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Why L^2 Topology in Quantum Physics

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Abstract

It is known that in quantum mechanics, the set S of all possible states coincides with the set of all the complex-valued functions $\psi(x)$ for which $\int |\psi(x)|^2\,dx = 1$. From the mathematical viewpoint, this set is a unit sphere in the space L^2 of all the functions for which the value $\|\psi\|^2 \stackrel{\text{def}}{=} \int |\psi(x)|^2\,dx$ is finite. Because of this mathematical fact, usually the set S is considered with the topology induced by L^2, i.e., topology in which the basis of open neighborhood of a state ψ is formed by the open balls $B_\varepsilon(\psi) = \{\varphi : \|\psi - \varphi\| < \varepsilon\}$. This topology seem to work fine, but since this is a purely mathematical definition, a natural question appears: does this topology have a physical meaning? In this paper, we show that a natural physical definition of closeness indeed leads to the usual L^2-topology.

Keywords: quantum physics, L^2 spaces, topology, foundations of quantum physics

1 Formulation of the Problem

States in quantum mechanics: a brief reminder. It is known that in quantum mechanics, the set S of all possible states coincides with the set of all the complex-valued functions $\psi(x)$ for which $\int |\psi(x)|^2\,dx = 1$; see, e.g., [1].

When we measure the spatial location of a particle, then for each spatial set $X \subseteq R^3$, the probability that the measured spatial location will be inside the set X is equal to $\int_X |\psi(x)|^2\,dx$. From this viewpoint, the above condition $\int |\psi(x)|^2\,dx = 1$ simply reflects the fact that all possible results of measuring location are in the space R^3, so the probability that the measured value belongs to the set $X = R^3$ is equal to 1.

States in quantum mechanics: a usual mathematical description. From the mathematical viewpoint, the set S can be described as follows. We take the set L^2 of all the functions $\psi(x)$ for which the integral $\int |\psi(x)|^2\,dx$ is finite. For functions from this set, we can define the *norm* $\|\psi\|$ as $\|\psi\| \stackrel{\text{def}}{=} \sqrt{\int |\psi(x)|^2\,dx}$, and we can define the distance $d(\psi, \varphi)$ between two functions ψ and φ as the norm of their difference: $d(\psi, \varphi) \stackrel{\text{def}}{=} \|\psi - \varphi\|$.

Once we have defined a metric, we can then define, for each point ψ and for each real number $r \ge 0$, the *ball* $B_r(\psi) = \{\varphi : d(\psi, \varphi) \le r\}$ and the sphere $S_r(\psi) = \{\varphi : d(\psi, \varphi) = r\}$ of radius r with a center in ψ.

In these terms, the above set S is a unit sphere in the set L^2 with a center in 0, i.e., the set

$$S_1(0) = \{\psi : \|\psi - 0\| = 1\} = \{\psi : \|\psi\| = 1\}.$$

Comment. To be more precise, for every real number α, the functions $\psi(x)$ and $e^{\alpha\cdot\mathrm{i}} \cdot \psi(x)$ describe the same physical state. So, the actual distance between the two states is defined as

$$d^2(\psi, \varphi) = \min_{\alpha,\beta} \int \left| e^{\alpha\cdot\mathrm{i}}\psi(x) - e^{\beta\cdot\mathrm{i}} \cdot \varphi(x) \right|^2\,dx.$$

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How topology is usually defined on the set of all quantum states. To describe closeness between states, it is necessary to describe a *topology*. One way to describe a topology is via the notion of a *neighborhood* of a point. In metric spaces (i.e., spaces with metrics), a neighborhood is usually defined as follows: a set U is a neighborhood of a point $x \in U$ if and only if U contains not only the point x itself, but also, for some $\varepsilon > 0$, the entire ball $B_\varepsilon(x)$ with a center in x.

Because of the above metric interpretation of the set S of all quantum states, usually, on this set S, we also consider the topology generated by the L^2 metric, i.e., topology in which neighborhood are defined by the balls

$$B_\varepsilon(\psi) = \{\varphi \in S : d(\psi, \varphi) \le \varepsilon\}.$$

What is the physical meaning of this topology? Pragmatically, this topology seem to work fine – at least it does not seem to lead to any physically meaningless conclusions. However, since this topology is based on a purely mathematical definition, a natural question appears: does this topology have a physical meaning?

