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# Generalized Derivative of Fuzzy Nonsmooth Functions

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**Abstract**

In this paper, we consider fuzzy nonsmooth functions which can be appeared in a large number of systems and problems. To the best of the authors' knowledge, there are not enough contributions about introducing suitable approximated derivatives for this class of fuzzy functions. For this purpose, we are going to define a generalized derivative (GD) for fuzzy nonsmooth functions. We first define a special functional optimization problem for crisp smooth functions which its optimal solution is the derivative of these functions. Then, we solve this problem for crisp nonsmooth functions and obtain the GD of these functions. Here, we apply the discretization method and introduce a linear programming problem for approximating the GD. In next step, we extend this definition of GD for interval-valued nonsmooth functions. Moreover, using $\alpha$ -levels of a fuzzy number, the GD of fuzzy nonsmooth functions is defined. In addition, we show that the result of our approach for fuzzy smooth functions is exactly similar to the generalized Hakuhara derivative defined by Bede. Finally, we obtain the GD of fuzzy nonsmooth functions in several illustrative examples.


**Keywords:** generalized derivative, smooth and nonsmooth functions, interval-valued function, fuzzy function, functional optimization

## 1 Introduction

Nonsmooth analysis had its origins in the early of 1970's when control theory experts and nonlinear programmers attempted to deal with necessary optimality conditions for problems with nonsmooth data or with nonsmooth functions. In the attempts to deal flexibly with such problems, various generalized derivative concepts were proposed to replace the nonexistent derivative. The main objective of these efforts was to define a generalized derivative for every point in the domain of functions belonging to a particular class. As a primary contribution to the canonical generalized derivatives one can point out to gradient introduced by Clarke in his pioneering work [12]. He applied this generalized gradient systematically to nonsmooth problems in a variety of problems [13, 14, 15]. Moreover, there are several definitions of generalized derivatives (or generalized differentiations) those are introduced by Mordukhovich [29, 30, 31, 32, 33, 34, 35] and Rockafellar and Wets [37, 38]. Indeed, the Gateaux derivative, Frechet derivative, and strict derivative as well as the Clarke generalized gradients are discussed by Clarke [16]. In addition, Warga's derivative container was introduced in [44, 45]. The notions of prederivatives were introduced and extensively studied by Ioffe [21, 22, 23, 24, 25], whereas H-differentials were given by Gowda and Ravendran [19]. Various definitions of subdifferentials can be found in books dealing with nonsmooth analysis as well as convex analysis [1, 8, 20, 47]. A survey of subdifferential calculus can also be found in [9]. See also [28] for Michel and Penot's subdifferentials and [42, 43] for Treiman's linear generalized gradients. A treatment of quasidifferentials can be found in [17].

Generalized derivative has also played an increasingly important role in several areas of application, notably in optimization, calculus of variations, differential equations, mechanics, and control theory. In the recent years, there are several works which uses the generalized derivative (see [9, 15, 17, 27, 33, 39, 40, 41, 46]).

It is important to know that one of the main concepts of nonsmooth analysis is fuzzy nonsmooth function which can appear in many problems in engineering, economic, physics phenomenon, mathematics, control theory, dynamical systems and other fields. Note that for solving these problems, we usually need to obtain an approximated derivative for fuzzy nonsmooth functions although there have not been enough contributions or any serious efforts in this domain up to now. However, there are several definitions for derivative of fuzzy smooth functions, first

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introduced by Chang and Zadeh [11] then followed by Dubois and Prade [18]. In 1983, Puri and Ralescu [36] defined the H-derivative of fuzzy functions which was discussed by Kaleva [26] in 1987. The H-derivative is the starting point in Hakuhara derivative of fuzzy functions. In addition, some of the shortcomings of H-derivative were solved by the concept of GH-derivative (strongly generalized derivative) and gH derivative (generalized Hakuhara derivative) which is discussed by Bede [4, 5, 6, 7]. Moreover, the $\pi$ - derivative of fuzzy functions is defined by Cano [10] which is equivalent to generalized derivative of this function.

But in above-mentioned works and definitions, there is not a definition for the derivative of fuzzy nonsmooth function. Hence, we are going to define a generalized derivative for fuzzy nonsmooth function. For this purpose, we introduce a special functional optimization problem, then having solved it, we derive the generalized derivative.

