Journal of Uncertain Systems
Vol.6, No.3, pp.186-199, 2012
Online at: www.jus.org.uk

Statistical Hypotheses Testing in the Fuzzy Environment

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Received 9 April 2011; Revised 22 February 2012

Abstract

The mean and the variance value of a random variable play an important role in statistical analysis. Sometimes the available observations are not precise, and we want to test the hypotheses of mean and variance in such environment. Hence, in this paper, we first extend and introduced L_2-metric based on imprecise (fuzzy) observations, and then, the concepts of fuzzy test statistics are defined based on the extend L_2-metric for testing the fuzzy hypotheses of mean and variance. Finally, we propose a method to evaluate the fuzzy hypotheses (for one-sample and two-sample) of interest.

Keywords: fuzzy canonical number, L_2-metric, fuzzy hypothesis, fuzzy data, testing hypotheses

1 Introduction

Statistical analysis, in traditional form, is based on crispness of data, random variable, point estimation, hypotheses, parameter and so on. As there are many different situations in which the above mentioned concepts are imprecise. On the other hand, the theory of fuzzy sets is a well known tool for formulation and analysis of imprecise and subjective concepts. Therefore the hypotheses testing for mean and variance with fuzzy data can be important. The problem of statistical inference in fuzzy environment is developed in different approaches.

Watanabe and Imaizumi [29] presented an approach for testing fuzzy hypotheses, in which they introduced fuzzy critical regions and produced a fuzzy conclusion. Arnold [6, 9] presented an approach to test fuzzy hypotheses, in which he considered fuzzy constraints on the type I and II errors. Holena [13] considered a fuzzy generalization of a sophisticated approach to exploratory data analysis, the general unary hypotheses automation. Holena [14] presented a principally different approach and motivated the observational logic and its success in automated knowledge discovery. Filzmoser and Viertl [11] investigated an approach for testing statistical hypotheses based on the fuzzy $p-$value. Taheri and Behboodian [24] and Torabi et. al. [27] studied a method on Neyman-Pearson Lemma for testing fuzzy hypotheses when the available data are crisp and vague, respectively. Some methods of statistical inference proposed by Buckley [7, 8] and Viertl [28] in a fuzzy environment. Thompson and Geyer [26] proposed the Fuzzy p-values in latent variable problems. Taheri and Arefi [25] studied an approach for testing fuzzy hypotheses based on fuzzy test statistic, (see also, Arefi and Taheri [5], when the available/observed data are fuzzy). Akbari and Rezaei [3] described a bootstrap method for variance that is designed directly for testing hypothesis in case of fuzzy data based on Yao-Wu signed distance. Parchami et al. [21] considered the problem of testing hypotheses, when the hypotheses are fuzzy and the data are crisp. They first introduce the notion of fuzzy p-value, by applying the extension principle and then present an approach for testing fuzzy hypotheses by comparing a fuzzy p-value and a fuzzy significance level, based on a comparison between two fuzzy sets.

The bootstrap using fuzzy data, is developed in different approaches.

Korner's asymptotic development [15] concerns general fuzzy random variables (taking on way-either finite or infinite-number of values in the space of compact convex fuzzy sets of a finite-dimensional Euclidean space). Montenegro et al. [18] have presented asymptotic one-sample procedure. Gonzalez et al. [12] have shown that the one-sample method of testing the mean of a fuzzy random variable can be extended to general ones (more precisely, to those whose range is not necessarily finite and whose values are fuzzy subsets of finite-dimensional

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Euclidean space). Akbari and Rezaei [1] describe a bootstrap method for variance that is designed directly for hypothesis testing in the case of fuzzy data based on Yao-Wu signed distance. Akbari and Rezaei [4] exhibit a method in order to bootstrap testing fuzzy hypotheses and observations on fuzzy statistics.

In this paper we construct a new method for testing hypotheses in fuzzy environment which is completely different from those mentioned above. For this purpose we organize the paper in the following way. In Section 2 we describe some basic concepts of canonical fuzzy numbers, L_2-metric, and fuzzy hypotheses. In Section 3, we come up with testing hypotheses for one-sample based on L_2-metric. Section 4, provides a testing hypothesis for two-sample based on L_2-metric. In Section 5, we compare our method with some other works. At last, a brief conclusion is provided in Section 6.

2 Preliminaries

In this section, we study canonical fuzzy numbers, L_2-metric,and fuzzy hypotheses.

2.1 Canonical Numbers

Let X be the universal space, then a fuzzy subset $\widetilde{x}$ of X is defined by its membership function $\mu_{\widetilde{x}} : X \to [0,1]$. We denote by $\widetilde{x}_\alpha = \{x : \mu_{\widetilde{x}}(x) \geq \alpha\}$ the $\alpha-$cut set of $\widetilde{x}$ and $\widetilde{x}_0$ is the closure of the set $\{x : \mu_{\widetilde{x}}(x) > 0\}$, and
(1) $\widetilde{x}$ is called normal fuzzy set if there exist $x \in X$ such that $\mu_{\widetilde{x}}(x) = 1$;
(2) $\widetilde{x}$ is called convex fuzzy set if $\mu_{\widetilde{x}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\widetilde{x}}(x)\ ,\ \mu_{\widetilde{x}}(y))$ for all $\lambda \in [0,1]$;
(3) the fuzzy set $\widetilde{x}$ is called a fuzzy number if $\widetilde{x}$ is normal convex fuzzy set and its $\alpha-$cut sets, is bounded $\forall \alpha \neq 0$;
(4) $\widetilde{x}$ is called a closed fuzzy number if $\widetilde{x}$ is fuzzy number and its membership function $\mu_{\widetilde{x}}$ is upper semicontinues;
(5) $\widetilde{x}$ is called a bounded fuzzy number if $\widetilde{x}$ is a fuzzy number and its membership function $\mu_{\widetilde{x}}$ has compact support.

If $\widetilde{x}$ is a closed and bounded fuzzy number with $x_\alpha^L = \inf\{x : x \in \widetilde{x}_\alpha\}$ and $x_\alpha^U = \sup\{x : x \in \widetilde{x}_\alpha\}$ and its membership function be strictly increasing on the interval $[x_\alpha^L, x_1^L]$ and strictly decreasing on the interval $[x_1^U, x_\alpha^U]$, then $\widetilde{x}$ is called canonical fuzzy number.

Let " $\odot$ " be a binary operation $\oplus$ or $\ominus$ between two canonical fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$. The membership function of $\widetilde{a} \odot \widetilde{b}$ is defined by

$$\mu_{\widetilde{a}\odot\widetilde{b}}(z) = \sup_{x \circ y = z} \min\{\mu_{\widetilde{a}}(x), \mu_{\widetilde{b}}(y)\}$$

for $\odot = \oplus$ or $\ominus$ and $\circ = +$ or $-$.

