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Existence and Uniqueness Theorem on Uncertain Differential Equations with Local Lipschitz Condition

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Abstract

It has been proved that uncertain differential equation (UDE) has a unique solution, under the conditions that the coefficients are global Lipschitz continuous. This paper extends this existence and uniqueness theorem from the following aspect: UDE has a unique solution, under the conditions that the coefficients are local Lipschitz continuous. Besides, it is also proved that UDE has at least one solution, under the conditions that the coefficients are continuous and linear growth.

Keywords: uncertainty theory, uncertain calculus, uncertain differential equations

1 Introduction

For a long time, nondeterministic phenomenon was mainly described by probability and fuzzy theory. However, lots of surveys show that some nondeterministic phenomenon behaves neither like randomness nor fuzziness. It gave the motivation to Liu [7, 10] to propose uncertainty theory. Many researchers have done a lot of work in this area, such as Gao *et al* [4], Peng *et al* [13], You [16] and Zhu [17]. To learn more, the readers may reference to Liu's works [7, 10].

In probability, the stochastic differential equation (SDE) is a type of differential equation driven by Wiener process. Similarly, in uncertainty theory, there exists uncertain differential equation (UDE), a type of differential equation driven by canonical process. Canonical process is one of the most important uncertain process [8].

As in the ordinary differential equation (ODE) [6, 14] and the stochastic differential equation (SDE) [2, 3, 12], existence and uniqueness are the fundamental problems in UDE. The first existence and uniqueness theorem was proved by Chen and Liu [1] in 2010. In their paper, they proved that if the coefficients are global Lipschitz continuous, UDE has a unique continuous solution on $[0, +\infty)$. Later, Liu *etc.* [11] gave another existence and uniqueness theorem for the homogeneous UDEs when the coefficients satisfy global Osgood conditions.

However, the global conditions of these existing theorems are too strict. As is known, there are few functions satisfying global Lipschitz continuity. The vast majority of functions are only local Lipschitz continuous. Does there exist an existence and uniqueness theorem for a UDE with local Lipschitz continuous coefficients? This paper gives the answer.

In this paper, a general existence and uniqueness theorem is presented. We obtain the same conclusion with Chen and Liu's existence and uniqueness theorem, but in our theorem the Lipschitz condition is only required a local one. This largely extends the existence and uniqueness theorem.

Sometimes, even the local Lipschitz condition is strict. The more general case is that we just know that the coefficients are continuous. Fortunately, this paper proves another conclusion: without Lipschitz condition, at least one continuous solution exists on $[0, +\infty)$. This is only an existence theorem, and it can not guarantee the uniqueness of a solution. The main application of this existence theorem is on numerical solution of UDE. Yao and Chen [15] can give a numerical solution under the condition that the UDE is solvable.

The remainder of this paper is organized as follows. In Section 2, some basic concepts and properties of uncertainty theory used throughout this paper are introduced. In Section 3, the existence and uniqueness

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theorem under the local Lipschitz condition is proved. Section 4 proves that the UDE have at least one solution under the condition that the coefficients are continuous. Section 5 gives a brief summary to this paper.

2 Preliminary Concepts and Definitions

In this section, we introduce some foundational concepts and properties of uncertainty theory, which are used throughout this paper.

Let Γ be a nonempty set, and Ł a σ-algebra over Γ. Each element $\Lambda \in$ Ł is assigned a number $\mathcal{M}\{\Lambda\} \in [0,1]$. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu [7, 10] presented the four following axioms: (1) normality, (2) self-duality, (3) countable subadditivity, and (4) product measure axioms. If the first three axioms are satisfied, the set function $\mathcal{M}\{\Lambda\}$ is called an uncertain measure.

Definition 1 [7] *Let Γ be a nonempty set, Ł a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, Ł, \mathcal{M})$ is called an uncertainty space.*

Definition 2 [7] *An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, Ł, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 3 [8] *Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, Ł, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set*

$$\{\xi_t \in B\} = \{\gamma \in \Gamma \mid \xi_t(\gamma) \in B\}$$

is an event.

That is, an uncertain process $X_t(\gamma)$ is a function of two variables such that the function $X_{t^*}(\gamma)$ is an uncertain variable for each t^*.