What we do in this paper. In this paper, we show that a natural physical definition of closeness indeed leads to the usual L^2-topology.

2 Towards a Physics-motivated Notion of Closeness

Physically meaningful closeness: a natural idea. From the physical viewpoint, two states are close if they lead to similar measurement results.

We would like to use this idea to describe a natural physical topology on the set of all quantum states. To do that, we need to recall the basics of the measurement process in quantum mechanics.

Comment. The above idea of a measurement-related topology has been proposed and analyzed in constructive mathematics – the study of computable mathematical objects – by K. Schultz; it is known as *Shultz topology* [2, 3, 4]. In these terms, what we want to define and analyze is the Schultz topology on the set of all quantum states.

Measurements in quantum mechanics: a brief reminder. In quantum mechanics (see, e.g., [1]), a generic measuring instrument is characterized by its *eigenstates* $e_1, \ldots, e_n, \ldots$, i.e., states that remain intact after the corresponding measurement. After the measurement, the original system is transformed into one of these eigenstates. Different eigenstates are usually characterized by different values of the corresponding measured quantity.

For example, in the case of discrete space-time, the corresponding states are states in which a particle is located with probability 1 at a certain spatial location. These states are characterized by different values of the corresponding measured quantity – spatial coordinates.

The eigenstates are *orthogonal* in the sense that $\langle e_i, e_j \rangle = 0$ for $i \ne j$, where

$$\langle \psi, \varphi \rangle \stackrel{\text{def}}{=} \int \psi(x) \cdot \varphi^*(x)\, dx,$$

and $\varphi^*(x)$ means a complex conjugate number (i.e., $(a + b \cdot \mathrm{i})^* = a - b \cdot \mathrm{i}$). Since e_i are states, we have $\langle e_i, e_i \rangle = \int |e_i(x)|^2\, dx = 1$. Thus, the states e_i for an *orthonormal basis* – and every other function $\psi \in L^2$ can be represented as $\psi = \sum\limits_{i=1}^{\infty} c_i \cdot e_i$. Due to orthonomality, we have $c_i = \langle \psi, e_i \rangle$.

The probability p_i that the state ψ will be transformed into the i-th eigenstate e_i is equal to $p_i = |c_i|^2 = |\langle \psi, e_i \rangle|^2$. This description fits well with the above description of the results of spatial measurements: for discrete space, and for eigenfunctions $e_i(x)$ which are equal to 1 when $x = i$ and to 0 otherwise, we have $\psi(x) = \sum\limits_{i=1}^{\infty} c_i \cdot e_i(x)$ with $c_i = \psi(i)$. Thus, the probability $|c_i|^2$ is exactly $|\psi(x)|^2$.

In general, we have $\langle \psi, \psi \rangle = \|\psi\|^2 = \sum\limits_{i=1}^{\infty} |c_i|^2$. Thus, for quantum states ψ, we have $\|\psi\|^2 = 1$, and hence, $\sum\limits_{i=1}^{\infty} |c_i|^2 = 1$.

A simplified formula for the distance between the states. In the general complex-valued case, a state is defined modulo a multiplicative factor $e^{\alpha\cdot\mathrm{i}}$. Thus, when we compute the distance between the two states, we have to take these factors into account. The corresponding formula can be simplified as follows:

$$d^2(\psi,\varphi)=2-2\cdot|\langle\psi,\varphi\rangle|.$$

Proof of the simplified formula: a brief reminder. Indeed, for every complex number z, we have $|z|^2=z\cdot z^*$, hence

$$\left|e^{\alpha\cdot\mathrm{i}}\cdot\psi(x)-e^{\beta\cdot\mathrm{i}}\cdot\varphi(x)\right|^2=\left(e^{\alpha\cdot\mathrm{i}}\cdot\psi(x)-e^{\beta\cdot\mathrm{i}}\cdot\varphi(x)\right)\cdot\left(e^{-\alpha\cdot\mathrm{i}}\cdot\psi^*(x)-e^{-\beta\cdot\mathrm{i}}\cdot\varphi^*(x)\right)$$
$$=|\psi(x)|^2+|\varphi(x)|^2-e^{(\alpha-\beta)\cdot\mathrm{i}}\cdot\psi(x)\cdot\varphi^*(x)-e^{-(\alpha-\beta)\cdot\mathrm{i}}\cdot\psi^*(x)\cdot\varphi(x).$$