In the very first step, we state several preliminaries which we need in the next sections. In Section 3, we put forward the concept of the GD of crisp nonsmooth functions and discuss the problems of characterizations and existence of the GD. In Section 4, we use the concept of $\alpha$ -level sets to define the GD of fuzzy nonsmooth functions and discuss the relation between derivative of interval-valued and fuzzy nonsmooth functions. In Section 5, we design several illustrative numerical examples and in Section 6, provide the conclusions of this paper.

# 2 Preliminaries

Let $I$ be a closed interval on real line. A general fuzzy set $u$ over $I$ is usually denoted by its membership function $\mu_u : I \to [0,1]$ and is uniquely characterized by the pairs $(x, \mu_u(x))$ for all $x \in I$ . By $F_I$ , we mean the collection of the fuzzy sets over $I$ . There are various definitions for the fuzzy numbers. Consider the following definition of fuzzy number:

**Definition 2.1** A fuzzy number is a fuzzy set $u \in F_I$ satisfying the following properties:

I) $u$ is normal, i.e. there is one $x_0 \in I$ with $\mu_u(x_0) = 1$ .

II) $u$ is a convex fuzzy set , i.e., $\mu_u(\lambda x + (1-\lambda) y) \ge \min\{\mu_u(x), \mu_u(y)\}$ for all $x, y \in I$ , and $\lambda \in [0,1]$ .

III) $u$ is an upper semi-continuous on $I$, i.e., for all $x_0 \in I$ and $\varepsilon > 0$ there is an $\delta > 0$ such that $\mu_u(x) \le \mu_u(x_0) + \varepsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$ .

IV) Closure of set $\{x \in I : \mu_u(x) > 0\}$ is compact.

**Definition 2.2** Let $u \in F_I$ . The $\alpha$ -level set of $u$ is a crisp nonempty compact convex subset of $I$ which we show by $[u]^\alpha$ for $\alpha \in (0,1]$ and define as set $\{x \in I : \mu_u(x) \ge \alpha\}$ .

It is obvious that the $\alpha$ -level set of $u$ is a closed and bounded interval which we show by $[u]^\alpha = [u_-(\alpha), u_+(\alpha)], \alpha \in [0,1]$ . Thus, a fuzzy number $u$ is determined by the initial and end points of interval $[u]^\alpha$ .

**Theorem 2.3** Let $u \in F_I$ . Then $[u]^\alpha, \alpha \in [0,1]$ are compact and convex subsets of $I$ and satisfied in the following conditions:

$(i)$ $\alpha_1 \le \alpha_2 \Rightarrow [u]^{\alpha_1} \supseteq [u]^{\alpha_2}$ ,

$(ii)$ $\left(\alpha_n \to \alpha \text{ and } \alpha_1 \le \alpha_2 \le ...\right) \Rightarrow [u]^\alpha = \bigcap_{n=1}^{\infty} [u]^{\alpha_n}$ .

Conversely, if the family $\{A_\alpha\}_\alpha$ , $\alpha \in [0,1]$ is the family of compact and convex subsets of $I$ such that are satisfied in the conditions *(i)* and *(ii)*, then the fuzzy set $u \in F_I$ defined by membership function

$$\mu_u(x) = \sup\{\alpha : \alpha \in [0,1] \text{ and } x \in A_\alpha\} , \quad x \in I$$

will be a fuzzy number such that $[u]^\alpha = A_\alpha , \alpha \in [0,1]$ .

Now suppose that $\mu_u(.)$ , $\mu_v(.)$ and $[u]^\alpha, [v]^\alpha$ , $\alpha \in [0,1]$ are membership functions and $\alpha$ -level sets of $u, v \in F_I$ , respectively. The fuzzy addition $u + v \in F_I$ for all $u, v \in F_I$ and the fuzzy scalar multiplication $k.u \in F_I$ for $k \in \mathbb{R} - \{0\}$ have membership functions (according to Zadeh's expansion principle) as follows:

$$\mu_{u+v}(z) = \sup_{z=x+y} \min\{\mu_u(x), \mu_v(y)\} \ , \quad \mu_{k.u}(x) = \mu_u(\tfrac{x}{k}) ,$$

also, $\alpha$ -levels are as follows:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha = \{x + y : x \in [u]^\alpha , y \in [v]^\alpha\}$$

$$[ku]^\alpha = k[u]^\alpha = \{kx : x \in [u]^\alpha\} .$$

The fuzzy subtraction $u-v\in F_I$ is defined as the fuzzy addition $u+(-v)$ where $-v=(-1).v$. Finally, the Hausdorff distance on $F_I$ is defined by

$$D(u,v)=\sup_{\alpha\in[0,1]}\left\{\max\{|u_-(\alpha)-v_-(\alpha)|,|u_+(\alpha)-v_+(\alpha)|\}\right\}$$

and $(F_I,D)$ is a complete metric space.