In the following, let $\odot_{int}$ be a binary operation $\oplus_{int}$ or $\ominus_{int}$ between two closed intervals $\widetilde{a}_\alpha = [a_\alpha^L, a_\alpha^U]$ and $\widetilde{b}_\alpha = [b_\alpha^L, b_\alpha^U]$. Then $\widetilde{a}_\alpha \odot_{int} \widetilde{b}_\alpha$ is defined by

$$\widetilde{a}_\alpha \odot_{int} \widetilde{b}_\alpha = \{z \in \mathcal{R} : z = x \circ y,\ x \in \widetilde{a}_\alpha,\ y \in \widetilde{b}_\alpha\}.$$

If $\widetilde{a}$ and $\widetilde{b}$ be two closed fuzzy numbers. Then $\widetilde{a} \oplus \widetilde{b}$ and $\widetilde{a} \ominus \widetilde{b}$ are also closed fuzzy numbers. Furthermore, we have

$$\begin{aligned}
(\widetilde{a} \oplus \widetilde{b})_\alpha &= \widetilde{a}_\alpha \oplus_{int} \widetilde{b}_\alpha = [a_\alpha^L + b_\alpha^L, a_\alpha^U + b_\alpha^U] \\
(\widetilde{a} \ominus \widetilde{b})_\alpha &= \widetilde{a}_\alpha \ominus_{int} \widetilde{b}_\alpha = [a_\alpha^L - b_\alpha^U, a_\alpha^U - b_\alpha^L].
\end{aligned}$$

2.2 L_2-Metric

Now we define a distance between fuzzy numbers which will be used later.

Several ranking methods have been proposed so far, by Cheng [10], Yao and Wu [31] Modarres and Sadi-Nezhad [17], Nojavan and Ghazanfari [20], Puri and Ralescu [22], and Akbari and Rezaei [2].

In this paper we use another metric for canonical fuzzy numbers that is nominated L_2-metric.

Given a real number $x \in \mathcal{R}$, we can induce a fuzzy number $\widetilde{x}$ with membership function $\mu_{\widetilde{x}}(r)$ such that $\mu_{\widetilde{x}}(x) = 1$ and $\mu_{\widetilde{x}}(r) < 1$ for $r \neq x$. We call $\widetilde{x}$ as a fuzzy real number induced by the real number x.

Let $\mathcal{F}(\mathcal{R})$ be the set of all fuzzy real numbers induced by the real numbers $\mathcal{R}$. We define the relation $\sim$ on $\mathcal{F}(\mathcal{R})$ as $\widetilde{x}_1 \sim \widetilde{x}_2$ iff $\widetilde{x}_1$ and $\widetilde{x}_2$ are induced by the same real number x. Then $\sim$ is an equivalence relation, which induces the equivalence classes $[\widetilde{x}] = \{\widetilde{a} : \widetilde{a} \sim \widetilde{x}\}$. The quotient set $\mathcal{F}(\mathcal{R})/\sim$ is the set of all equivalence classes. We call $\mathcal{F}(\mathcal{R})/\sim$ as the fuzzy real number system. In practice, we take only one element $\widetilde{x}$ from each equivalence class $[\widetilde{x}]$ to from the fuzzy real number system $(\mathcal{F}(\mathcal{R})/\sim)$ that is,

$$(\mathcal{F}(\mathcal{R})/\sim) = \{\widetilde{x} : \widetilde{x} \in [\widetilde{x}], \widetilde{x} \ \ is\ the\ only\ element\ from\ [\widetilde{x}]\}.$$

If the fuzzy real number system $(\mathcal{F}(\mathcal{R})/\sim)$ consists all of canonical fuzzy real numbers then we call $(\mathcal{F}(\mathcal{R})/\sim)$ as the canonical fuzzy real number system.

For each $\alpha-$cuts of $\widetilde{a} \in \mathcal{F}(\mathcal{R}^n)$ the support function $S_{\widetilde{a}_\alpha}$ is defined as $S_{\widetilde{a}_\alpha}(t) = \sup_{x\in\widetilde{a}_\alpha} \ll x, t \gg$, $t \in S^{n-1}$, S^{n-1} the $(n-1)-$dimensional unit sphere in $\mathcal{R}^n$. Using support function we define L_2-metric

$$\delta_2(\widetilde{a}, \widetilde{b}) = (n \int_0^1 (\rho_2(\widetilde{a}_\alpha, \widetilde{b}_\alpha))^2 d\alpha)^{\frac{1}{2}} \qquad \widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathcal{R}^n),$$

where

$$\rho_2(\widetilde{a}_\alpha, \widetilde{b}_\alpha) = (\int_{S^{n-1}} |S_{\widetilde{a}_\alpha}(t) - S_{\widetilde{b}_\alpha}(t)|^2 \mu(dt))^{\frac{1}{2}}.$$

Note that μ is the normalized Lebesgue measure on S^{n-1}.

Lemma 2.1 *Let $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ be the intuitionistic fuzzy numbers. The L_2-metric of $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ satisfies the following properties*

(i) $\delta_2^2(\widetilde{x}, \widetilde{x}) = 0.$

(ii) $\delta_2^2(\widetilde{x}, \widetilde{y}) = \delta_2^2(\widetilde{y}, \widetilde{x}).$

(iii) $\delta_2^2(\widetilde{x}, \widetilde{z}) \leq \delta_2^2(\widetilde{x}, \widetilde{y}) + \delta_2^2(\widetilde{y}, \widetilde{z}).$

Example 1: As an example of a canonical fuzzy set on $\mathcal{R}$ consider so-called LR-fuzzy numbers $\widetilde{a} = (\mu, l, r)_{LR}$ with central value $\mu \in \mathcal{R}$, left and right spread $l \in \mathcal{R}^{\geq 0}$, $r \in \mathcal{R}^{\geq 0}$, decreasing left and right shape functions $L : \mathcal{R}^{\geq 0} \to [0, 1]$, $R : \mathcal{R}^{\geq 0} \to [0, 1]$ with $L(0) = R(0) = 1$, i.e., a fuzzy set $\widetilde{a}$ with

$$\mu_{\widetilde{a}}(x) = \begin{cases} L(\frac{\mu - x}{l}) & x \leq \mu \\ \\ R(\frac{x-\mu}{r}) & x \geq \mu. \end{cases}$$

An LR-fuzzy number $\widetilde{a} = (\mu, l, r)_{LR}$ with $L = R$ and $l = r = \varepsilon$ is called symmetric LR-fuzzy number and abbreviated by $\widetilde{a} = (\mu - \varepsilon, \mu, \mu + \varepsilon)$.

Let $\widetilde{a}_i = (\mu_i, l_i, r_i)_{LR}$; $i = 1, 2$. We have

$$\widetilde{a}_{i\alpha} = [\mu_i - L^{-1}(\alpha) l_i, \mu_i + R^{-1}(\alpha) r_i] \qquad i = 1, 2,$$

furthermore

$$S_{\widetilde{a}_{i\alpha}}(t) = \begin{cases} -\mu_i + L^{-1}(\alpha) l_i & t = -1 \\ \\ \mu_i + R^{-1}(\alpha) r_i & t = 1. \end{cases}$$

Thus

$$\delta_2^2(\widetilde{a}_1, \widetilde{a}_2) = (\mu_1 - \mu_2)^2 + \frac{1}{2}\int_0^1 (L^{-1}(\alpha))^2 d\alpha \ (l_1 - l_2)^2 + \frac{1}{2}\int_0^1 (R^{-1}(\alpha))^2 d\alpha \ (r_1 - r_2)^2$$

$$- \int_0^1 (L^{-1}(\alpha)) d\alpha \ (\mu_1 - \mu_2)(l_1 - l_2) + \int_0^1 (R^{-1}(\alpha)) d\alpha \ (\mu_1 - \mu_2)(r_1 - r_2).$$

For symmetric fuzzy numbers $\widetilde{a}_i = (\mu_i - \varepsilon_i, \mu_i, \mu_i + \varepsilon_i)$; $i = 1, 2$. We have

$$\delta_2^2(\widetilde{a}_1, \widetilde{a}_2) = (\mu_1 - \mu_2)^2 + \int_0^1 (L^{-1}(\alpha))^2 d\alpha \ (\varepsilon_1 - \varepsilon_2)^2.$$

2.3 Fuzzy Hypotheses

We define some models, as fuzzy sets of real numbers, for modeling the extended versions of the simple, the one-sided, and the two-sided ordinary (crisp) hypotheses to the fuzzy ones [4].