Definition 4 [9] *An uncertain process C_t is said to be a canonical process if*
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \mathcal{M}\{\xi \le x\} = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \in \Re$$

whose expected value is e and variance is σ^2.

Definition 5 *Let X_t be an uncertain process and let C_t be a canonical process. For any partition of closed interval $[a,b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the uncertain integral of X_t with respect to C_t is

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. For this case, the uncertain process X_t is said to be integrable.

In 2010, Chen and Liu [1] gave the following existence and uniqueness theorem on UDEs.

Theorem 1 *([1] Existence and Uniqueness Theorem) The UDE*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

has a unique solution if the coefficients $f(t,x)$ and $g(t,x)$ satisfy the global Lipschitz condition

$$|f(t,x_1) - f(t,x_2)| + |g(t,x_1) - g(t,x_2)| \le L|x_1 - x_2|, \quad \forall x, y \in \Re, t \ge 0$$

and the linear growth condition

$$|f(t,x)| + |g(t,x)| \le L(1 + |x|), \quad \forall x \in \Re, t \ge 0$$

for some constant L. Moreover, the solution is sample-continuous.

Indeed, the global Lipschitz condition in Theorem (1) is too strict, and few equations can satisfy these conditions. In this paper, we will extend this theorem by replacing global Lipschitz condition by local Lipschitz condition. For the sake of simplicity, our discussion is not based on the form of UDE (1) but on its equivalent integral form, or say, uncertain integral equation

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s. \tag{2}$$

3 Existence and Uniqueness Theorem with the Local Lipschitz Condition

Since almost all sample paths of canonical process C_t are Lipschitz continuous functions, there exists a set Γ_0 in Γ with $\mathcal{M}\{\Gamma_0\} = 1$ such that for any $\gamma \in \Gamma_0$, $C_t(\gamma)$ is Lipschitz continuous. For the sake of simplicity, in this paper we set $\Gamma_0 = \Gamma$. Thus, for each path γ, there exists a positive number $K(\gamma)$ such that

$$|C_s(\gamma) - C_t(\gamma)| \le K(\gamma)|s - t|, \quad \forall s, t \ge 0.$$

The uncertain integral of C_t is equivalent to Riemann-Stieltjes integral from the point of each sample path. Hence, we can focus on the integral equations as follows

$$X_t(\gamma) = X_0 + \int_0^t f(s, X_s(\gamma))\mathrm{d}s + \int_0^t g(s, X_s(\gamma))\mathrm{d}C_s(\gamma). \tag{3}$$

We will prove that under some conditions, for each sample path γ, the integral equation (3) has a unique solution on $[0, +\infty)$.

We first prove the existence and uniqueness theorem on a local interval $[t_0, t_0 + c]$ for some positive c. The integral equations (3) becomes

$$X_t(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^t f(s, X_s(\gamma))\mathrm{d}s + \int_{t_0}^t g(s, X_s(\gamma))\mathrm{d}C_s(\gamma). \tag{4}$$

Condition 1 *The functions $f(t,x)$ and $g(t,x)$ satisfy a local Lipschitz condition on $(t,x) \in [0,+\infty) \times \Re$.*

Condition 1 indicates that for any $\gamma \in \Gamma$ and $t_0 \in [0, +\infty)$, there exists a rectangular region

$$\mathcal{R} = \mathcal{R}((t_0, X_{t_0}(\gamma)), a, b) = \{(t,x) : t_0 \le t \le t_0 + a, |x - X_{t_0}(\gamma)| \le b\},$$

on which $f(t,x)$ and $g(t,x)$ satisfy a Lipschitz condition, *i.e.*, there exists a positive constant L such that

$$|f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2|, \quad |g(t,x_1) - g(t,x_2)| \le L|x_1 - x_2|$$

whenever (t, x_1) and $(t, x_2) \in \mathcal{R}$. Obviously, L is dependent on the region $\mathcal{R}$.

Since $f(t,x)$ and $g(t,x)$ are continuous on $\mathcal{R}$, $f(t,x)$ and $g(t,x)$ are bounded. Set

$$H = \max_{\mathcal{R}} \Big(|f(t,x)| + K(\gamma) \cdot |g(t,x)| \Big) + 1,$$

where $K(\gamma)$ is the Lipschitz constant to the sample path $C_t(\gamma)$. Now, set $\alpha = \min(a, b/H)$.

