

# A New Answer to Pauli's Question: Almost All Quantum States can be Uniquely Determined by Measuring Location and Momentum

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## Abstract

In classical mechanics, we can uniquely reconstruct the state of each particle by measuring its spatial location and momentum. In his 1958 paper, W. Pauli, one of the founders of quantum mechanics, conjectured that the same should be true in the quantum case as well: that every quantum state can be uniquely determined by measuring spatial location and momentum. This conjecture was disproven: there are pairs of physically different states that cannot be distinguished if we only measure spatial location and momentum. A natural question is: how frequent are such pairs? In this paper, we show that almost all quantum states can be uniquely reconstructed by measuring spatial location and momentum. Thus, in practice, Pauli's conjecture is true.

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## 1 Standard Quantum Description of a Particle: A Brief Reminder

**Classical (pre-quantum) description.** In the traditional classical (pre-quantum) description of particles, the state of each elementary particle can be characterized by its location  $x \in R^3$  in the 3-D space  $R^3$ , by its momentum  $p = m \cdot v$  (where  $m$  denotes the particle's mass and  $v$  its velocity), etc.

**Quantum description is probabilistic.** In quantum physics, a particle does not have a certain location  $x$  or a certain momentum  $p$ : if we measure location of several particles prepared in the same state, we get different locations with different probabilities. Similarly, we get different values of the momentum with different probabilities.

In contrast to the classical (pre-quantum) case, a *state* of a quantum particle does not enable us to determine the exact location or the exact momentum; instead, a quantum state uniquely determine the *probabilities* of different locations and/or different momenta.

**Case of a single particle.** In the traditional quantum mechanics, a state is described by a complex-valued function  $\psi(x)$  called a *wave function*.

For example, a state of a single particle is described by a complex-valued function  $\psi(x)$  defined on the 3-D space ( $x \in R^3$ ). Under this description, for every set  $S$ , the probability to find this particle in an area  $S \subseteq R^3$  is equal to the integral  $\int_S |\psi(x)|^2 dx$ , where  $|\psi(x)|$  denotes the absolute value (magnitude) of the complex number  $\psi(x)$ . In other words, the function  $|\psi(x)|^2$  is the probability density function of the probability function which describes the particle's location.

The total probability to get any value  $x \in R^3$  should be equal to 1, so we must have  $\int_{R^3} |\psi(x)|^2 dx = 1$ .

The probability to find the moment  $p$  within a certain area  $S$  is similarly equal to  $\int_S |F(\psi)(\omega)|^2 d\omega$ , where  $F(\psi)$  denotes the Fourier transform of the wave function  $\psi(x)$ .

## 2 Pauli's Question: What is Known and What is Still Open

**Pauli's original question: how uniquely can we determine the quantum state based on the measurements?** For a single particle, if all we measure is spatial location, then we can only determine the

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absolute values  $|\psi(x)|$  of the wave function  $\psi(x)$ . By measuring other characteristics such as momentum, we can gain more information about the wave function.

W. Pauli asked the following natural question (see, e.g., [15]): to what extent can we determine the wave function from the measurements? In particular, he asked whether we can uniquely reconstruct a state by measuring spatial location and momentum.

*Clarification.* It is known that from the physical viewpoint, for every real value  $\alpha$ , the functions  $\psi(x)$  and  $\psi'(x) = e^{i\alpha} \cdot \psi(x)$  describe the exact same physical state. Thus, we mean uniqueness modulo such a factor.

**Pauli's question: what is known.** The original Pauli's conjecture was that every quantum state can be uniquely determined by measuring location and momentum. This conjecture turned out to be false: there are cases when two physically different states lead to the same measurements of location and momentum: i.e., to the same probabilities of different values of location and momentum.

It has also been shown that if we allow additional measurements – not just measurements of location and momentum – then it is already possible to uniquely reconstruct the quantum state; see, e.g., [2, 5, 6, 7, 8, 9, 10, 12, 13, 14]. Specifically, in addition to measuring location and momenta at the initial moment of time, we can place the particle in some (physically meaningful) potential fields and re-measure location and momentum after a certain time.

*Comment.* It is worth mentioning that even when the reconstruction is theoretically possible, it is not always computationally feasible. In general, the reconstructing problem is computationally difficult (NP-hard); see, e.g., [11]. Crudely speaking, this means that in some cases, for this reconstruction, we need computation time which exceeds the lifetime of the Universe.