Definition 2.1 *Let θ_0 be a real number and known.*

i) Any hypothesis of the form ($H : \theta \;\; is \;\; approximately \;\; \theta_0$) is called to be a fuzzy simple hypothesis.

ii) Any hypothesis of the form ($H : \theta \;\; is \; not \;\; approximately \;\; \theta_0$) is called to be a fuzzy two-sided hypothesis.

iii) Any hypothesis of the form ($H : \theta \;\; is \; essentially \; smaller \; than \;\; \theta_0$) is called to be a fuzzy left one-sided hypothesis.

iv) Any hypothesis of the form ($H_1 : \theta \;\; is \; essentially \; larger \; than \;\; \theta_0$) is called to be a fuzzy right one-sided hypothesis.

We denote the above definitions by

$$a) \quad \begin{cases} H_0 : \theta \;\; is \;\; approximately \;\; \theta_0 \\ H_1 : \theta \;\; is \; not \;\; approximately \;\; \theta_0 \end{cases}$$

$$b) \quad \begin{cases} H_0 : \theta \;\; is \;\; essentially \;\; larger \;\; than \;\; \theta_0 \\ H_1 : \theta \;\; is \; not \;\; essentially \; larger \; than \;\; \theta_0 \end{cases}$$

$$c) \quad \begin{cases} H_0 : \theta \;\; is \;\; essentially \;\; smaller \;\; than \;\; \theta_0 \\ H_1 : \theta \;\; is \; not \;\; essentially \;\; than \;\; \theta_0 \end{cases}$$

The above areas are shown in Figures 1, 2 and 3.

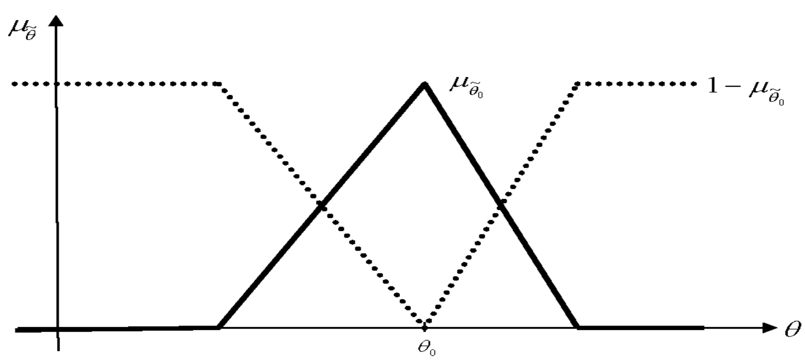

Figure 1: The fuzzy hypotheses of the form a)

3 Testing Fuzzy Hypotheses for One-Sample

We introduce a method to get testing hypotheses with one-sample of fuzzy data.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A compact convex random set (Cr.s.) $\mathbf{X}$ is a Borel measurable function from $(\Omega, \mathcal{F}, P)$ to $(\mathbf{X}, \mathcal{B}, P_{\mathbf{X}})$, where $P_{\mathbf{X}}$ is the probability measure induced by $\mathbf{X}$ and is called the distribution of the Cr.s. X , i.e.,

$$P_{\mathbf{X}}(A) = P(\mathbf{X} \in A) = \int_{\mathbf{X} \in A} dP \qquad\qquad \forall A \in \mathcal{B}.$$

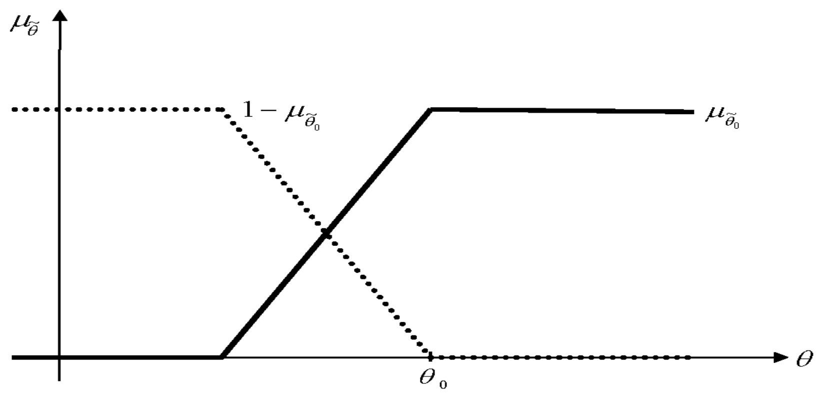

Figure 2: The fuzzy hypotheses of the form b)

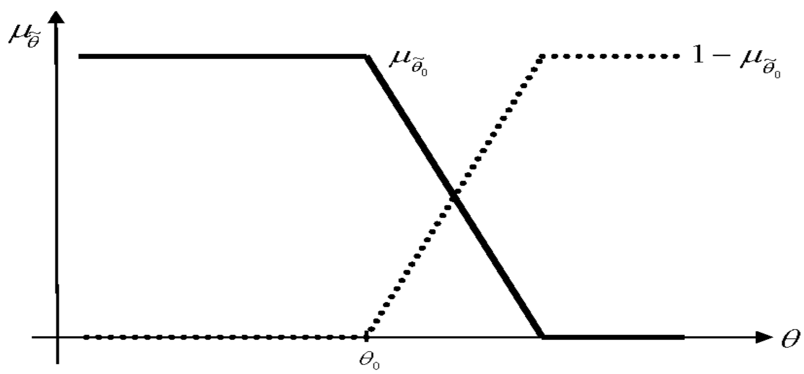

Figure 3: The fuzzy hypotheses of the form c)

Definition 3.1 *A fuzzy random variable (Fr.v.) is a Borel measurable function* $\widetilde{\mathbf{X}} : \Omega \to \mathcal{F}(\mathcal{R}^n)$ *where*

$$\{(\omega, x) : \omega \in \Omega \ , \ x \in \widetilde{\mathbf{X}}_\alpha(\omega)\} \in \ \mathcal{F} \times \mathcal{B} \qquad\qquad \forall \ \alpha \in [0,1].$$

Then, all $\alpha-$cuts of $\widetilde{\mathbf{X}}$ are Cr.s. and further more the above definition used here is equivalent to the often used definition by Puri and Ralescu [23], and for $n = 1$ to the definition by Kwakernaak [16].

Lemma 3.1 *Let* $\mathcal{F}(\mathcal{R})$ *be a canonical fuzzy real number system. Then* $\widetilde{X}$ *is a Fr.v. iff* X_α^L *and* X_α^U *are random variables for all* $\alpha \in [0,1]$.