Now, before using the above-mentioned preliminaries we need to define the GD of crisp nonsmooth function.

# 3 The GD of Crisp Nonsmooth Functions

Before defining the fuzzy nonsmooth derivatives, first, we are going to introduce a functional optimization problem the solution of which is the derivative of smooth crisp function on an interval. For solving this problem, we introduce a linear programming problem. First of all, we state the following Lemma.

**Lemma 3.1** Let $h:(0,1)\to\mathbb{R}$ be a function such that $\lim_{x\to c}h(x)=L$ where $L\in\mathbb{R}$ and $c\in(0,1)$. Then for all $K\in\mathbb{R}$ there exists $\rho_c>0$ such that $|h(x)-L|\le|h(x)-K|$ for all $x\in(c-\rho_c,c+\rho_c)\setminus\{c\}$.

**Proof:** The proof, by attention to concept of limit, is trivial. □

We assume that $PC(0,1)$, $C(0,1)$ and $C^1(0,1)$ are space of piecewise continuous, continuous and continuous differentiable functions on $(0,1)$, respectively.

**Proposition 3.2** Let $f(.)\in C^1(0,1)$, $g(.)\in PC(0,1)$ and $m\in\mathbb{N}$. Then there exists $\delta>0$ such that for all $s_i\in((i-1)/m,i/m)$, $i=1,2,...,m.$ We have

$$\int_{s_i-\delta}^{s_i+\delta}|f(x)-f(s_i)-(x-s_i)f'(s_i)|dx\le\int_{s_i-\delta}^{s_i+\delta}|f(x)-f(s_i)-(x-s_i)g(s_i)|dx. \tag{1}$$

**Proof:** Let $s_i\in((i-1)/m,i/m)$, $i=1,2,...,m$. Since

$$f'(s_i)=\lim_{x\to s_i}\frac{f(x)-f(s_i)}{x-s_i},$$

by Lemma 3.1, there is $\rho_{s_i}>0$ such that for all $x\in(s_i-\rho_{s_i},s_i+\rho_{s_i})\setminus\{s_i\}$, we have

$$\left|\frac{f(x)-f(s_i)}{x-s_i}-f'(s_i)\right|\le\left|\frac{f(x)-f(s_i)}{x-s_i}-g(s_i)\right|,$$

$$|f(x)-f(s_i)-(x-s_i)f'(s_i)|\le|f(x)-f(s_i)-(x-s_i)g(s_i)|. \tag{2}$$

Suppose that $\delta=\min\left\{\rho_{s_i}:1,2,...,m\right\}$. Thus $(s_i,s_i+\delta)\subseteq(s_i-\rho_{s_i},s_i+\rho_{s_i})\setminus\{s_i\}$ for $i=1,2,...,m$ and by (2) we have

$$\int_{s_i}^{s_i+\delta}|f(x)-f(s_i)-(x-s_i)f'(s_i)|dx\le\int_{s_i}^{s_i+\delta}|f(x)-f(s_i)-(x-s_i)g(s_i)|dx. \tag{3}$$

In addition, $(s_i-\delta,s_i)\subseteq(s_i-\rho_{s_i},s_i+\rho_{s_i})\setminus\{s_i\}$ and by (2)

$$\int_{s_i-\delta}^{s_i}|f(x)-f(s_i)-(x-s_i)f'(s_i)|dx\le\int_{s_i-\delta}^{s_i}|f(x)-f(s_i)-(x-s_i)g(s_i)|dx \tag{4}$$

Hence, using (3) and (4)

$$\int_{s_i-\delta}^{s_i+\delta}|f(x)-f(s_i)-(x-s_i)f'(s_i)|dx\le\int_{s_i-\delta}^{s_i+\delta}|f(x)-f(s_i)-(x-s_i)g(s_i)|dx. \ \square$$

Let $f\in C^1(0,1)$ and $m\in\mathbb{N}$ be a given large number. Also, assume that $s_i\in((i-1)/m,i/m)$, for all $i=1,2,...,m$ be arbitrary numbers. Define the functional optimization problem:

$$\begin{aligned}&\textit{Minimize}\quad L(g(.))=\sum_{i=1}^{m}\int_{s_i-\delta}^{s_i+\delta}|f(x)-f(s_i)-(x-s_i)g(s_i)|dx\\&\textit{subject to}\quad g(.)\in PC(0,1)\end{aligned} \tag{5}$$

where $\delta > 0$ is a given sufficiently small number.