For every real number z, we have $z+z^*=2\cdot\mathrm{Re}(z)$, hence

$$|e^{\alpha\cdot\mathrm{i}}\cdot\psi(x)-e^{\beta\cdot\mathrm{i}}\cdot\varphi(x)|^2=|\psi(x)|^2+|\varphi(x)|^2-2\cdot\mathrm{Re}\left(e^{(\alpha-\beta)\cdot\mathrm{i}}\cdot\psi(x)\cdot\varphi^*(x)\right).$$

Thus,

$$d^2(\psi,\varphi)=\int\left|e^{\alpha\cdot\mathrm{i}}\cdot\psi(x)-e^{\beta\cdot\mathrm{i}}\cdot\varphi(x)\right|^2\,dx$$
$$=\int|\psi(x)|^2\,dx+\int|\varphi(x)|^2\,dx-2\cdot\mathrm{Re}\left(e^{(\alpha-\beta)\cdot\mathrm{i}}\cdot\int\psi(x)\cdot\varphi^*(x)\,dx\right).$$

Since $\|\psi\|=1$, and $\|\psi\|^2=\int|\psi(x)(x)|^2\,dx$, we thus get $\int|\psi(x)|^2\,dx=1$ and similarly, $\int|\varphi(x)|^2\,dx=1$. Thus, we have

$$d^2(\psi,\varphi)=\int\left|e^{\alpha\cdot\mathrm{i}}\cdot\psi(x)-e^{\beta\cdot\mathrm{i}}\cdot\varphi(x)\right|^2\,dx=2-2\cdot\mathrm{Re}\left(e^{(\alpha-\beta)\cdot\mathrm{i}}\cdot\langle\psi,\varphi\rangle\right).$$

This value is the smallest if and only if the subtracted real part is the largest. Since $\left|e^{(\alpha-\beta)\cdot\mathrm{i}}\right|$, the real part is the largest when it is equal to the absolute value of $\langle\psi,\varphi\rangle$, hence $d^2(\psi,\varphi)=2-2\cdot|\langle\psi,\varphi\rangle|$.

For simplicity, let us first restrict ourselves to real-valued states. In reality, a general state contains complex-values coefficients c_i. For clarity of a physical argument, let us first restrict ourselves to the case when the values are real and non-negative. In this case, $p_i=c_i^2$, and, vice versa, $c_i=\sqrt{p_i}$ – and there is no need to consider factors $e^{\alpha\cdot\mathrm{i}}$.

We first show that the natural physical definition leads to L^2 topology for such real-valued states; a similar argument for the general complex-valued case will be describe in the following section.

Towards the definition of closeness. It is natural to define a neighborhood of a state ψ as the set of all the states φ for which the results of some measurement are close – e.g., ε-close for some ε.

To escribe a measurement, we need to select a basis $e_1,\ldots,e_n,\ldots$ This basis describes an idealized measurement, in which, in principle, we can get infinitely many possible measurement results. In practice, every real measuring instrument can only distinguish between finally many measurement results. Thus, it is reasonable to only consider finite sequences $e_1,\ldots,e_n$.

As a result of the measurement, we get different probabilities c_i. Thus, we say that the states ψ and φ are close if the corresponding probabilities differ by $\le\varepsilon$. Thus, we arrive at the following definition.

Resulting definition. Let $e_1,\ldots,e_n,\ldots$ be an orthonormal basis in the state L^2. We say that a set U is a *physical neighborhood* of a state $x\in U$ if for some n and ε, the set U contains all the states $\varphi\in S$ for which

$$\left||\langle\psi,e_i\rangle|^2-|\langle\varphi,e_i\rangle|^2\right|\le\varepsilon$$

for all $i=1,\ldots,n$.

3 Main Result: Real-valued Case

Two notions of closeness. Now, we have two different notions of a neighborhood on the set S of all quantum states:

- the traditional L^2 definition of a neighborhood, and
- the new definition of a physical neighborhood.

Our main result is that these notions coincide.