The expected value $\widetilde{E}(\widetilde{\mathbf{X}})$ of the Fr.v. $\widetilde{\mathbf{X}}$ is defined by

$$\widetilde{E}_\alpha(\widetilde{\mathbf{X}}) = \{E(\mathbf{X}) | \mathbf{X} : \Omega \to \mathcal{R}^n, \ \mathbf{X}(\omega) = \widetilde{\mathbf{X}}_\alpha(\omega)\}.$$

Definition 3.2 *The variance of a Fr.v.* $\widetilde{\mathbf{X}}$ *is defined as* $\nu(\widetilde{\mathbf{X}}) = E[\delta_2^2(\widetilde{\mathbf{X}}, \ E(\widetilde{\mathbf{X}}))]$. *Using* $E_\alpha(\widetilde{\mathbf{X}}) = E(\widetilde{\mathbf{X}}_\alpha)$ *and* $S_{E(\widetilde{\mathbf{X}}_\alpha)}(t) = E(S_{\widetilde{\mathbf{X}}_\alpha}(t))$ *this can be written as*

$$\nu(\widetilde{\mathbf{X}}) = n \int_0^1 \int_{S^{n-1}} Var(S_{\widetilde{\mathbf{X}}_\alpha}(t)) \mu(dt) d\alpha.$$

Näther (2006) defined an scalar multiplication between $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ given by

$$< \widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} >= n \int_0^1 \int_{S^{n-1}} S_{\widetilde{\mathbf{X}}_\alpha}(t) S_{\widetilde{\mathbf{Y}}_\alpha}(t) \mu(dt) d\alpha,$$

thus

$$\nu(\widetilde{\mathbf{X}}) = E < \widetilde{\mathbf{X}}, \widetilde{\mathbf{X}} > - < E(\widetilde{\mathbf{X}}), E(\widetilde{\mathbf{X}}) >$$

and similarly

$$\begin{aligned} Cov(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) &= n \int_0^1 \int_{S^{n-1}} Cov(S_{\widetilde{\mathbf{X}}_\alpha}(t), S_{\widetilde{\mathbf{Y}}_\alpha}(t))\mu(dt)d\alpha \\ &= E < \widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} > - < E(\widetilde{\mathbf{X}}), E(\widetilde{\mathbf{Y}}) > . \end{aligned}$$

Definition 3.3 *Let $\widetilde{X}$ and $\widetilde{Y}$ be two Fr.v.'s. We say that $\widetilde{X}$ and $\widetilde{Y}$ are independent iff each random variable in the set $\{X_\alpha^L, X_\alpha^U : 0 \le \alpha \le 1\}$ is independent with any random variable in the set $\{Y_\alpha^L, Y_\alpha^U : 0 \le \alpha \le 1\}$.*

Definition 3.4 *We say $\widetilde{X}$ and $\widetilde{Y}$ are identically distributed iff X_α^L, Y_α^L are identically distributed, and X_α^U, Y_α^U are identically distributed for all $\alpha \in [0,1]$.*

Definition 3.5 *We say $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n)$ is a fuzzy random sample iff $\widetilde{X}_i$'s are independent and identically distributed.*

Lemma 3.2 *Let $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n)$ is a fuzzy random sample. The sample fuzzy mean value $\widetilde{\overline{\mathbf{X}}} = \frac{1}{n} \oplus_{i=1}^n \widetilde{X}_i$ is an unbiased estimator of the parameter $\widetilde{E}(\widetilde{X})$; and*

$$\lim_{n\to\infty} \delta_2^2 \left(\widetilde{\overline{\mathbf{X}}}, \widetilde{E}(\widetilde{X}) \right) = 0.$$

Proof. We have

$$\begin{aligned} \lim_{n\to\infty} \delta_2^2 \left(\widetilde{\overline{\mathbf{X}}}, \widetilde{E}(\widetilde{X}) \right) &= \lim_{n\to\infty} \int_0^1 \int_{-1}^1 [S_{\widetilde{\overline{\mathbf{X}}}_\alpha}(t) - S_{\widetilde{E}(\widetilde{X}_\alpha)}(t)]^2 dt d\alpha \\ &= \lim_{n\to\infty} \int_0^1 \int_{-1}^1 [\frac{1}{n} \sum_{i=1}^n S_{\widetilde{X}_{i\alpha}}(t) - S_{\widetilde{E}(\widetilde{X}_\alpha)}(t)]^2 dt d\alpha \\ &= \int_0^1 \int_{-1}^1 \lim_{n\to\infty} [\frac{1}{n} \sum_{i=1}^n S_{\widetilde{X}_{i\alpha}}(t) - S_{\widetilde{E}(\widetilde{X}_\alpha)}(t)]^2 dt d\alpha \\ &= 0. \end{aligned}$$

The latest equality is obtained from $S_{\widetilde{E}(\widetilde{X}_\alpha)}(t) = E(S_{\widetilde{X}_\alpha}(t))$ and strong law of large number.

Lemma 3.3 *Let $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n)$ is a fuzzy random sample. The sample fuzzy variance value $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n \delta_2^2 \left(\widetilde{X}_i, \widetilde{\overline{\mathbf{X}}} \right)$ is an unbiased estimator of the parameter $\nu(\widetilde{X})$; where $\widetilde{\overline{\mathbf{X}}}$ is the sample fuzzy mean value $\frac{1}{n} \oplus_{i=1}^n \widetilde{X}_i$.*

Proof. We have

$$\begin{aligned} E[S_n^2] &= \frac{1}{n-1} \sum_{i=1}^n \int_0^1 \int_{-1}^1 E[S_{\widetilde{X}_{i\alpha}}(t) - S_{\widetilde{\overline{\mathbf{X}}}_\alpha}(t)]^2 dt d\alpha \\ &= \frac{1}{n-1} \sum_{i=1}^n \int_0^1 \int_{-1}^1 E[S_{\widetilde{X}_{i\alpha}}(t) - E(S_{\widetilde{X}_{i\alpha}}(t)) + E(S_{\widetilde{X}_{i\alpha}}(t)) - S_{\widetilde{\overline{\mathbf{X}}}_\alpha}(t)]^2 dt d\alpha \\ &= \frac{1}{n-1} \sum_{i=1}^n \left[\nu(\widetilde{X}_i) + \int_0^1 \int_{-1}^1 Var[S_{\widetilde{\overline{\mathbf{X}}}_\alpha}(t)] dt d\alpha - 2 \int_0^1 \int_{-1}^1 Cov(S_{\widetilde{X}_{i\alpha}}(t), S_{\widetilde{\overline{\mathbf{X}}}_\alpha}(t)) dt d\alpha \right] \\ &= \frac{1}{n-1} \sum_{i=1}^n \left[\nu(\widetilde{X}_i) + \frac{\nu(\widetilde{X}_i)}{n} - 2 \frac{\nu(\widetilde{X}_i)}{n} \right] \\ &= \nu(\widetilde{X}). \end{aligned}$$

Lemma 3.4 *Consider Lemma 3.3. For given crisp value* S_n^2

$$\lim_{n\to\infty} S_n^2 = \nu(\widetilde{X}).$$

Proof. It is a special condition of strong law of large numbers.

Lemma 3.5 *If* $Var(X)$ *be a variance of the crisp random variable* X*, then we have*

$$\nu(X) = Var(X).$$

Proof. According to Example 2.1 it is obvious.

