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A New Justification for Weighted Average Aggregation in Fuzzy Techniques

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Abstract

In many practical situations, we need to decide whether a given solution is good enough, based on the degree a_i to which different criteria are satisfied. In this paper, we show that natural requirements lead to the weighted average decision, according to which a solution is acceptable if $\sum w_i \cdot a_i \geq t$ for some weights w_i and threshold t.

Keywords: fuzzy degrees, weighted average, decision making

1 Formulation of the Problem

In many practical situations, we need to decide whether to accept or to continue improving. In many practical situations, we want to come up with a good solution, so we start with some solution and keep improving it until we decide that this solution is good enough.

For example, this is how software is designed: we design the first version, test it, if the results are satisfactory, we release it, otherwise, if this version still has too many bugs, we continue improving it. Similarly, when a legislature works on a law (e.g., on an annual state budget), it starts with some draft version. If the majority of the legislators believe that this budget is good enough, the budget is approved, otherwise, the members of the legislature continue working on it until the majority is satisfied. Yet another example is home remodeling: the owners hire a company, the company produces a remodeling plan. If the owners are satisfied with this plan, the remodeling starts, if not, the remodeling company makes changes and adjustments until the owners are satisfied.

In many such situations, we only have fuzzy evaluations of the solution's quality. In some cases, the requirements are precisely formulated. For example, for software whose objective is to control critical systems such as nuclear power plants or airplanes, we usually have very precise specifications, and we do not release the software until we are 100% sure that the software satisfies all these specifications.

However, in most other situations, the degree of satisfaction is determined subjectively. Usually, there are several criteria that we want the solution to satisfy. For example, the budget must not contain too many cuts in important services, not contain drastic tax increases, be fair to different parts of the population and to different geographic areas. In many situations, these criteria are not precise, so the only way to decide to what extent each of these criteria is satisfied it to ask experts.

It is natural to describe the experts' degree of satisfaction in each criterion by a real number from the interval $[0, 1]$ so that 0 means no satisfaction at all, 1 means perfect satisfaction, and intermediate values mean partial satisfaction. This is exactly what fuzzy techniques start with.

Many methods are known to elicit the corresponding values from the experts; see, e.g., [1]. For example, if each expert is absolutely confident about whether the given solution satisfies the given criterion or not, we can take, as degree of satisfaction, the proportion of experts who considers this solution satisfactory. For example, if 60% of the experts considers the given aspect of the solution to be satisfactory, then we say that the expert's degree of satisfaction is 0.6. This is how decisions are usually made in legislatures.

In many practical situations, however, experts are not that confident; each expert, instead of claiming that the solution is absolutely satisfactory or absolutely unsatisfactory, feels much more comfortable marking his or her degree of satisfaction on a scale – e.g., on a scale from 0 to 5. This is, e.g., how in the US universities, students evaluate their professors. If a student marks 4 on a scale from 0 to 5 as an answer to the question

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"Is a professor well organized?", then we can say that the student's degree of satisfaction with the professor's organization of the class is $4/5 = 0.8$. The degrees corresponding to several students are then averaged to form the class evaluation. Similarly, in general, the experts' estimates are averaged.

Formulation of the problem. Let us assume that we have several (n) criteria. For a given solution, for each of these criteria, we ask the experts and come up with a degree a_i to which – according to the experts — this particular criterion is satisfied. We need to come up with a criterion that enables us, based on these n numbers $a_1, \ldots, a_n \in [0, 1]$, to decide whether solution as a whole is satisfactory to us.

2 Solution

Towards a formal description of the problem. We need to divide the unit cube $[0, 1]^n$ – the set of all possible values of the tuple $a = (a_1, \ldots, a_n)$ – into two complimentary sets: the set S of all the tuples for which the solution is accepted as satisfactory, and the set U of all the tuples for which the solution is rejected as unsatisfactory.

Natural requirements. Let us assume that we have two groups of experts whose tuples are a and b, and that, according to both tuples, we conclude that the solution is satisfactory, i.e., that $a \in S$ and $b \in S$. It is then reasonable to require that if we simply these two groups of experts together, we will still come up with a satisfactory decision.

Similarly, it is reasonable to conclude that if two groups decide that the solution is unsatisfactory, then by combining their estimates, we should still be able to conclude that the solution is unsatisfactory.

According to our description, when we have two groups of experts consisting of n_a and n_b folks, then, to form a joint tuple, we combine the original tuples with the weights proportional to these numbers, i.e., we consider the tuple

$$c = \frac{n_a}{n_a + n_b} \cdot a + \frac{n_b}{n_a + n_b} \cdot b.$$

Thus, we conclude that if $a, b \in S$ and $r \in [0, 1]$ is a rational number, then $r \cdot a + (1 - r) \cdot b \in S$.

It is also reasonable to require that if, instead of simply averaging, we use arbitrary weights to take into account that some experts are more credible, we should also be able to conclude that the combined group of experts should lead to a satisfactory decision. In other words, we conclude that if $a, b \in S$ and r is an arbitrary number from the interval $[0, 1]$ is a rational number, then we should have $r \cdot a + (1 - r) \cdot b \in S$. In mathematical terms, this means that the set S is *convex*.

Similarly, if $a, b \in U$ and $r \in [0, 1]$, then $r \cdot a + (1 - r) \cdot b \in U$. Thus, the complement U to the set S should be convex.

Analysis of the requirement. Two disjoint convex sets can always be separated by a half-plane; see, e.g., [2]. In this case, all the satisfactory tuples are on one side, all unsatisfactory points are on the other side. A general hyper-plane can be described by linear equations $\sum w_i \cdot x_i = t$, so all the S-points correspond to $\sum w_i \cdot x_i \geq t$ and all the U points to $\sum w_i \cdot x_i \leq t$.

Conclusion. We have shown that reasonable conditions on decision making indeed leads to the weighted average.

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