Lemma 1 *Fix $\gamma \in \Gamma$, under Condition 1, the integral equation*

$$X_t(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, X_s(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s(\gamma))\mathrm{d}C_s(\gamma) \tag{4}$$

has a unique solution on the interval $[t_0, t_0 + \alpha]$. Moreover, the solution is continuous.

Proof: We will prove this lemma in three steps by using the following successive approximations

$$\begin{cases} X_t^{(0)}(\gamma) = X_{t_0}(\gamma), \\ X_t^{(n+1)}(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, X_s^{(n)}(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s^{(n)}(\gamma))\mathrm{d}C_s(\gamma), \quad n \geq 0. \end{cases} \tag{5}$$

Obviously, for each $n \geq 0$, $\{X_t^{(n)}(\gamma)\}$ is continuous in t.

Step 1. In this step, we will prove that $(t, X_t^{(n)}(\gamma)) \in \mathcal{R}$, $\forall n \geq 0$ when $t \in [t_0, t_0 + \alpha]$.

Here, we employ mathematical induction. The conclusion is obvious when $n = 0$. Assume that when $t \in [t_0, t_0 + \alpha]$, $(t, X_t^{(n)}(\gamma)) \in \mathcal{R}$. Then,

$$\begin{aligned} |X_t^{(n+1)}(\gamma) - X_0(\gamma)| &= \left| \int_{t_0}^{t} f(s, X_s^{(n)}(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s^{(n)}(\gamma))\mathrm{d}C_s(\gamma) \right| \\ &\leq \int_{t_0}^{t} |f(s, X_s^{(n)}(\gamma))|\mathrm{d}s + \int_{t_0}^{t} |g(s, X_s^{(n)}(\gamma))|\mathrm{d}C_s(\gamma) \\ &\leq \int_{t_0}^{t} |f(s, X_s^{(n)}(\gamma))|\mathrm{d}s + K(\gamma) \cdot \int_{t_0}^{t} |g(s, X_s^{(n)}(\gamma))|\mathrm{d}s \\ &\leq H|t - t_0| \leq H\alpha \leq b. \end{aligned}$$

This means that $(t, X_t^{(n+1)}(\gamma)) \in \mathcal{R}$ when $t \in [t_0, t_0 + \alpha]$.

Step 2. In this step, we will prove that the sequence $\{X_t^{(n)}(\gamma)\}_{n=0}^{\infty}$ given by (5) converges uniformly to the solution of integral equation (4) on $[t_0, t_0 + \alpha]$ as $n \to \infty$.

First, as in Step 1, mathematical induction is employed. When $n = 0$, we have

$$\begin{aligned} |X_t^{(1)}(\gamma) - X_0(\gamma)| &= \left| \int_{t_0}^{t} f(s, X_s^{(0)}(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s^{(0)}(\gamma))\mathrm{d}C_s(\gamma) \right| \\ &\leq \int_{t_0}^{t} |f(s, X_s^{(0)}(\gamma))|\mathrm{d}s + \int_{t_0}^{t} |g(s, X_s^{(0)}(\gamma))|\mathrm{d}C_s(\gamma) \\ &\leq \int_{t_0}^{t} |f(s, X_s^{(0)}(\gamma))|\mathrm{d}s + K(\gamma) \cdot \int_{t_0}^{t} |g(s, X_s^{(0)}(\gamma))|\mathrm{d}s \\ &\leq H|t - t_0|. \end{aligned}$$

Second, assume that when $t \in [t_0, t_0 + \alpha]$, $|X_t^{(n)}(\gamma) - X_t^{(n-1)}(\gamma)| \leq \dfrac{H(L + K(\gamma) \cdot L)^{n-1}}{n!}|t - t_0|^n$. Then