3.1 Simple Hypothesis Against the Two-Sided Hypothesis

Suppose that we have canonical fuzzy data $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_n)$. We want to test the following fuzzy hypotheses

$$\widetilde{H}_0 : \ the \quad mean \quad of \quad population \quad is \quad approximately \quad \theta_0$$

versus

$$\widetilde{H}_1 : \ the \quad mean \quad of \quad population \quad is \quad not \quad approximately \quad \theta_0,$$

and

$$\widetilde{H}_0 : \ the \quad variance \quad of \quad population \quad is \quad approximately \quad \theta_0$$

versus

$$\widetilde{H}_1 : \ the \quad variance \quad of \quad population \quad is \quad not \quad approximately \quad \theta_0.$$

We obtain the $\alpha-$cuts of the so-called fuzzy test statistics for mean and variance

$$\widetilde{T}_\alpha = \frac{\widetilde{\overline{\mathbf{x}}}_\alpha \ominus_{int} \widetilde{\theta}_{0\alpha}}{\frac{S_n}{\sqrt{n}}} = [\frac{\overline{\mathbf{x}}_\alpha^L - \theta_{0\alpha}^U}{\frac{S_n}{\sqrt{n}}}, \frac{\overline{\mathbf{x}}_\alpha^U - \theta_{0\alpha}^L}{\frac{S_n}{\sqrt{n}}}]$$

and

$$\widetilde{\chi}_\alpha^2 = \frac{(n-1)S_n^2}{\widetilde{\theta}_{0\alpha}} = [\frac{(n-1)S_n^2}{\theta_{0\alpha}^U}, \frac{(n-1)S_n^2}{\theta_{0\alpha}^L}],$$

respectively, where
(1) $\widetilde{\overline{\mathbf{x}}} = \frac{1}{n} \oplus_{i=1}^n \widetilde{x}_i$
(2) $S_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^n \delta_2^2(\widetilde{x}_i, \widetilde{\overline{\mathbf{x}}})}$.

We use the fuzzy test statistics to provide an approach for testing above fuzzy hypotheses based on the following assumptions (see Figure 4).

- **ASSUMPTIONS for mean(or variance)**

1. C_T is the total area under $\widetilde{T}$ (or $\widetilde{\chi}^2$).
2. C_1 and C_2 are the areas according to Figure 4.
3. $C_R = C_1 + C_2$.

t^γ is the $100(1-\gamma)$ percentile of the T distribution with $n-1$ degree of freedom and χ_γ^2 is the $100(1-\gamma)$ percentile of the $Chi-Square$ distribution with $n-1$ degree of freedom.

- **DECISION RULE**

- If $\frac{C_R}{C_T} \le 2\gamma$, then we accept $\widetilde{H}_0$.
- If $\frac{C_R}{C_T} \ge 2\gamma$, then we reject $\widetilde{H}_0$.

We choose a small probability γ (significant level), like $0.01,\ 0.05$ *or* 0.1, and observe $\frac{C_R}{C_T}$ as an evident against $\widetilde{H}_1$ according to the following conventions:

$$\frac{C_R}{C_T} < 0.1 \ extremity \ evident \ against \ \widetilde{H}_1,$$

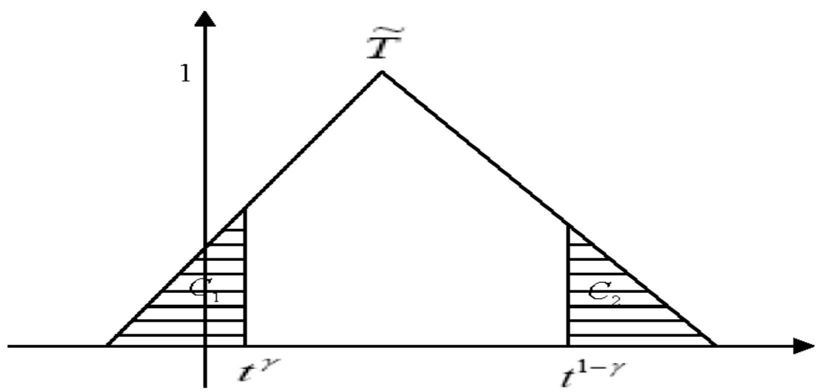

Figure 4: $\widetilde{T}$ (or $\widetilde{\chi}^2$) in testing fuzzy simple hypothesis versus fuzzy two-sided hypothesis

$$\frac{C_R}{C_T} < 0.05 \; appealability \; evident \; against \; \widetilde{H}_1,$$
$$\frac{C_R}{C_T} < 0.05 \; strong \; evident \; against \; \widetilde{H}_1.$$

Example 2: Suppose that we have taken a fuzzy random sample of size $n = 9$ from a population and we observed the following triangular fuzzy data:

Table 1: Fuzzy random sample of size $n = 9$ from a population

N	Observation	N	Observation	N	Observation
1	$(32, 35, 40)$	4	$(60, 63, 63)$	7	$(70, 73, 75)$
2	$(80, 82, 82)$	5	$(41, 45, 47)$	8	$(54, 56, 59)$
3	$(60, 60, 60)$	6	$(93, 95, 96)$	9	$(34, 35, 36)$

Now suppose that we want to test the following fuzzy hypotheses

$$\begin{cases} \widetilde{H}_0 : mean \quad is \quad (69, 70, 72) \\ \widetilde{H}_1 : mean \quad is \quad not \quad (69, 70, 72). \end{cases}$$

Here, $\widetilde{H}_0$ suggests that *mean* is approximately 70, and $\widetilde{H}_1$ suggests that *mean* is away from 70.

For significance level $\gamma = 0.05$ ($t^{1-\gamma} = -t^{1-\gamma} = 1.86$), we have $\widetilde{T}_\alpha = [-2.01 + 0.63\alpha, -1 - 0.38\alpha]$, $C_R = C_1 + C_2 = 0.018 + 0 = 0.018$, $C_T = 0.51$. Since $\frac{C_R}{C_T} = 0.04 \le 0.1$, thus we certainly accept $\widetilde{H}_0$ (see Figure 5).

Example 3: Consider Table 1. Now suppose that we want to test the following fuzzy hypotheses

$$\begin{cases} \widetilde{H}_0 : variance \quad is \quad (1000, 1200, 1500) \\ \widetilde{H}_1 : variance \quad is \quad not \quad (1000, 1200, 1500). \end{cases}$$

For significance level $\gamma = 0.05$ ($\chi^2_{0.05} = 2.73$ and $\chi^2_{0.95} = 15.51$), we have $\widetilde{\chi}^2_\alpha = [\frac{3304.2}{1500-300\alpha}, \frac{3304.2}{1000+200\alpha}]$, $C_R = 0.26 + 0 = 0.26$, $C_T = 0.58$. Since $\frac{C_R}{C_T} = 0.41 > 0.1$, thus we reject $\widetilde{H}_0$ (see Figure 6).