**Theorem 3.3** Let $f \in C^1(0,1)$. Then there is a sufficiently small number $\delta > 0$ such that the function $f'(.)$ on interval $(0,1)$ is an optimal solution of the functional optimization problem (3).

**Proof:** Let $g(.)$ be an arbitrary function in $PC(0,1)$. By Theorem 3.2 and relation (1) there is $\delta > 0$ such that

$$L(f'(.)) \le L(g(.)). \tag{6}$$

The left side of inequality (6) is a lower bound for all values of $L(g(.))$. Thus

$$L(f'(.)) \le \underset{g(.)\in PC(0,1)}{Minimize}\, L(g(.)).$$

On the other hand $f'(.) \in C(0,1)$ and $C(0,1) \subset PC(0,1)$. Thus $f'(.) \in PC(0,1)$ and optimal solution of the functional optimization problem (5) is $f'(.)$. So $L(f'(.)) = \underset{g(.)\in PC(0,1)}{Minimize}\, L(g(.))$.□

**Definition 3.4** Let $f(.) \in PC(0,1)$ and $m \in \mathbb{N}$ be a given large number. Moreover, suppose that $g^*(.)$ be the optimal solution of the functional optimization problem (5), the GD of $f(.)$ on $(0,1)$ denoted by $\partial f(.)$ is defined as $\partial f(.) = g^*(.)$ on $(0,1)$.

We solve the functional optimization problem (5) by discretization method. For this goal, let $\delta > 0$ be a given small number and select arbitrary points $s_i \in ((i-1)/m, i/m)$, for $i = 1,2,...,m$. Suppose that

$$\varphi_i(x) = \left| f(x) - f(s_i) - (x - s_i)\, g(s_i) \right|, \quad x \in [s_i - \delta, s_i + \delta]$$

$$x_{i1} = s_i - \delta, \quad x_{i2} = s_i + \delta, \quad \varphi_{ij} = \varphi_i(x_{ij}), \quad f_i = f(s_i),$$

$$f_{ij} = f(x_{ij}), \quad g_i = g(s_i), \quad i = 1,2,...,m, \quad j = 1,2.$$

By trapezoidal approximation method and techniques of linear and nonlinear programming [2, 3], the functional optimization problem (5) is approximated with the following linear programming problem where decision variables are $g_i$ and $\varphi_{ij}$ for $i = 1,2,...,m$ and $j = 1,2$:

$$\begin{array}{ll} Minimize & \delta \sum_{i=1}^{m} \left( \varphi_{i1} + \varphi_{i2} \right) \\ subject\ to & -\varphi_{ij} + (x_{ij} - s_i)\, g_i \le f_{ij} - f_i \\ & -\varphi_{ij} - (x_{ij} - s_i)\, g_i \le -f_{ij} + f_i \\ & \varphi_{ij} \ge 0, \quad i = 1,...,m, \ j = 1,2. \end{array} \tag{7}$$

By solving problem (7), we obtain optimal solutions $g_i^*$ and $\varphi_{ij}^*$ for all $i = 1,2,...,m$ and $j = 1,2$. Note that, we have $\partial f(s_i) = g_i^*$ for all $i = 1,2,...,m$.

# 4 The GD of Interval-Valued and Fuzzy Nonsmooth Functions

In this section, we are going to extend the definition of GD of crisp nonsmooth functions to the fuzzy nonsmooth functions. We first define the GD of interval-valued nonsmooth functions. Then, we propose the GD of fuzzy nonsmooth functions which is based on $\alpha$ -level sets of fuzzy numbers.

Let $F(.)$ be an interval-valued function on $[0,1]$. By $F(x) = [F_1(x), F_2(x)]$ for each $x \in [0,1]$, we mean $F(.)$.