$$\begin{aligned} &|X_t^{(n+1)}(\gamma) - X_t^{(n)}(\gamma)| \\ =&\left| \int_{t_0}^{t} \Big(f(s, X_s^{(n)}(\gamma)) - f(s, X_s^{(n-1)}(\gamma)) \Big) \mathrm{d}s + \int_{t_0}^{t} \Big(g(s, X_s^{(n)}(\gamma)) - g(s, X_s^{(n-1)}(\gamma)) \Big) \mathrm{d}C_s(\gamma) \right| \\ \leq& \int_{t_0}^{t} \left| \Big(f(s, X_s^{(n)}(\gamma)) - f(s, X_s^{(n-1)}(\gamma)) \Big) \right| \mathrm{d}s + \int_{t_0}^{t} \left| \Big(g(s, X_s^{(n)}(\gamma)) - g(s, X_s^{(n-1)}(\gamma)) \Big) \right| \mathrm{d}C_s(\gamma) \\ \leq& (L + K(\gamma) \cdot L) \int_{t_0}^{t} |X_s^{(n)}(\gamma) - X_s^{(n-1)}(\gamma)|\mathrm{d}s \\ \leq& \frac{H(L + K(\gamma) \cdot L)^n}{n!} \int_{t_0}^{t} |s - t_0|^n \mathrm{d}s = \frac{H(L + K(\gamma) \cdot L)^n}{(n+1)!}|t - t_0|^{n+1}. \end{aligned} \tag{6}$$

The above inequality (6) gives an upper bound of $|X_t^{(n+1)}(\gamma)-X_t^{(n)}(\gamma)|$ on $[t_0, t_0+\alpha]$, for $n \geq 1$. Obviously, $\forall\epsilon > 0$, there exists a integer N such that

$$\sum_{n\geq N}|X_t^{(n+1)}(\gamma)-X_t^{(n)}(\gamma)| \leq \frac{H}{L+K(\gamma)\cdot L}\sum_{n\geq N}\frac{(L+K(\gamma)\cdot L)^{n+1}|t-t_0|^{n+1}}{(n+1)!}$$
$$\leq \frac{H}{L+K(\gamma)\cdot L}\sum_{n\geq N}\frac{(L+K(\gamma)\cdot L)^{n+1}\alpha^{n+1}}{(n+1)!} < \epsilon. \tag{7}$$

Since $X_t^{(n)}(\gamma) = X_t^{(0)}(\gamma) + \sum_{i=1}^{n}(X_t^{(i)}(\gamma) - X_t^{(i-1)}(\gamma))$, inequality (7) indicates that $X_t^{(n)}(\gamma)$ converges uniformly on $[t_0, t_0+\alpha]$ as $n \to \infty$. Denote $X_t(\gamma) = \lim_{n\to\infty} X_t^{(n)}(\gamma)$.

It is known that

$$X_t^{(n+1)}(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, X_s^{(n)}(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s^{(n)}(\gamma))\mathrm{d}C_s(\gamma), \quad n \geq 0.$$

Since $f(t,x)$ and $g(t,x)$ are continuous, and $C_t(\gamma)$ is Lipschitz continuous, we have

$$X_t(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, X_s(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s(\gamma))\mathrm{d}C_s(\gamma).$$

In short, the sequence $\{X_t^{(n)}\}, n \geq 0$ given by (5) converges uniformly to the solution of integral equation (4) on $[t_0, t_0+\alpha]$ as $n \to \infty$. Since for each $n \geq 0$, $\{X_t^{(n)}(\gamma)\}$ is continuous, $X_t(\gamma)$ is also continuous on $[t_0, t_0+\alpha]$. This completes the proof of existence.

Step 3. In this step, we will prove that $X_t(\gamma)$ obtained in Step 2 is the unique solution of integral equation (4) on $[t_0, t_0+\alpha]$.

Suppose that $\tilde{X}_t(\gamma)$ is another solution of of integral equation (4), *i.e.*,

$$\tilde{X}_t(\gamma) = \tilde{X}_{t_0}(\gamma) + \int_{t_0}^{t} f(s, \tilde{X}_s(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, \tilde{X}_s(\gamma))\mathrm{d}C_s(\gamma) \quad t \in [t_0, t_0+\beta],$$

where $0 < \beta \leq \alpha$.

