xi_i x_i,\ t = 1, 2, \ldots, 20\}$.

If the parameters of $\tilde{\xi}_i$ are known as Table 2, then for the given $\alpha = 0.8125$, the model (12) is a linear programming problem, Lingo software provides the optimal decision-makings of distribution of funds

$$\begin{aligned} x = (&0, 0.0642136, 0.6519550, 0, 0.0894621, 0.1021039, 0, 0.0160934, 0.0045299, \\ &0, 0.0190554, 0.0118926, 0.0020362, 0, 0, 0, 0, 0, 0, 0.0386580), \end{aligned} \tag{13}$$

and the level to the regret degree is $\bar{r} = 0.1947703$.

Table 1: The type-2 triangular fuzzy return rates $\tilde{\xi}_i,\ i = 1, 2, \ldots, 20$

i	1	2	3	4	5	6	7	8	9	10
$r_{1,i}$	-0.105	-0.12	-0.105	-0.09	-0.123	-0.121	-0.094	-0.127	-0.118	-0.096
$r_{2,i}$	0.125	0.14	0.124	0.112	0.14	0.14	0.114	0.14	0.139	0.125
$r_{3,i}$	0.185	0.23	0.182	0.176	0.24	0.24	0.18	0.22	0.22	0.19
i	11	12	13	14	15	16	17	18	19	20
$r_{1,i}$	-0.116	-0.116	-0.115	-0.106	-0.11	-0.11	-0.108	-0.113	-0.098	-0.118
$r_{2,i}$	0.138	0.136	0.136	0.13	0.132	0.13	0.13	0.135	0.126	0.138
$r_{3,i}$	0.217	0.217	0.215	0.2	0.21	0.21	0.205	0.213	0.19	0.217

Table 2: The parameters of type-2 triangular fuzzy return rates $\tilde{\xi}_i,\ i = 1, 2, \ldots, 20$

i	1	2	3	4	5	6	7	8	9	10
$\theta_{l,i}$	0.1	0.5	0.9	0.8	0.7	0.2	0.3	0.9	1.0	0.8
$\theta_{r,i}$	0.8	0.7	0.2	0.3	0.9	1.0	0.8	0.5	0.4	0.6
i	11	12	13	14	15	16	17	18	19	20
$\theta_{l,i}$	0.5	0.4	0.6	0	0.3	0.4	0.6	0.7	0.8	0.1
$\theta_{r,i}$	0.5	0.3	0.4	0.6	0.7	0.8	0.1	0.1	0.5	0.9

If parameters $\theta_{l,i} = \theta_l$ and $\theta_{r,i} = \theta_r$ for any $i = 1, 2, \ldots, 20$ are unknown, we use domain decomposition method to find the solution to the model (12). The corresponding four disjoint feasible subregions D_1, D_2, D_3 and D_4 are defined by the equations (6)-(9) in Section 3. In subregion D_1, Lingo software provides the investment proportions

$$\begin{aligned} x^1 = (&0, 0.1152966, 0, 0, 0.1387157, 0.1394842, 0, 0.0874749, 0.0868456, 0, 0.0761376, 0.0701316, \\ &0.0647085, 0.0011562, 0.0386056, 0.0323556, 0.0171048, 0.0563000, 0, 0.0756830) \end{aligned} \tag{14}$$

with $\bar{r} = 0.1551408$, $\alpha = 0.8125$ and objective value $\bar{r}/\alpha = 0.1909426$.

In subregion D_2, Lingo software provides the investment proportions

$$x^2 = (0, 0, 0, 0, 0.5181146, 0.0095068, 0, 0.4723786, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \tag{15}$$

with $\bar{r} = 0.0000008$, $\alpha = 0.5$ and objective value $\bar{r}/\alpha = 0.0000016$.

In subregion D_3, Lingo software provides the investment proportions

$$\begin{aligned} x^3 = (&0.0557038, 0.0440142, 0.0564466, 0.0608338, 0.0418269, 0.0420808, 0.0588172, \\ &0.0449190, 0.0461710, 0.0562415, 0.0470262, 0.0469241, 0.0474900, 0.0520273, \\ &0.0491517, 0.0490455, 0.0505030, 0.0481737, 0.0558822, 0.0467216) \end{aligned} \tag{16}$$

with $\bar{r} = 0$, $\alpha = 0.1961272$ and objective value $\bar{r}/\alpha = 0$.

In subregion D_4, Lingo software provides the investment proportions

$$\begin{aligned} x^4 = (&0.0555503, 0.0455764, 0.0559763, 0.0516329, 0.0409057, 0.0408921, 0.0511171, \\ &0.0505004, 0.0496816, 0.0537544, 0.0503760, 0.0489335, 0.0498896, 0.0526432, \\ &0.0494039, 0.0479729, 0.0503697, 0.0501177, 0.0543143, 0.0503921) \end{aligned} \tag{17}$$

with $\bar{r} = 0$, $\alpha = 0.3125$ and objective value $\bar{r}/\alpha = 0$.

Consequently, by comparing the four solutions, the investor obtains the investment proportions $x = x^4$.

The numerical experiments demonstrate that model (12) can provide diversification investments to assets for this portfolio problem.

5 Conclusions

In the current development, we studied the portfolio selection problem from a new viewpoint. The major new results include the following four aspects:

(*i*) Based on generalized credibility measure of fuzzy event, we deduce some useful VaR formulas about reduced fuzzy variables of type-2 triangular fuzzy variables;

(*ii*) We model the portfolio selection problem based on fuzzy possibility theory, in which the return rates are characterized by type-2 fuzzy variables with known secondary possibility distributions. We use the mean reduction methods to reduce the type-2 fuzzy return rates, then in the case that the investors expect to invest according to a portfolio with minimum regret, we formulate a generalized credibility constrained portfolio selection model with VaR objective;

(*iii*) When the regret degree of a portfolio is specified by some mathematical expression, and the return rates are mutually independent type-2 triangular fuzzy variables, with the obtained VaR formulas, the established fuzzy portfolio selection model can be transformed into the crisp equivalent parametric programming (4). When the parameters are known, for any given α, the model (4) is a linear programming problem. If the parameters are unknown, then we suggested a domain decomposition method to separate the model (5) into four subproblems, and solved the subproblems by Lingo software;

(*iv*) For the proposed portfolio selection models, we provide one numerical example to illustrate the modeling idea and the efficiency of the proposed method.

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