**Remaining open problem.** Counter-examples to the original Pauli's conjecture are rare and exotic. It is therefore reasonable to ask how frequent are these examples? If almost all states (in some reasonable sense) can be uniquely reconstructed, then in practice, the chance of encountering such states are close to 0, so, in effect, we can always reconstruct a state by measuring location and momentum.

**What we do in this paper.** In this paper, we show that indeed, almost all states can be uniquely determined by measuring location and momentum. Thus, in practice, Pauli's conjecture is true.

### 3 Main Result

**How to meaningfully formalize “almost all”: a need for discretization.** In a finite-dimensional space  $R^n$ , there are natural notion of “almost all” – e.g., “almost all” in the sense of the Lebesgue measure on  $R^n$ , when, crudely speaking, the volume of the set of all the states that do not satisfy the given property is 0.

The space of all possible quantum states is the space of all possible wave functions. This set is infinite-dimensional, in the sense that one needs infinitely many parameters to describe a general element from this set. On such spaces, it is difficult to select a physically meaningful notion of “almost all”. To be more precise, there are such notion but they have weird properties: for example, if we define “almost all” in terms of a natural measure on the set of all the functions (such as Wiener's measure), then we conclude that almost all functions are nowhere differentiable etc.

A reasonable solution to this problem is to take into account that in practice, at every moment of time, we only know the values of finitely many parameters. As a result, it is reasonable to consider a finite-dimensional approximation to the original problem.

To describe a general wave function  $\psi(x)$ , we need infinitely many parameters – the values of  $\psi(x)$  at different spatial locations  $x$ . Thus, if we want to restrict ourselves to a finite-dimensional case, we should consider only finitely many spatial locations.

The simplest case is to consider an  $n \times n \times n$  3-D grid of spatial locations.

**Precise formulation of the problem.** Functions on a grid and their Fourier transforms have a natural polynomial interpretation (see, e.g., [3]). Namely, on a grid, coordinates along each direction can be labeled as  $0, 1, \dots, n-1$ . As a result, every point is labeled by a triple of integers  $\langle i_1, i_2, i_3 \rangle$ ,  $i_j = 0, 1, \dots, n-1$ . Thus, the values of each function  $\psi(x)$  on grid points  $x = \langle i_1, i_2, i_3 \rangle$  form a 3-D array with values  $a_{i_1, i_2, i_3} = \psi(\langle i_1, i_2, i_3 \rangle)$ . We can form a polynomial with these coefficients

$$P(z_1, z_2, z_3) = \sum_{i_1, i_2, i_3} a_{i_1, i_2, i_3} \cdot z_1^{i_1} \cdot z_2^{i_2} \cdot z_3^{i_3}.$$

The Fourier transform  $F(\omega)$  consists of the values of this polynomial at points  $z_j = \exp(i \cdot m_j \cdot (2 \cdot \pi)/n)$ . Based on these values, we can uniquely reconstruct the original coefficients  $a_{i_1, i_2, i_3}$  – and thus, the original polynomial.

The complex conjugate  $F^*(\omega)$  to the Fourier transform is also a Fourier transform – corresponding to the function  $\psi_-(x) \stackrel{\text{def}}{=} \psi(-x)$  and a related polynomial  $P_*(z)$ . The product  $|F(\omega)|^2 = F(\omega) \cdot F^*(\omega)$  corresponds to the product  $P(z) \cdot P_*(z)$  of the corresponding polynomials.

When is reconstruction not unique? when we have a different state  $\psi'(x)$ , with a different Fourier transform  $F'(\omega)$  that leads to the exact same polynomial, i.e.,  $P(z) \cdot P_*(z) = P'(z) \cdot P'_*(z)$ . If a polynomial  $P(z)$  can be factorized into a product of two or more polynomial of lower order – which is usual for polynomials of one variable – then, in principle, such an equality is possible. However, if a polynomial is irreducible – i.e., cannot be represented as a product of polynomials of lower order – then the factors  $P'(z)$  and  $P'_*(z)$  must, in effect, coincide with the original factors  $P(z)$  and  $P_*(z)$  – modulo constants. Thus, we have two options:

- $P'(z) = c \cdot P(z)$  for some constant  $c$ ; in this case,  $\psi'(x) = c \cdot \psi(x)$ , so we get the exact same physical state;
- $P'(z) = c \cdot P_*(z)$ , meaning that  $\psi'(x) = c \cdot \psi(-x)$ ; since we also assume that the results of measuring location coincide, we also have  $|\psi'(x)| = |\psi(x)|$  hence  $|\psi(-x)| = |\psi(x)|$ , i.e., the modulus of the wave function is symmetric; this is almost never true and thus, in almost all cases, the state can be uniquely reconstructed by measuring location and momentum.