Following the local Lipchitz condition, we have

$$\begin{aligned}
&|X_t(\gamma) - \tilde{X}_t(\gamma)| \\
=&\left|\int_{t_0}^{t}\Big(f(s, X_s(\gamma)) - f(s, \tilde{X}_s(\gamma))\Big)\,\mathrm{d}s + \int_{t_0}^{t}\Big(g(s, X_s(\gamma)) - g(s, \tilde{X}_s(\gamma))\Big)\,\mathrm{d}C_s(\gamma)\right| \\
\leq&\int_{t_0}^{t}\left|\Big(f(s, X_s(\gamma)) - f(s, \tilde{X}_s(\gamma))\Big)\right|\mathrm{d}s + \int_{t_0}^{t}\left|\Big(g(s, X_s(\gamma)) - g(s, \tilde{X}_s(\gamma))\Big)\right|\mathrm{d}C_s(\gamma) \\
\leq&(L+K(\gamma)\cdot L)\int_{t_0}^{t}|X_s(\gamma) - \tilde{X}_s(\gamma)|\mathrm{d}s.
\end{aligned} \tag{8}$$

By Gronwell inequality [5], expression (8) leads to

$$|X_t(\gamma) - \tilde{X}_t(\gamma)| \leq 0\cdot\exp((L+K(\gamma)\cdot L)t) = 0,$$

that is, on $[t_0, t_0+\tilde{\alpha}]$, $X_t(\gamma) = \tilde{X}_t(\gamma)$. This completes the proof of uniqueness. □

According to Lemma 1, the integral differential equation (3) has a unique solution on some local interval $[0, \alpha]$. We will show that the solution can be extended to $[0, +\infty)$ under the following condition.

Condition 2 *The functions $f(t,x)$ and $g(t,x)$ satisfy a local linear growth condition on $(t,x) \in [0,+\infty)\times\Re$, i.e., for each $T > 0$, there exists a constant G_T such that*

$$|f(t,x)| \leq G_T(1+|x|), \quad |g(t,x)| \leq G_T(1+|x|), \quad \forall x \in \Re, t \in [0,T].$$

Lemma 2 *Fix $\gamma \in \Gamma$, under Condition 1 and Condition 2, the solution of integral equation (3) can be extended uniquely to* $[0, +\infty)$.

Proof: Define $\mathcal{T} = \Big\{t$: integral equation (3) has a unique continuous solution on $[0,t)\Big\}$, and $\tau = \sup \mathcal{T}$. According to Lemma 1, the set $\mathcal{T}$ is nonempty. We will prove that $\tau = +\infty$.

Assume that $\tau < +\infty$, and a contradiction will be derived. As the definition, $X_t(\gamma)$ is the unique solution of integral equation (3) on $[0,\tau)$. Then,

$$\begin{aligned}|X_t(\gamma)| &\le |X_0(\gamma)| + \int_0^t |f(s, X_s(\gamma))|\mathrm{d}s + \int_0^t |g(s, X_s(\gamma))|\mathrm{d}C_s(\gamma)\\ &\le |X_0(\gamma)| + \int_0^t |f(s, X_s(\gamma))|\mathrm{d}s + K(\gamma)\cdot \int_0^t |g(s, X_s(\gamma))|\mathrm{d}s\\ &\le |X_0(\gamma)| + \int_0^t G_\tau(1+|X_s(\gamma)|)\mathrm{d}s + K(\gamma)\cdot \int_0^t G_\tau(1+|X_s(\gamma)|)\mathrm{d}s\\ &\le (|X_0(\gamma)| + \tau(1+K(\gamma))G_\tau) + (1+K(\gamma))G_\tau\int_0^t |X_s(\gamma)|\mathrm{d}s, \quad \forall t \in [0,\tau).\end{aligned}$$

Setting $A = |X_0(\gamma)| + \tau(1+K(\gamma))G_\tau$, by Gronwell inequality [5], we have

$$|X_t(\gamma)| \le A\cdot \exp[(1+K(\gamma))G_\tau t] \le A\cdot\exp[(1+K(\gamma))G_\tau \tau] = H_1 < +\infty, \ \forall t\in[0,\tau),$$

that is, $X_t(\gamma)$ is bounded on $[0,\tau)$.

Moreover, $X_t(\gamma)$ is uniformly continuous on $[0,\tau)$. Setting

$$H_2 = \max_{0\le t\le\tau, |x|\le H_1}\Big(|f(t,x)| + K(\gamma)\cdot|g(t,x)|\Big),$$

we have

$$\begin{aligned}|X_{t_1}(\gamma) - X_{t_2}(\gamma)| &\le \left|\int_{t_1}^{t_2} |f(s, X_s(\gamma))|\mathrm{d}s + \int_{t_1}^{t_2} |g(s, X_s(\gamma))|\mathrm{d}C_s(\gamma)\right|\\ &\le \left|\int_{t_1}^{t_2} |f(s, X_s(\gamma))|\mathrm{d}s + K(\gamma)\cdot\int_{t_1}^{t_2} |g(s, X_s(\gamma))|\mathrm{d}s\right|\\ &\le H_2|t_1 - t_2|, \quad \forall t_1, t_2 \in [0,\tau).\end{aligned}$$