Main result. *For every state $\psi \in S$ and every set $U \ni \psi$, the following two properties are equivalent to each other:*

- *the set U is an L^2 neighborhood of the state ψ, and*
- *the set U is a physical neighborhood of the state ψ.*

Proof.

1°. Let us first assume that a set U is an L^2 neighborhood of the state ψ, and let us prove that in this case, this set U is also a physical neighborhood of ψ.

1.1°. By definition of an L^2 neighborhood, our assumption means that there exists some value $r > 0$ for which $\|\psi - \varphi\| \le r$ implies that $\varphi \in U$. To, to prove our conclusion, it is sufficient to find n and ε for which

$$\left| |\langle \psi, e_i \rangle|^2 - |\langle \varphi, e_i \rangle|^2 \right| \le \varepsilon$$

for all $i = 1, \ldots, n$ implies that $\|\psi - \varphi\| \le r$.

1.2°. Let us denote the values $|\langle \psi, e_i \rangle|^2$ by p_i and the values $|\langle \varphi, e_i \rangle|^2|$ by q_i. In these terms, we have $\sum\limits_{i=1}^{\infty} p_i = 1$, $\sum\limits_{i=1}^{\infty} q_i = 1$, $\psi = \sum\limits_{i=1}^{\infty} \sqrt{p_i} \cdot e_i$ and $\varphi = \sum\limits_{i=1}^{\infty} \sqrt{q_i} \cdot e_i$, hence $\psi - \varphi = \sum\limits_{i=1}^{\infty} (\sqrt{p_i} - \sqrt{q_i}) \cdot e_i$, and thus, $\|\psi - \varphi\|^2 = \sum\limits_{i=1}^{\infty} (\sqrt{p_i} - \sqrt{q_i})^2$. In these terms, the desired statement has the following form:

Let $p_i \ge 0$ be a sequence for which $\sum\limits_{i=1}^{\infty} p_i = 1$, and let $r > 0$ be a real number. We want to find the values n and ε for which, for every sequence $q_i \ge 0$ for which the conditions $\sum\limits_{i=1}^{\infty} q_i = 1$, $|p_1 - q_1| \le \varepsilon$, $\ldots$, and $|p_n - q_n| \le \varepsilon$ imply that $\sum\limits_{i=1}^{\infty} (\sqrt{p_i} - \sqrt{q_i})^2 \le r^2$.

1.3°. If $p_i \ge q_i$, then $0 \le \sqrt{p_i} - \sqrt{q_i} \le \sqrt{p_i} + \sqrt{q_i}$ and thus,

$$(\sqrt{p_i} - \sqrt{q_i})^2 \le (\sqrt{p_i} - \sqrt{q_i}) \cdot (\sqrt{p_i} + \sqrt{q_i}) = p_i - q_i = |p_i - q_i|.$$

Similarly, when $p_i \ge q_i$, we also have

$$(\sqrt{p_i} - \sqrt{q_i})^2 \le |p_i - q_i|.$$

Thus, this inequality holds for all possible values p_i and q_i.

So, we have

$$\sum_{i=1}^{\infty} (\sqrt{p_i} - \sqrt{q_i})^2 \le \sum_{i=1}^{\infty} |p_i - q_i|.$$

Therefore, to prove the above result, it is sufficient to find the values n and ε for which, for every sequence $q_i \ge 0$ for which the conditions $\sum\limits_{i=1}^{\infty} q_i = 1$, $|p_1 - q_1| \le \varepsilon$, $\ldots$, and $|p_n - q_n| \le \varepsilon$ imply that $\sum\limits_{i=1}^{\infty} |p_i - q_i| \le r^2$.

We assumed that $\sum\limits_{i=1}^{\infty} p_i = 1$. By definition of an infinite sum, this means that $\sum\limits_{i=1}^{n} p_i \to 1$ as $n \to \infty$. Thus, there exists an n for which $\sum\limits_{i=1}^{n} p_i \ge 1 - r^2/4$. Since

$$\sum_{i=1}^{\infty} p_i = \sum_{i=1}^{n} p_i + \sum_{i=n+1}^{\infty} p_i = 1,$$

this inequality implies that

$$\sum_{i=n+1}^{\infty} p_i \le \frac{r^2}{4}.$$

Let us now take $\varepsilon \stackrel{\text{def}}{=} r^2/(4 \cdot n)$. For this choice, the inequality $|p_i - q_i| \le \varepsilon$ implies that $q_i \ge p_i - \varepsilon$ hence

$$\sum_{i=1}^{n} q_i \ge \sum_{i=1}^{n} p_i - n \cdot \varepsilon = \sum_{i=1}^{n} p_i - \frac{r^2}{4}.$$