So, to prove the uniqueness in the general case, it is sufficient to prove that almost all polynomials are irreducible.

**Almost all polynomials of 3 variables are irreducible: a proof.** For each monomial  $z_1^{i_1} \cdot z_2^{i_2} \cdot z_3^{i_3}$ , its degree is usually defined as the sum  $i_1 + i_2 + i_3$ . A degree of a polynomial is defined as the largest of the degrees of its monomials. When two polynomials are multiplied, their degrees add.

Let us prove that for every overall degree  $d$ , almost all polynomials of degree  $\leq d$  are irreducible, i.e., cannot be represented as a product of two polynomials of smaller degrees  $d' > 0$  and  $d - d'$ . Specifically, we will prove that the dimension of the set of all reducible polynomials is smaller than the dimension of the set of all polynomials of degree  $\leq d$  – and thus, almost all polynomials are irreducible (because the volume of a set of smaller dimension is always 0).

Let us first count the dimension of the set of all polynomials of degree  $\leq d$ , i.e., the total number of all coefficients  $a_{i_1, i_2, i_3}$  with  $i_1 + i_2 + i_3 \leq d$ . Each such triple can be described if we first list  $i_1$  0s, then place a separating 1, then place  $i_2$  zeros, then a separating 1, then  $i_3$  zeros, then separating 1. Totally, we have  $d + 3$  locations:  $d$  original locations plus 3 separators. Vice versa, if we place three 1s in a list of  $d + 3$  locations, then we can take, as  $i_1$ , the number of 0s before the first 1, etc. So, the desired dimension is equal to the number of different ways to place three 1s in  $d + 3$  different locations, i.e.,

$$\binom{d+3}{3} = \frac{(d+3) \cdot (d+2) \cdot (d+1)}{1 \cdot 2 \cdot 3}.$$

If we denote  $D = d+2$ , then the numerator of this expression takes the form  $(D+1) \cdot D \cdot (D-1) = (D^2 - 1) \cdot D = D^3 - D = (d+2)^3 - (d+2)$ . Thus, the desired dimension is equal to  $((d+2)^3 - (d+2))/6$ .

For every  $d'$ , each factorization consists of a polynomial of degree  $d'$  and a polynomial of degree  $d - d'$ . To describe such factorization, we thus need to describe all  $((d' + 2)^3 - (d' + 2))/6$  coefficients of the first

polynomial and all  $((d - d' + 2)^3 - (d - d' + 2)) / 6$  coefficients of the second polynomial. Thus, the total number of parameters needed to describe such factorizations is equal to  $F(d')/6$ , where we denoted

$$F(d') \stackrel{\text{def}}{=} (d' + 2)^3 - (d' + 2) + (d - d' + 2)^3 - (d - d' + 2).$$

We want to prove that for every  $d' \in (0, d)$ , this value is smaller than  $((d + 2)^3 - (d + 2)) / 6$  – this will show that the dimension is indeed smaller.

We will actually prove that this inequality holds not only for integer values  $d' \in (0, d)$ , but for all possible real values  $d' \in (0, d)$  as well. To prove it, let us consider the corresponding function  $F(d')$ . Its derivative is equal to 0 when  $3 \cdot (d' + 2)^2 - 1 - 3 \cdot (d - d' + 2)^2 + 1 = 0$ , i.e., when  $(d' + 2)^2 = (d - d' + 2)^2$ , hence  $d' + 2 = d - d' + 2$ ,  $d' = d - d'$ , and  $d' = d/2$ . For this value, this function  $F(d')$  clearly has the minimum. Thus, its maxima are at one the endpoints of the interval  $[0, d]$ . Since the values at both endpoints coincide, the maximum is attained at both endpoints. Hence, all intermediate values are smaller than the values at the endpoints – exactly what we tried to prove.

Thus, almost all polynomials of 3 variables are irreducible – and hence, we can almost always uniquely reconstruct a quantum state by measuring location and momentum.

*Comment.* Similar real-valued results about almost always unique reconstruction of a function from the modulus of its Fourier transform are known for images; see, e.g., [1, 4].

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