**Definition 4.1** Interval-valued function $F(.) = [F_1(.), F_2(.)]$ is a smooth function when the both functions $F_1(.)$ and $F_2(.)$ are smooth functions; otherwise it is called a nonsmooth function.

**Definition 4.2** The GD of interval-valued nonsmooth function $F(.)$ when $F(x) = [F_1(x), F_2(x)]$, $x \in [0,1]$ is denoted by $\partial F(x)$ and defined as follows:

$$\partial F(x) = \begin{cases} [\partial F_1(x), \partial F_2(x)] & \varphi(x) \ge 0 \\ [\partial F_2(x), \partial F_1(x)] & \varphi(x) < 0 \end{cases} \tag{8}$$

where $\varphi(x) = \partial F_2(x) - \partial F_1(x)$ for any $x \in [0,1]$.

Note that each $\alpha$ -level set of a fuzzy function is an interval-valued function.

**Definition 4.3** The function $f:[0,1]\to F_I$ is called fuzzy function and its $\alpha$ -level sets are denoted by $[f(x)]^{\alpha}=[f_-(\alpha,x), f_+(\alpha,x)]$ for $\alpha\in[0,1]$ and $x\in I$ .

**Remark 4.4** Note that for each $f:[0,1]\to F_I$ and fixed $\alpha\in[0,1]$ , the function $P(.)$ defined by $P(x)=[f_-(\alpha,x), f_+(\alpha,x)]$, $x\in[0,1]$ is a interval-valued function. Also $f_-(\alpha,.)$ and $f_+(\alpha,.)$ are crisp functions on interval $I$ .

**Definition 4.5** Function $f:[0,1]\to F_I$ which is defined by $[f(x)]^{\alpha}=[f_-(\alpha,x), f_+(\alpha,x)]$ , $\alpha\in[0,1]$ for $x\in I$ is a fuzzy smooth function when for each $\alpha\in[0,1]$ functions $f_-(\alpha,.)$ and $f_+(\alpha,.)$ are smooth; otherwise it is called fuzzy nonsmooth function.

**Definition 4.6** The GD of fuzzy nonsmooth function $f:[0,1]\to F_I$ is denoted by $\partial f(.)$ on $[0,1]$ and defined as follows:

$$[\partial f(x)]^{\alpha}=\begin{cases}[\partial f_-(\alpha,x),\partial f_+(\alpha,x)] & \varphi_\alpha(x)\ge 0\\ [\partial f_+(\alpha,x),\partial f_-(\alpha,x)] & \varphi_\alpha(x)<0\end{cases} \tag{9}$$

where $\varphi_\alpha(x)=\partial f_+(\alpha,x)-\partial f_-(\alpha,x)$ for all $x\in[0,1]$.

**Theorem 4.7** For every fixed number $x$ in $[0,1]$, if the family $[\partial f(x)]^{\alpha}$, $\alpha\in[0,1]$ is a convex and compact subset of $I$ and is satisfied in the conditions *(i)* and *(ii)* of Theorem 2.3, then $\partial f_\alpha(x)$ will be a fuzzy number and there exists $\partial f(x)$ .

**Proof:** The proof can immediately be got from Theorem 2.3. □

**Theorem 4.8** Let function $f:(0,1)\to F_I$ be defined by $[f(x)]^{\alpha}=[f_-(\alpha,x), f_+(\alpha,x)]$ , $\alpha\in(0,1]$ for $x\in I$ be a smooth fuzzy function. Then $\partial f(.)=f'_{gH}(.)$ where $f'_{gH}(.)$ is the generalized Hakuhara derivative defined by Bede [7].

**Proof:** By attention to (9), if $\varphi_\alpha(x)\ge 0$ then $[\partial f(x)]^{\alpha}=[\partial f_-(\alpha,x),\partial f_+(\alpha,x)]$ . Moreover, if $\varphi_\alpha(x)<0$, then $[\partial f(x)]^{\alpha}=[\partial f_+(\alpha,x),\partial f_-(\alpha,x)]$ . On the other hand, by Theorem 3.3 and Definition 3.4, we have

$$\partial f_-(\alpha,x)=\frac{d}{dx}f_-(\alpha,x) \text{ and } \partial f_+(\alpha,x)=\frac{d}{dx}f_+(\alpha,x).$$