It follows that $\lim\limits_{x\to\tau-} X_t(\gamma)$ exists. Set

$$X_\tau(\gamma) = \lim_{t\to\tau-} X_t(\gamma).$$

Then $X_t(\gamma)$ is continuous on $[0,\tau]$, and

$$X_t(\gamma) = X_0(\gamma) + \int_0^t f(s, X_s(\gamma))\mathrm{d}s + \int_0^t g(s, X_s(\gamma))\mathrm{d}C_s(\gamma), \quad \forall t\in[0,\tau].$$

Consider the following integral equation

$$X_t(\gamma) = X_\tau(\gamma) + \int_\tau^t f(s, X_s(\gamma))\mathrm{d}s + \int_\tau^t g(s, X_s(\gamma))\mathrm{d}C_s(\gamma), \quad t > \tau. \tag{9}$$

Lemma 1 says that there exists a positive number c such that integral equation (9) has a unique continuous solution $\tilde{X}_t(\gamma)$ on $[\tau, \tau + c]$.

Hence, functin

$$Y_t(\gamma) = \begin{cases} X_t(\gamma), & if \quad t\in[0,\tau],\\ \tilde{X}_t(\gamma), & if \quad t\in[\tau,\tau+c]\end{cases}$$

is the unique continuous solution of integral equation (4) on $[0, \tau + c]$.

It is a contradiction from $\tau = \sup\mathcal{T} < +\infty$. Thus, $\tau = +\infty$, and the solution of integral equation (3) can be extended uniquely to $[0, +\infty)$. □

Since γ is an arbitrary sample path, we have

Theorem 2 *Under Condition 1 and Condition 2, the uncertain integral equation*

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s$$

has a unique solution. Moreover, the solution is sample-continuous.

Or, equivalently, we have

Theorem 3 *Under Condition 1 and Condition 2, the UDE*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

has a unique solution. Moreover, the solution is sample-continuous.

Example 1. Consider the UDE

$$\mathrm{d}X_t = tX_t\mathrm{d}t + tX_t\mathrm{d}C_t.$$

Obviously, the coefficients $f(s, x) = tx$ and $g(t, x) = tx$ are local Lipschitz continuous but not global Lipschitz continuous. For this UDE, Chen and Liu's existence and uniqueness theorem does not work. However, according to Theorem 3, it has a unique continuous solution. In fact, the solution is

$$X_t = X_0\exp(\frac{t^2}{2} + \int_0^t s\mathrm{d}C_s).$$

4 An Additional Conclusion: Existence Theorem Without the Lipschitz Condition

In this section, we only study the existence theorem. The Lipschitz condition is removed, and it only requires that the coefficients $f(t, x)$ and $g(t, x)$ are continuous. Under the condition without the Lipschitz condition, we can find a solution to the integral equation (4) on $[0, +\infty)$. However, we cannot guarantee that this solution is unique.

Condition 3 *The functions $f(t, x)$ and $g(t, x)$ are continuous on $(t, x) \in [0, +\infty) \times \Re$.*

Under Condition 3, for any $\gamma \in \Gamma$ and $t_0 \in [0, +\infty)$, we can find a rectangular region

$$\mathcal{R} = \mathcal{R}((t_0, X_{t_0}(\gamma)), a, b) = \{(t, x) : t_0 \le t \le t_0 + a, |x - X_{t_0}(\gamma)| \le b\},$$

on which $f(t, x)$ and $g(t, x)$ are continuous. We will prove that on some subinterval, say $[t_0, t_0 + \alpha]$, of the interval $[t_0, t_0 + a]$, there exists a continuous solution of integral equation (4).

As in Section 3, set

$$H = \max_{\mathcal{R}} \Big(|f(t, x)| + K(\gamma) \cdot |g(t, x)|\Big) + 1,$$

where $K(\gamma)$ is the Lipschitz constant to the sample path $C_t(\gamma)$, and set $\alpha = \min(a, b/H)$.