3.2 Right One-Sided Hypothesis Against the Left One-Sided Hypothesis

We want to test the fuzzy null hypotheses

$$\widetilde{H}_0 : \; the \quad mean \; (or \; variance) \quad of \quad population \quad is \quad essentially \quad larger \quad than \quad \theta_0$$

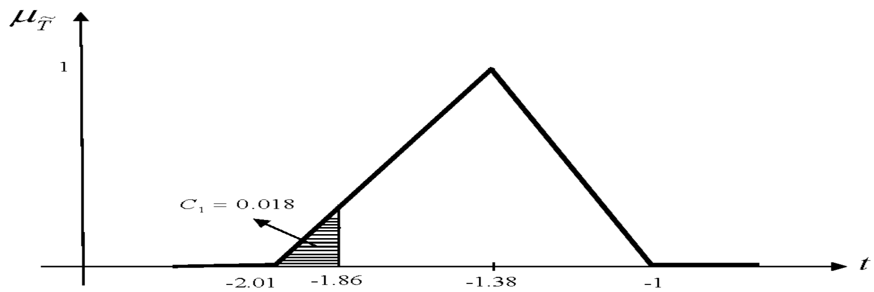

Figure 5: $\widetilde{H}_0$ versus $\widetilde{H}_1$ for mean

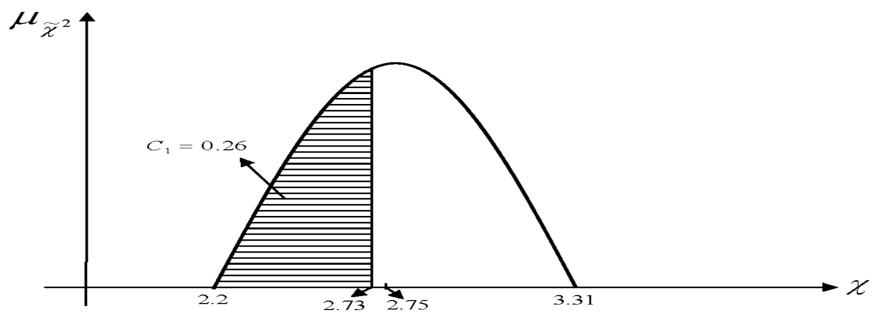

Figure 6: $\widetilde{H}_0$ versus $\widetilde{H}_1$ for variance

versus

$$\widetilde{H}_1 : \textit{the mean (or variance) of population is not essentially larger than } \theta_0.$$

We obtain the $\alpha-$cuts of fuzzy test statistics $\widetilde{T}$ $(or\ \widetilde{\chi}^2)$ and use the fuzzy test statistics to provide an approach for testing above fuzzy hypotheses, based on the following assumptions (see Figure 7).

- **ASSUMPTIONS for mean (or variance)**

1. C_T is the total area under $\widetilde{T}$ $(or\ \widetilde{\chi}^2)$.
2. C_1 is the area according to Figure 7.
3. $C_R = C_1$.

- **DECISION RULE for mean (or variance)**

• If $C_R/C_T \leq \gamma$, then we accept $\widetilde{H}_0$.
• If $C_R/C_T \geq \gamma$, then we reject $\widetilde{H}_0$.

Example 4: Consider Table 1. Now suppose that we want to test the following fuzzy hypotheses

$$\begin{cases} \widetilde{H}_0 : \textit{the mean is essentially larger than } 87 \\ \widetilde{H}_1 : \textit{the mean is not essentially larger than } 87 \end{cases}$$

where the fuzzy hypothesis $\widetilde{H}_0$ has the following membership function

$$\mu_{\widetilde{H}_0}(y) = \begin{cases} 0 & y < 85 \\ \frac{y-85}{2} & 85 \leq y \leq 87 \\ 1 & y > 87. \end{cases}$$

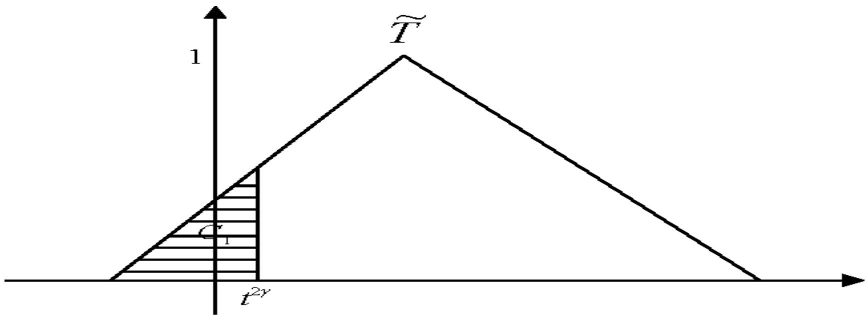

Figure 7: $\widetilde{T}$ (*or* $\widetilde{\chi}^2$) in testing fuzzy right one-sided hypothesis against the left one-sided hypothesis

For significance level $2\gamma = 0.05$ ($t^{2\gamma} = -1.86$), we have $\widetilde{T}_\alpha = [-3.22+0.72\alpha, -1.7-0.8\alpha]$, $C_R = C_1 = 0.69$, $C_T = 0.76$. Since $C_R/C_T = 0.91 \geq 0.05$, thus we reject $\widetilde{H}_0$. Similarly, we can use the above method for testing the following hypotheses

$$\widetilde{H}_0: \textit{the variance of population is essentially larger than } \theta_0$$

versus

$$\widetilde{H}_1: \textit{the variance of population is not essentially larger than } \theta_0.$$

According to Figure 8, the problem of left one-sided hypothesis against the right one-sided hypothesis for mean and variance is similar.

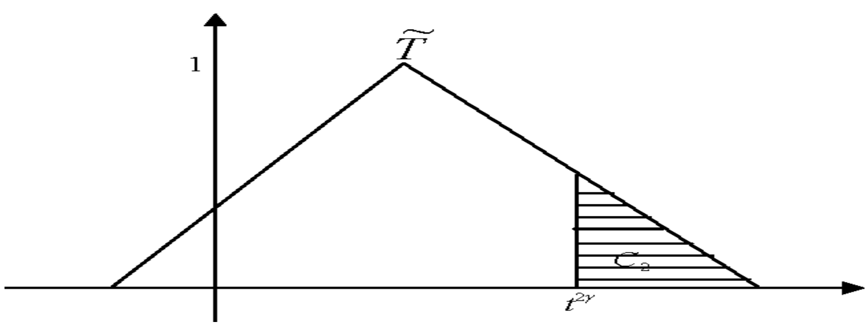

Figure 8: $\widetilde{T}$ (*or* $\widetilde{\chi}^2$) in testing fuzzy left one-sided hypothesis against the right one-sided hypothesis

4 Testing Hypotheses for Two-Sample

In this section we describe a method to testing hypotheses for two-sample of fuzzy data.

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y, ..., y_m)$ are two crisp random sample from normal populations. In non-fuzzy form, if we are not willing to assume that the variances in the two populations are equal and wanted to test only whether their means are equal, we could use the two-sample *Student's* $T - distribution$ with

$$\frac{(\frac{\hat{\sigma}_n^2}{n} + \frac{\hat{\sigma}_m^2}{m})^2}{\frac{1}{n-1}(\frac{\hat{\sigma}_n^2}{n})^2 + \frac{1}{m-1}(\frac{\hat{\sigma}_m^2}{m})^2}$$

degrees of freedom as follows:

$$T(\mathbf{x}, \mathbf{y}) = \frac{\overline{\mathbf{x}} - \overline{\mathbf{y}}}{\sqrt{\frac{\hat{\sigma}_n^2}{n} + \frac{\hat{\sigma}_m^2}{m}}}$$

where $\hat{\sigma}_n^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{\mathbf{x}})^2$ and $\hat{\sigma}_m^2 = \frac{1}{m-1}\sum_{i=1}^m (y_i - \overline{\mathbf{y}})^2$.