Since we selected n as the value for which $\sum\limits_{i=1}^{n} p_i \ge 1 - r^2/4$, we thus conclude that

$$\sum_{i=1}^{n} q_i \ge \sum_{i=1}^{n} p_i - \frac{r^2}{4} \ge (1 - \frac{r^2}{4}) - \frac{r^2}{4} = 1 - \frac{r^2}{2}.$$

Similarly, $|p_i - q_i| \le \varepsilon$ implies that

$$\sum_{i=1}^{\infty} |p_i - q_i| \le n \cdot \varepsilon = \frac{r^2}{4}.$$

We want to prove that $\sum\limits_{i=1}^{\infty} |p_i - q_i| \le r^2$. This infinite sum can be represented as

$$\sum_{i=1}^{\infty} |p_i - q_i| = \sum_{i=1}^{n} |p_i - q_i| + \sum_{i=n+1}^{\infty} |p_i - q_i|.$$

For every $i > n$, we have $|p_i - q_i| \le p_i + q_i$, hence

$$\sum_{i=1}^{\infty} |p_i - q_i| = \sum_{i=1}^{n} |p_i - q_i| + \sum_{i=n+1}^{\infty} p_i + \sum_{i=n+1}^{\infty} q_i.$$

We know that the first sum in the right-hand side is bounded by $r^2/4$, the second sum by $r^2/4$, and the third sum by $r^2/2$. Thus,

$$\sum_{i=1}^{\infty} |p_i - q_i| \le \frac{r^2}{4} + \frac{r^2}{4} + \frac{r^2}{2} = r^2.$$

The statement is proven.

2°. Let us now assume that a set U is a physical neighborhood of ψ. Let us prove that in this case, this set U is also an L^2-neighborhood of ψ.

In terms of the values p_i and q_i, this means the following. Let $p_i \ge 0$ be a sequence for which $\sum\limits_{i=1}^{\infty} p_i = 1$, let n be an integer, and let $\varepsilon > 0$ be a real number. We want to find the value r such that if $\sum\limits_{i=1}^{\infty} q_i = 1$ and $\sum\limits_{i=1}^{\infty} (\sqrt{p_i} - \sqrt{q_i})^2 \le r^2$, then we have $|p_i - q_i| \le \varepsilon$ for all $i = 1, \ldots, n$.

It turns out that we can take $r = \varepsilon/2$. Indeed, for each i, we have

$$(\sqrt{p_i} - \sqrt{q_i})^2 \le \sum_{i=1}^{\infty} (\sqrt{p_i} - \sqrt{q_i})^2 \le r^2,$$

hence $|\sqrt{p_i} - \sqrt{q_i}| \le r$. For the desired difference $p_i - q_i$, we have

$$p_i - q_i = (\sqrt{p_i} - \sqrt{q_i}) \cdot (\sqrt{p_i} + \sqrt{q_i}),$$

hence

$$|p_i - q_i| = |\sqrt{p_i} - \sqrt{q_i}| \cdot (\sqrt{p_i} + \sqrt{q_i}).$$

Since $p_1 \le 1$ and $q_i \le 1$, we have $\sqrt{p_i} \le 1$, $\sqrt{q_i} \le 1$, and $\sqrt{p_i} + \sqrt{q_i} \le 2$. Thus, $|p_i - q_i| \le 2 \cdot |\sqrt{p_i} - \sqrt{q_i}|$. We already know that $|\sqrt{p_i} - \sqrt{q_i}| \le r$, so we conclude that $|p_i - q_i| \le 2 \cdot r = \varepsilon$. The statement is proven.

4 Main Result: General Complex-valued Case

General complex-valued case: why we need a different definition. In the real-valued case, it was sufficient to consider functions e_i corresponding to a single measurement procedure. In the complex-valued case, this is not sufficient, since by these measurements, we will not be able to distinguish between the state $(\sqrt{2}/2) \cdot e_1 + (\sqrt{2}/2) \cdot e_2$ and the state $(\sqrt{2}/2) \cdot \mathrm{i} \cdot e_1 + (\sqrt{2}/2) \cdot e_2$, for which the probabilities $p_i = |c_i|^2$ are the same.