Before go further, we must introduce two concepts and Ascoli-Arzelà Theorem.

Definition 6 *(Equicontinuity) On a bounded interval $[a, b]$, a sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to be equicontinuous if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that*

$$|f_n(t_1) - f_n(t_2)| < \epsilon, \quad \forall n \ge 1, \ |t_1 - t_2| < \delta(\epsilon).$$

Definition 7 *(Uniform Boundedness) On a bounded interval $[a, b]$, a sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to be bounded if there exists a non-negative number B such that*

$$|f_n(t)| < B, \quad \forall n \ge 1, \ \forall t \in [a, b].$$

Theorem 4 *(Ascoli-Arzelà Theorem) On a bounded interval $[a, b]$, every bounded and equicontinuous sequence of functions $\{f_n\}_{n=1}^{\infty}$ contains a subsequence which is uniformly convergent on $[a, b]$.*

Based on Ascoli-Arzelà Theorem, we can prove the following existence theorem on a local interval.

Lemma 3 *Fix $\gamma \in \Gamma$, under Condition 1, the integral equation*

$$X_t(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, X_s(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s(\gamma))\mathrm{d}C_s(\gamma) \tag{4}$$

has at least one continuous solution on the interval $[t_0, t_0+\alpha]$.

Proof: We construct a sequence of continuous functions $\{Y_n(t)\}_{n=1}^{\infty}$ by

$$Y_n(t) = \begin{cases} X_{t_0}(\gamma), & if \quad t \le t_0, \\ X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, Y_n(s-\frac{1}{n}))\mathrm{d}s + \int_{t_0}^{t} g(s, Y_n(s-\frac{1}{n}))\mathrm{d}C_s(\gamma), & if \quad t \in [t_0, t_0+\alpha]. \end{cases}$$

For each n, $Y_n(t)$ is well defined. Indeed, if $t_0 \le t \le t_0 + 1/n$, then $t - 1/n \le t_0$ and $Y_n(t-1/n) = X_{t_0}(\gamma)$. Also, $Y_n(t)$ is continuous on $[t_0, t_0+1/n]$. Moreover, on the interval $t_0 \le t \le t_0 + 1/n$,

$$\begin{aligned} |Y_n(t) - X_{t_0}(\gamma)| &\le \int_{t_0}^{t} |f(s, Y_n(s-\frac{1}{n}))|\mathrm{d}s + \int_{t_0}^{t} |g(s, Y_n(s-\frac{1}{n}))|\mathrm{d}C_s(\gamma) \\ &= \int_{t_0}^{t} |f(s, X_{t_0}(\gamma))|\mathrm{d}s + \int_{t_0}^{t} |g(s, X_{t_0}(\gamma))|\mathrm{d}C_s \\ &\le \int_{t_0}^{t} |f(s, X_{t_0}(\gamma))|\mathrm{d}s + K(\gamma) \cdot \int_{t_0}^{t} |g(s, X_{t_0}(\gamma))|\mathrm{d}s \\ &\le H|t - t_0| \le H\alpha \le b. \end{aligned}$$

Hence $(t, Y_n(t)) \in \mathcal{R}$, when $t_0 \le t \le t_0 + 1/n$. If $t_0 + 1/n \le t \le t_0 + 2/n$, then $t_0 \le t - 1/n \le t_0 + 1/n$. So, $Y_n(t-1/n)$ is determined from the previous step and $|Y_n(t-1/n) - X_{t_0}(\gamma)| \le b$. Also, $Y_n(t)$ is continuous on $[t_0, t_0+2/n]$. Moreover, on the interval $t_0 + 1/n \le t \le t_0 + 2/n$,

$$\begin{aligned} |Y_n(t) - X_{t_0}(\gamma)| &\le \int_{t_0}^{t} |f(s, Y_n(s-\frac{1}{n}))|\mathrm{d}s + \int_{t_0}^{t} |g(s, Y_n(s-\frac{1}{n}))|\mathrm{d}C_s(\gamma) \\ &\le \int_{t_0}^{t} |f(s, Y_n(s-\frac{1}{n}))|\mathrm{d}s + \int_{t_0}^{t} |g(s, Y_n(s-\frac{1}{n}))|\mathrm{d}C_s(\gamma) \\ &\le \int_{t_0}^{t} |f(s, Y_n(s-\frac{1}{n}))|\mathrm{d}s + K(\gamma) \cdot \int_{t_0}^{t} |g(s, Y_n(s-\frac{1}{n}))|\mathrm{d}s \\ &\le H|t - t_0| \le H\alpha \le b. \end{aligned}$$

After a finite number steps, we construct $Y_n(t)$. Meanwhile, $Y_n(t)$ is bounded on $[t, t_0+\alpha]$, since $|Y_n(t) - X_{t_0}(\gamma)| \le b$.