If we are willing to assume that the variances in the two populations are equal, and wanted to test whether their means are equal, we could use the two-sample *Student's* $T-distribution$ with with $n+m-2$ degrees of freedom. It uses the pooled estimate of standard error σ. We could base the test on

$$T(\mathbf{x},\mathbf{y}) = \frac{\overline{\mathbf{x}} - \overline{\mathbf{y}}}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}}$$

where

$$S_p = \sqrt{\frac{(n-1)\hat{\sigma}_n^2 + (m-1)\hat{\sigma}_m^2}{n+m-2}}.$$

If we want to test the whether their variances are equal, under the null hypothesis (H_0 : variances in the two populations are equal), we could use the *Fisher's* $F-distribution$ with $n-1$ and $m-1$ degrees of freedom as follows,

$$F(\mathbf{x},\mathbf{y}) = \frac{(n-1)\hat{\sigma}_n^2}{(m-1)\hat{\sigma}_m^2}.$$

Lemma 4.1 *Let* $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n)$ *and* $\widetilde{\mathbf{Y}} = (\widetilde{Y}_1, \widetilde{Y}_2, ..., \widetilde{Y}_n)$ *are two fuzzy random sample. The pooled variance value* $S_p^2 = ((n-1)S_n^2 + (m-1)S_m^2)/(n+m-2)$ *is an unbiased estimator of the parameter* $\nu(\widetilde{X})$*; where* $\nu(\widetilde{X}) = \nu(\widetilde{Y})$*.*

Let we have fuzzy random data $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_n)$ and $\widetilde{\mathbf{y}} = (\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_m)$ from possibly different probability distribution, and we wish to test the null hypothesis
H_0 : the mean (or variance) of first population is equal to the mean (or variance) of second population.
H_1 : the mean (or variance) of first population is not equal to the mean (or variance) of second population.

Without any loss generation, let the variance of first population is equal to the variance of second population. We obtain the $\alpha-$cuts of the fuzzy test statistics of means as follows

$$\widetilde{T}_\alpha = \frac{\widetilde{\overline{\mathbf{x}}}_\alpha \ominus_{int} \widetilde{\overline{\mathbf{y}}}_\alpha}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}} = [\frac{\overline{\mathbf{x}}_\alpha^L - \overline{\mathbf{y}}_\alpha^U}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}}, \frac{\overline{\mathbf{x}}_\alpha^U - \overline{\mathbf{y}}_\alpha^L}{S_p\sqrt{\frac{1}{n}+\frac{1}{m}}}]$$

and crisp test statistics of variances is

$$F = \frac{(n-1)S_n^2}{(m-1)S_m^2}$$

where
(1) $\widetilde{\overline{\mathbf{x}}} = \frac{1}{n} \oplus_{i=1}^n \widetilde{x}_i$,
(2) $S_p^2 = \frac{(n-1)S_n^2 + (m-1)S_m^2}{n+m-2}$,
(3) $S_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^n \delta_2^2(\widetilde{x}_i, \widetilde{\overline{\mathbf{x}}})}$ and $S_m = \sqrt{\frac{1}{m-1}\sum_{i=1}^m \delta_2^2(\widetilde{y}_i, \widetilde{\overline{\mathbf{y}}})}$.

We use the fuzzy test statistics to provide an approach to testing above fuzzy hypotheses based on Figure 4.

- **DECISION RULE**

- For mean, if $C_R/C_T \leq 2\gamma$, then we accept $\widetilde{H}_0$.
- For mean, if $C_R/C_T \geq 2\gamma$, then we reject $\widetilde{H}_0$.
- For variance, if $F_{n-1,m-1}^{\gamma} < F < F_{n-1,m-1}^{1-\gamma}$, then we accept $\widetilde{H}_0$.
- For variance, if $F_{n-1,m-1}^{\gamma} \geq F$ or $F \geq F_{n-1,m-1}^{1-\gamma}$, then we reject $\widetilde{H}_0$, where $F_{n-1,m-1}^{\gamma}$ is the $100(1-\gamma)$ percentile of the Fisher's F distribution with $n-1$ and $m-1$ degrees of freedom.

Example 5: Suppose that we have taken two fuzzy random samples of size $n = 9$ and $m = 7$ from two populations and we observed the following triangular fuzzy data:

Table 2: Fuzzy random sample of size $n = 9$ and $m = 7$ from two populations

The first population	The second population
$(51, 52, 54)$	$(92, 94, 95)$
$(101, 104, 107)$	$(197, 197, 199)$
$(146, 146, 146)$	$(15, 16, 17)$
$(80, 10, 11)$	$(36, 38, 40)$
$(49, 50, 51)$	$(99, 99, 99)$
$(29, 31, 32)$	$(140, 141, 143)$
$(39, 40, 40)$	$(23, 23, 23)$
$(25, 27, 28)$	
$(46, 46, 46)$	

Let we want to test

H_0 : the mean of first population is equal to the mean of second population.

H_1 : the mean of first population is not equal to the mean of second population.

For significance level $\gamma = 0.05$, we have $\widetilde{T}_\alpha = [-1.92 + 0.7\alpha, -0.34 - 0.88\alpha]$, $C_R = C_1 + C_2 = 0 + 0.04 = 0.04$, $C_T = 0.79$. Since $\frac{C_R}{C_T} = 0.051 \leq 0.1$, thus we accept H_0 certainly.

Furthermore, we have $F_{8,6}^{0.05} = 0.28 < F = 0.53 < F_{8,6}^{0.95} = 4.15$, thus we accept H_0 certainly.

5 A Comparison Study

In this section, we want to compare our method with Arefi et al.'s [5] and Wu's [30] approaches.

- Comparison with Arefi et al.'s approach

Arefi et al.'s studied the problem of testing fuzzy hypotheses based on the fuzzy test statistic, when the available data are fuzzy. They first introduced a method for obtaining a point estimation based on fuzzy data, called the fuzzy point estimation. Then, the fuzzy test statistic could be defined based on the $\alpha-$cuts of the fuzzy point estimation and the $\alpha-$cuts of the fuzzy null hypothesis. Finally, they introduced a credit level to evaluate the fuzzy hypotheses of interest. In the following, we list some comments between our proposed approach and Arefi et al.'s approach.

1. We extended the L_2-metric based on the $\alpha-$cuts of fuzzy data for constructing the fuzzy test statistic, but Arefi et al. used the interval arithmetic between the $\alpha-$cuts of fuzzy hypotheses and fuzzy data for obtaining the fuzzy test statistic.

2. We introduce a nonparametric statistic for testing statistical hypotheses, but Arefi et al.'s method is constructed based on a parametric statistic.

- Comparison with Wu's approach

Wu presented an approach for testing the fuzzy mean based on fuzzy data. He introduced a notation for testing fuzzy hypotheses as follows:

$$x_\alpha^L = \frac{1}{n}\sum_{i=1}^{n} \widetilde{x}_{i\alpha}^L - core(\widetilde{\mu}_0) \qquad\qquad x_\alpha^U = \frac{1}{n}\sum_{i=1}^{n} \widetilde{x}_{i\alpha}^U - core(\widetilde{\mu}_0)$$

where $x_\alpha^L = \inf\{t : \widetilde{x}_i(t) \geq \alpha\}$ and $x_\alpha^U = \sup\{t : \widetilde{x}_i(t) \geq \alpha\}$, and $core(\widetilde{\mu}_0)$ is the center of the fuzzy number. Then, he proposed to accept $\widetilde{H}_0$ in the α-cut sense if $x_\alpha^U < z_{1-\beta}\frac{\sigma}{\sqrt{n}}$, and to accept $\widetilde{H}_1$ in the α-cut sense if $x_\alpha^L \geq z_{1-\beta}\frac{\sigma}{\sqrt{n}}$.