Hence

$$\begin{aligned}[\partial f(x)]^{\alpha}&=\left[\min\left\{\partial f_-(\alpha,x),\partial_+ f(\alpha,x)\right\},\max\left\{\partial f_-(\alpha,x),\partial f_+(\alpha,x)\right\}\right]\\ &=\left[\min\{\frac{d}{dx}f_-(\alpha,x),\frac{d}{dx}f_-(\alpha,x)\},\max\{\frac{d}{dx}f_-(\alpha,x),\frac{d}{dx}f_+(\alpha,x)\}\right]\\ &=[f'_{gH}(x)]^{\alpha}.\end{aligned}$$

Thus we have $\partial f(.)=f'_{gH}(.)$ .□

In Section 5, we obtain the GD of some fuzzy smooth and fuzzy nonsmooth function $F(.)$ .

# 5 Simulation Results

In this section, we conduct several numerical simulations to illustrate the efficiency of our approach for obtaining the GD of fuzzy smooth and nonsmooth functions. It is important to emphasize that our approach for fuzzy smooth functions, can be compared to the other approaches (which in here is compared to the gH-derivative by Bede [7]), But, when we deal with the fuzzy nonsmooth functions, since there is not any approach, we cannot compare it. In fact, the H-derivative, Hakuhara derivative, GH-derivative, gH-derivative and $\pi-$derivative do not exist for fuzzy nonsmooth functions, even though by our approach we can obtain an approximate derivative for this class of fuzzy functions.

Here, the LP problem (7) for obtaining the GD is solved by the Simplex method in MATLAB software.

**Example 5.1:** Consider the fuzzy smooth function $f:(0,1)\to F_I$ defined by

$$[f(x)]^{\alpha}=\left[xe^{-x}+\alpha^2(e^{-x^2}+x-e^{-x}),\, e^{-x^2}+x+(1-\alpha^2)(e^{x}-x+e^{-x^2})\right]$$

and discussed by Bede [7]. The function $f(.)$ is illustrated in Figure 1. Using LP problem (7) and Definition 4.6, we obtain $[\partial f(x)]^{\alpha}$, $x\in[0,1]$ which has been shown in Figure 2. By Theorem 4.7 and Figure 2, $\partial f(x)$ for

$x \in (0, 610) \cup (0.710, 1)$ exist and for $x \in [0.610, 0.710]$ do not exist. Note that the result of our approach is exactly similar to the obtained gH-derivative of $f(.)$ by Bede [7].

**Example 5.2:** Consider the fuzzy nonsmooth function $f:(0,1) \to F_I$ defined by

$$[f(x)]^\alpha = \Big[(\alpha - 1)\big|\,|2x-1| - 0.5\big|\,,\, 0.5(1-\alpha)\big|\sin(3\pi x)\big|\Big], \qquad x \in (0,1).$$

The functions $f(.)$ is illustrated in Figure 3. By LP problem (7) and Definition 4.6, we obtain $[\partial f(x)]^\alpha$ for $x \in (0,1)$ which has been shown in Figure 4. Here $\partial f(x)$ for all $x \in (x_1, x_2) \cup (x_3, x_4) \cup (x_5, x_6)$ exist and for other points of interval $(0,1)$ do not exist where $x_1 = 0.167, x_2 = 0.250, x_3 = 0.333, x_4 = 0.666, x_5 = 0.750$ and $x_6 = 0.834$.