From the construction, it is found that $Y_n(t)$ is Lipschtiz continuous on $[t_0, t_0+\alpha]$. Indeed,

$$\begin{aligned} |Y_n(t_1) - Y_n(t_2)| &\le \left| \int_{t_1}^{t_2} |f(s, Y_n(s-\frac{1}{n}))|\mathrm{d}s + \int_{t_1}^{t_2} |g(s, Y_n(s-\frac{1}{n}))|\mathrm{d}C_s(\gamma) \right| \\ &\le \left| \int_{t_1}^{t_2} |f(s, Y_n(s-\frac{1}{n}))|\mathrm{d}s + K(\gamma) \cdot \int_{t_1}^{t_2} |g(s, Y_n(s-\frac{1}{n}))|\mathrm{d}s \right| \\ &\le H|t_1 - t_2|, \quad \forall n \ge 1. \end{aligned} \tag{10}$$

Thus, $\{Y_n(t)\}_{n=1}^{\infty}$ is equicontinuous and bounded. According to Ascoli-Arzelà Theorem, there exists a subsequence of $\{Y_n(t)\}_{n=1}^{\infty}$ that converges uniformly on $[t_0, t_0+\alpha]$. For the sake of simplicity, we still denote this subsequence as $\{Y_n(t)\}_{n=1}^{\infty}$.

Denote $X_t(\gamma) = \lim_{n\to\infty} Y_n(t)$. $X_t(\gamma)$ is continuous on $[t_0, t_0+\alpha]$. From the definition of $Y_n(t)$ and inequality (10), we have

$$\begin{aligned} |Y_n(t-\frac{1}{n}) - X_t(\gamma)| &\le |Y_n(t-\frac{1}{n}) - Y_n(t)| + |Y_n(t) - X_t(\gamma)| \\ &\le \frac{H}{n} + |Y_n(t) - X_t(\gamma)|, \quad \forall n \ge 1, \ \forall t \in [t_0, t_0+\alpha]. \end{aligned}$$

Hence, $Y_n(t-\frac{1}{n})$ converges uniformly to $X_t(\gamma)$.

Since $f(t,x)$ and $g(t,x)$ are continuous, from the definition of $Y_n(t)$,

$$Y_n(t) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, Y_n(s-\frac{1}{n}))\mathrm{d}s + \int_{t_0}^{t} g(s, Y_n(s-\frac{1}{n}))\mathrm{d}C_s(\gamma), \tag{11}$$

we obtain

$$X_t(\gamma) = X_{t_0}(\gamma) + \int_{t_0}^{t} f(s, X_s(\gamma))\mathrm{d}s + \int_{t_0}^{t} g(s, X_s(\gamma))\mathrm{d}C_s(\gamma)$$

by taking limit on both sides of expression (11). The proof of Lemma 3 is completed. □

As in Section 3, we can extend the solution to $[0,+\infty)$ under the linear growth condition.

Lemma 4 *Fix $\gamma \in \Gamma$, under Condition 3 and Condition 2, the solution of integral equation (3) can be extended to* $[0,+\infty)$.

Proof: The proof of this lemma is similar to that of Lemma 2. □

Since γ is an arbitrary sample path, we have

Theorem 5 *Under Condition 3 and Condition 2, the uncertain integral equation*

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s$$

has at least one sample-continuous solution.

Or, equivalently, we have

Theorem 6 *Under Condition 3 and Condition 2, the UDE*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

has at least one sample-continuous solution.

5 Conclusion

This paper mainly gave an existence and uniqueness theorem for UDEs under the local Lipschitz condition. This theorem is more general than the existence and uniqueness theorem proved by Chen and Liu [1]. Then, an existence theorem without Lispchitz condition was given. Although only under continuous condition, the uniqueness of solution is not guaranteed, this theorem largely extends the set of solvable UDEs.

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