Some advantages of our method can be listed as follows:

1. We introduce a nonparametric fuzzy test statistic based on the $\alpha-$cuts of the fuzzy hypothesis and fuzzy data, but Wu used the center of the fuzzy null hypotheses. Hence, with different withes and similar centers, the our method has the different results as compared with Wu's method.

2. For testing fuzzy hypotheses, we use all the $\alpha-$cuts of fuzzy data for obtaining the fuzzy test statistic, but Wu only used the lower and upper of the cuts of fuzzy data.

6 Conclusions

In this paper, we proposed a technique in order to get testing hypotheses with one-sample or two-sample fuzzy observations based on L_2-metric. As for this paper, it sounds that the introduced method is more simple and convenient than Buckley and Taheri. This metric is very realistic because

- which implies very good statistical properties in connection with variance;
- it involves distances between extreme points;
- it is distance with convenient statistical features.

Extension of the proposed method in order to hypotheses testing for the fuzzy coefficient of linear models such as regression models, and design of experiment is a potential area for the future work. Furthermore, for the hypotheses testing of the fuzzy hypothesis, we can apply methods introduced by Akbari et. al. [1] and Akbari and Rezaei [4].

Acknowledgements

The authors wish to express their thanks to the referees for valuable comments which improved the paper. Also, The authors would like to thank Dr. M. Arefi for reading the manuscript and for his suggestions.

References

[1] Akbari, M.G., and A. Rezaei, Bootstrap statistical inference for the variance based on fuzzy data, *Austrian Journal of Statistics*, vol.38, no.2, pp.121–130, 2009.

[2] Akbari, M.G., and A. Rezaei, Order statistics using fuzzy random variables, *Statistics and Probability Letters*, vol.79, no.8, pp.1031–1037, 2009.

[3] Akbari, M.G., A. Rezaei, and Y. Waghei, Statistical inference about the variance of fuzzy random variables, *Sankhyā*, vol.71, no.2, pp.206–221, 2009.

[4] Akbari, M.G., and A. Rezaei, Bootstrap testing fuzzy hypotheses and observations on fuzzy statistic, *Expert Systems with Applications*, vol.37, no.8, pp.5782–5787, 2010.

[5] Arefi, M., and S.M. Taheri, Testing fuzzy hypotheses using fuzzy data based on fuzzy test statistic, *Journal of Uncertain Systems*, vol.5, no.1, pp.45–61, 2011.

[6] Arnold, B.F., An approach to fuzzy hypothesis testing, *Metrika*, vol.44, no.1, pp.119–126, 1996.

[7] Arnold, B.F., Testing fuzzy hypothesis with crisp data, *Fuzzy Sets and Systems*, vol.94, no.3, pp.323–333, 1998.

[8] Buckley, J.J., *Fuzzy Probabilities: New Approach and Applications*, Springer-Verlag, Berlin, Heidelberg, 2005.

[9] Buckley, J.J., *Fuzzy Probability and Statistics*, Springer-Verlag, Berlin, Heidelberg, 2006.

[10] Cheng, C., A new approach for ranking fuzzy numbers by distance method, *Fuzzy Sets and Systems*, vol.95, no.3, pp.307–317, 1998.

[11] Filzmoser, P., and R. Viertl, Testing hypotheses with fuzzy data: the fuzzy $p-$value, *Metrika*, vol.59, no.1, pp.21–29, 2004.

[12] Gonzalez-Rodriguez, G., M. Montenegro, A. Colubi, and M.A. Gil, Bootstrap techniques and fuzzy random variables: synergy in hypothesis testing with fuzzy data, *Fuzzy Sets and Systems*, vol.157, no.19, pp.2608–2613, 2006.

[13] Holena, M., Fuzzy hypotheses for GUHA implications, *Fuzzy Sets and Systems*, vol.98, no.1, pp.101–125, 1998.

[14] Holena, M., Fuzzy hypotheses testing in the framework of fuzzy logic, *Fuzzy Sets and Systems*, vol.145, no.1, pp.229–252, 2004.

[15] Korner, R., An asymptotic $\alpha-$cut for the expectation of random fuzzy variables, *Journal of Statistical Planning and Inferences*, vol.50, no.5, pp.331–346, 2000.

[16] Kwakernaak, H., Fuzzy random variables I, *Information Sciences*, vol.15, no.1, pp.1–29, 1978.

[17] Modarres, M., and S. Sadi-Nezhad, Ranking fuzzy numbers by preference ratio, *Fuzzy Sets and Systems*, vol.118, no.3, pp.429–436, 2001.

[18] Montenegro, M., A. Colubi, M.R. Casals, and M.A. Gil, Asymptotic and bootstrap techniques for testing the expected value of a fuzzy random variable, *Metrika*, vol.59, no.1, pp.31–49, 2004.

[19] Näther, W., Regression with fuzzy data, *Computational Statistics and Data Analysis*, vol.51, no.1, pp.235–252, 2006.

[20] Nojavan, M., and M. Ghazanfari, A fuzzy ranking method by desirability index, *Journal of Intelligent and Fuzzy Systems*, vol.17, no.1, pp.27–34, 2006.

[21] Parchami, A., S.M. Taheri, and M. Mashinchi, Fuzzy p-value in testing fuzzy hypotheses with crisp data, *Statistical Papers*, vol.51, no.1, pp.209–226, 2009.

[22] Puri, M.L., and D.A. Ralescu, Differential of fuzzy functions, *Journal of Mathematical Analysis and Applications*, vol.91, no.2, pp.552–558, 1983.

[23] Puri, M.L., and D.A. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, vol.114, no.2, pp.409–422, 1986.

[24] Taheri, S.M., and J. Behboodian, Neyman-Pearson Lemma for fuzzy hypotheses testing, *Metrika*, vol.49, no.1, pp.3–17, 1999.

[25] Taheri, S.M., and M. Arefi, Testing fuzzy hypotheses based on fuzzy statistics, *Soft Computing*, vol.13, no.6, pp.617–625, 2009.

[26] Thompson, E.A., and C.J. Geyer, Fuzzy p-values in latent variable problems, *Biometrika*, vol.94, no.1, pp.49–60, 2007.

[27] Torabi, H., J. Behboodian, and S.M. Taheri, Neyman-Pearson Lemma for fuzzy hypotheses testing with vague data, *Metrika*, vol.64, no.1, pp.289–304, 2006.

[28] Viertl, R., Univariate statistical analysis with fuzzy data, *Computational Statistics and Data Analysis*, vol.51, no.1, pp.133–147, 2006.

[29] Watanabe, N., and T. Imaizumi, A fuzzy statistical test of fuzzy hypotheses, *Fuzzy Sets and Systems*, vol.53, no.2, pp.167–178, 1993.

[30] Wu, H.C., Statistical hypotheses testing for fuzzy data, *Information Sciences*, vol.175, nos.1-2, pp.30–56, 2005.

[31] Yao, J.S., and K. Wu, Ranking fuzzy numbers based on decomposition principle and signed distance, *Fuzzy Sets and Systems*, vol.116, no.2, pp.275–288, 2000.