Definition of closeness: general complex-valued case. In the general case, let us assume that we have functions $e_1, e_2, \ldots$ that are everywhere dense in the state of all the states S.

For example, we can assume that we have a fixed orthonormal basis $f_1, f_2, \ldots$, and we have functions of the type $e_i = \sum\limits_{j=1}^{n} r_{ij} \cdot f_j$ for rational values r_{ij}.

In this case, we can define a physical neighborhood in a similar way: a set U is a *physical neighborhood* of a state $x \in U$ if for some n and ε, the set U contains all the states $\varphi \in S$ for which

$$\left| |\langle \psi, e_i \rangle|^2 - |\langle \varphi, e_i \rangle|^2 \right| \le \varepsilon$$

for all $i = 1, \ldots, n$.

In this case, we have a similar result:

Main result. *For every state $\psi \in S$ and every set $U \ni \psi$, the following two properties are equivalent to each other:*

- *the set U is an L^2 neighborhood of the state ψ, and*
- *the set U is a physical neighborhood of the state ψ.*

Proof.

$1°$. Let us first assume that a set U is an L^2-neighborhood of ψ, and let us show that it is also a physical neighborhood of ψ.

In other words, we assume that we have a state ψ and a real number $r > 0$. We want to show that there exists n and $\varepsilon > 0$ such that any state φ with $\|\varphi\| = 1$ and $\left| |\langle \psi, e_i \rangle|^2 - |\langle \varphi, e_i \rangle|^2 \right| \le \varepsilon$ for $i = 1, \ldots, n$ satisfies the condition $d(\psi, \varphi) \le r$.

We want to impose the condition on the distance $d(\psi, \varphi)$. Due to the triangle inequality, we have

$$d(\psi, \varphi) \le d(\psi, e_n) + d(e_n, \varphi).$$

So, if we choose n and ε in such a way that $d(\psi, e_n) \le r/2$ and $d(e_n, \varphi) \le r/2$, then we will guarantee that $d(\psi, \varphi) \le r$.

Since the sequence e_i is everywhere dense, for every $\delta > 0$, there exists an n for which $d(\psi, e_n) \le \delta$. To achieve the desired inequality $d(\psi, e_n) \le r/2$, we must select δ in such a way that $\delta \le r/2$. For example, we can take $\delta = r/4$.

Using the above simplified formula for the distance, we conclude that

$$d^2(\psi, e_n) = 2 - 2|\langle \psi, e_n \rangle| \le \delta^2,$$

hence $|\langle\psi,e_n\rangle|\ge 1-(\delta^2)/2$ and $|\langle\psi,e_n\rangle|^2\ge\left(1-(\delta^2)/2\right)^2$. So, if

$$\left||\langle\psi,e_n\rangle|^2-|\langle\varphi,e_n\rangle|^2\right|\le\varepsilon,$$

then we will be able to conclude that

$$|\langle\varphi,e_n\rangle|^2\ge|\langle\psi,e_n\rangle|^2-\varepsilon\ge\left(1-\frac{\delta^2}{2}\right)^2-\varepsilon.$$

As a result, we get

$$|\langle\varphi,e_n\rangle|\ge\sqrt{\left(1-\frac{\delta^2}{2}\right)^2-\varepsilon}.$$

We want to make sure that $d(\varphi,e_n)\le r/2$, i.e., that $d^2(\varphi,e_n)\le(r^2)/4$. From the simplified formula for the distance and from the above inequality, we conclude that

$$d^2(\varphi,e_n)=2-2\cdot|\langle\varphi,e_n\rangle|\le 2-2\cdot\sqrt{\left(1-\frac{\delta^2}{2}\right)^2-\varepsilon}.$$

Thus, to prove our result, we must select the values δ and ε in such a way that $\delta\le r/2$ and

$$2-2\cdot\sqrt{\left(1-\frac{\delta^2}{2}\right)^2-\varepsilon}\le\frac{r^2}{4}.$$