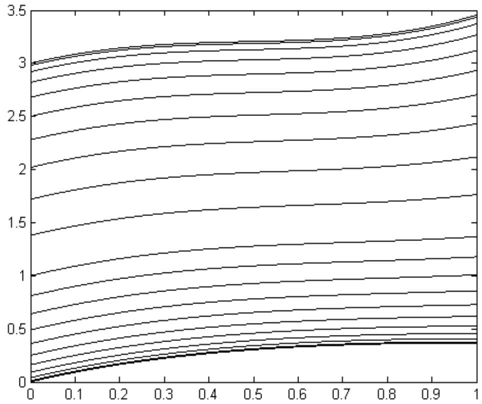


Figure 1: Fuzzy smooth function $f(.)$ for Ex.5.1

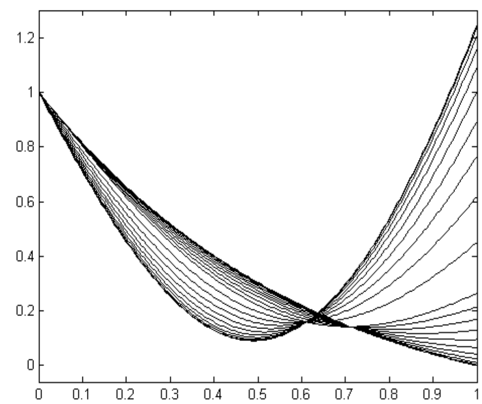


Figure 2: Function $\partial f(.)$ for Ex.5.1

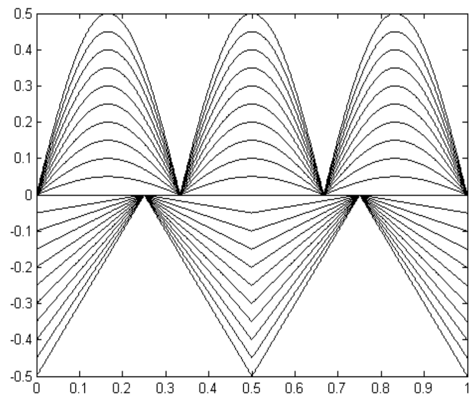


Figure 3: Fuzzy nonsmooth function $f(.)$ for Ex.5.2

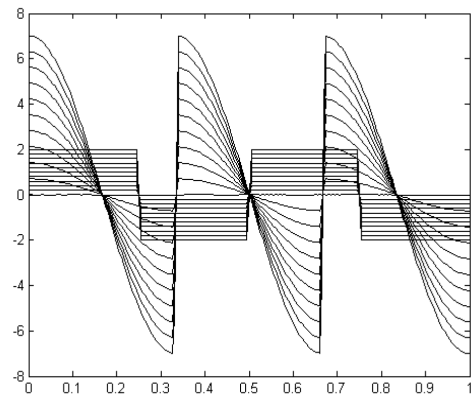


Figure 4: Function $\partial f(.)$ for Ex.5.2

**Example 5.3:** Consider the fuzzy nonsmooth function $f:(0,1) \to F_I$ defined by

$$[f(x)]^\alpha = \Big[\tfrac{1}{4}(\alpha - 1)\big|\cos(2\pi x)\big|\,,\,(1-\alpha)\big|\,x - \tfrac{1}{2}\big|\Big], \qquad x \in (0,1).$$

The functions $f(.)$ is illustrated in Figure 5. According to the last example, we obtain $[\partial f(x)]^\alpha$, $x \in [0,1]$ which has been shown in Figure 6. But $\partial f(x)$ for $x \in (0, 0.25) \cup (0.75, 1)$ exist, and for $x \in [0.25, 0.75]$ do not exist.

**Example 5.4:** Consider the fuzzy nonsmooth function $f:(0,1) \to F_I$ defined by

$$[f(x)]^\alpha = [-0.3\alpha\, e^{4|x-0.5|}\,,\,(1-\alpha)\, e^{4|(x-0.25)(x-0.75)|}], \qquad x \in (0,1).$$

The function $f(.)$ is illustrated in Figure 7. We obtain $[\partial f(x)]^\alpha$, $x \in [0,1]$ which has been shown in Figure 8. Here, $\partial f(x)$ for $x \in [0, 0.25] \cup [0.75, 1]$ exist, and for $x \in (0.25, 0.75)$ do not exist.

**Example 5.5:** Consider the fuzzy nonsmooth function $f:(0,1)\to F_I$ defined by

$$[f(x)]^{\alpha}=\left[0.6(\alpha-1)Arctan(|x-0.8|),0.6(1-\alpha)Arc\cot(1+|x-0.2|)\right],\qquad x\in(0,1).$$

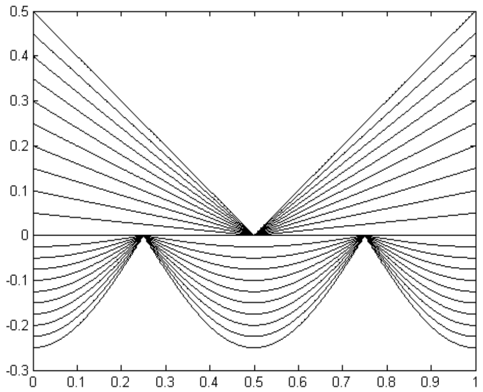


Figure 5: Fuzzy nonsmooth function $f(.)$ for Ex.5.3

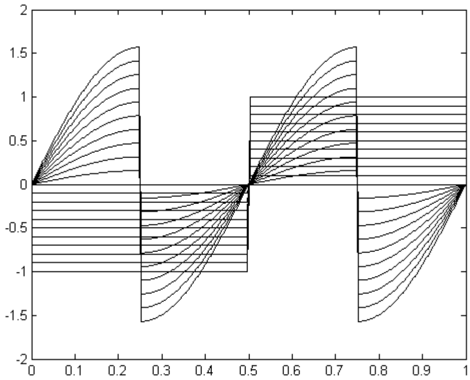


Figure 6: Function $\partial f(.)$ for Ex.5.3

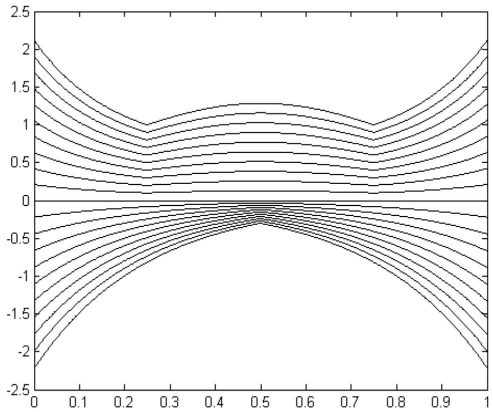


Figure 7: Fuzzy nonsmooth function $f(.)$ for Ex.5.4

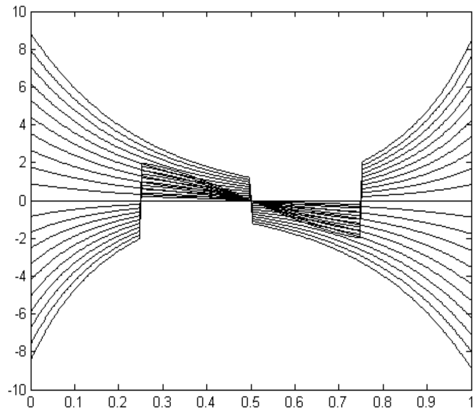


Figure 8: Function $\partial f(.)$ for Ex.5.4

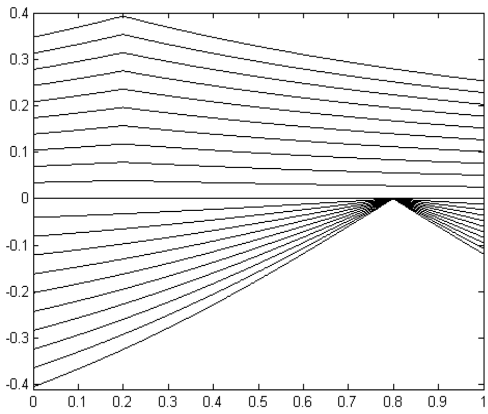


Figure 9: Fuzzy nonsmooth function $f(.)$ for Ex.5.5

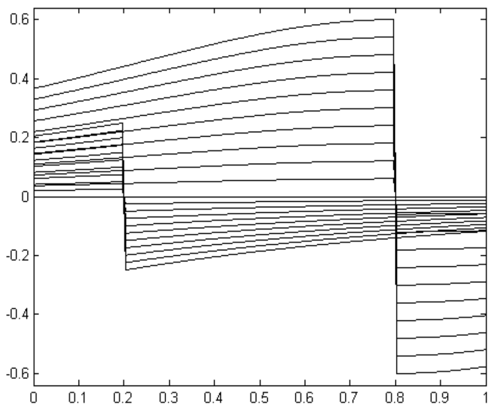


Figure 10: Function $\partial f(.)$ for Ex.5.5

The function $f(.)$ is shown in Figure 9. We obtain $[\partial f(x)]^{\alpha}$, $x\in(0,1)$ which has been shown in Figure 10. Here, $\partial f(x)$ for $x\in(0.2,0.8)$ exist, and for $x\in(0,0.2]\cup[0.8,1)$ do not exist.

# 6 Conclusion

In this paper, we firstly proposed the GD of crisp nonsmooth functions which is as an optimal solution of a special functional optimization. We show that this GD for crisp smooth functions is the usual derivative. In the next step, we extended this GD to the fuzzy nonsmooth functions by the concepts of interval-valued functions and $\alpha$ - cuts of fuzzy numbers. Here, this GD for fuzzy smooth functions is equivalent to gH-derivative of Bede [7].

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