Dividing both sides of this inequality by 2 and moving the square to the other side, we get an equivalent inequality

$$\sqrt{\left(1-\frac{\delta^2}{2}\right)^2-\varepsilon}\ge 1-\frac{r^2}{8}.$$

By taking squares of both sides, we get an equivalent inequality

$$\left(1-\frac{\delta^2}{2}\right)^2-\varepsilon\ge\left(1-\frac{r^2}{8}\right)^2,$$

or, equivalently,

$$\varepsilon\le\left(1-\frac{\delta^2}{2}\right)^2-\left(1-\frac{r^2}{8}\right)^2.$$

Since we selected $\delta=r/4$, we thus need to take

$$\varepsilon\le\left(1-\frac{r^2}{32}\right)^2-\left(1-\frac{r^2}{8}\right)^2.$$

The statement is proven.

2°. Let us now assume that a set U is a physical neighborhood of ψ. Let us prove that in this case, this set U is also an L^2-neighborhood of ψ.

In other words, we are given a state ψ, an integer n, and a real number $\varepsilon>0$. We want to find the value r such that if for some φ for which $\|\varphi\|=1$, we have $d(\psi,\varphi)\le r$, then $\left||\langle\psi,e_i\rangle|^2-|\langle\varphi,e_i\rangle|^2\right|\le\varepsilon$ for all $i=1,\ldots,n$.

It turns out that, similarly to the real-valued case, we can take $r=\varepsilon/2$.

By definition, the distance $d(\psi,\varphi)$ is the smallest of the values $\left\|e^{\alpha\cdot\mathrm{i}}\cdot\psi-e^{\beta\cdot\mathrm{i}}\cdot\varphi\right\|$. Let us select α and β for which this minimum is attained, and let us replace the original functions $\psi(x)$ and $\varphi(x)$ with functions $e^{\alpha\cdot\mathrm{i}}\cdot\psi$ and $e^{\beta\cdot\mathrm{i}}\cdot\varphi$ that represent the same states. After this replacement, we get $d(\psi,\varphi)=\|\psi-\varphi\|=r$.

We have

$$|\langle\psi,e_i\rangle|^2-|\langle\varphi,e_i\rangle|^2=(|\langle\psi,e_i\rangle|-|\langle\varphi,e_i\rangle|)\cdot(|\langle\psi,e_i\rangle|+|\langle\varphi,e_i\rangle|),$$

hence

$$\left||\langle\psi, e_i\rangle|^2 - |\langle\varphi, e_i\rangle|^2\right| = (||\langle\psi, e_i\rangle| - |\langle\varphi, e_i\rangle||) \cdot (|\langle\psi, e_i\rangle| + |\langle\varphi, e_i\rangle|).$$

Here, $|\langle\psi, e_i\rangle| \le \|\psi| \cdot \|e_i\| = 1 \cdot 1 = 1$ and similarly, $|\langle\varphi, e_i\rangle| \le 1$, thus, $|\langle\psi, e_i\rangle| + |\langle\varphi, e_i\rangle| \le 2$ and

$$\left||\langle\psi, e_i\rangle|^2 - |\langle\varphi, e_i\rangle|^2\right| \le 2 \cdot |(|\langle\psi, e_i\rangle| - |\langle\varphi, e_i\rangle|)|\,.$$

For all complex numbers a and b, we have $|a| \le |b| + |a - b|$ and $|a| \le |b| + |b - a| = |b| + |a - b|$, hence $|a| - |b| \le |a - b|$ and $|a| - |b| \le |a - b|$. Since the absolute value $|(|a| - |b|)|$ is equal to either $|a| - |b|$ or to $|b| - |a|$, and both these numbers are $\le |a - b|$, we thus conclude that $||a| - |b|| \le |a - b|$. Thus,

$$|(|\langle\psi, e_i\rangle| - |\langle\varphi, e_i\rangle|)| \le |\langle\psi - \varphi, e_i\rangle|.$$

Here,

$$|\langle\psi - \varphi, e_i\rangle| \le \|\psi - \varphi\| \cdot \|e_i\| \le r \cdot 1 = 1,$$

hence

$$\left||\langle\psi, e_i\rangle|^2 - |\langle\varphi, e_i\rangle|^2\right| \le 2 \cdot r = \varepsilon.$$

The statement is proven.

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