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The Uncertainty Principle and Large-Genus Surfaces

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Abstract

The uncertainty principle is examined in string theory with regard to a lower bound on the length of closed geodesics on the Euclideanized string worldsheet. A connection between the string uncertainty relation and curvature contributions is found.

Keywords: generalized uncertainty, minimal length uncertainty, curvature, string theory, boundary states

1 Introduction

It has been noted that, in contrast to particle theories, the string does not appear to be localized as a result of quantum fluctuations. The result follows from the curvature of the string worldsheet which introduces an extra amount of uncertainty in the proper distance in the embedding space-time between adjacent points on the manifold.

Amongst the different derivations of the minimal uncertainty in the length in string theory is the derivation of nontrivial commutators as Poisson brackets from an action that is equivalent to the Polyakov action at the quantum level. It follows that the derivation can be extended to arbirary genus surfaces. A contribution to the commutator of the position and momentum coordinates in curved space is known to include the curvature. The curvature is defined with respect to the target space rather than the Riemann surface, which is consistent with an uncertainty relation for coordinates in the embedding space-time.

The effect of the space-time curvature on the uncertainty relation follows from its derivation from the equivalent term in the commutator. Evaluation of the trace of this inequality and the average over coordinate axes yields a quadratic equation for the uncertainty in the position, Δx^A. The two solutions to this inequality produce different types of uncertainty relations. The first has a flat-space limit with additional terms decreasing with respect to the curvature. The second has a similar form to that found in string theory, and a minimum length uncertainty would be predicted.

The extension of the uncertainty relations to fermions reflects a connection between the basic parameters of the spin system and the curvature of the space-time. Instead of a conventional inequality for the product of $\Delta x \Delta p$, a constant term is added in the formula for $\overline{(\Delta x)^2}$. This term has the effect of producing an inequality for $\langle \Delta x \rangle$, which is analogous to the string theory uncertainty relation. The coefficient is related to the mass of the fermion, and the equivalence with the inequality in curved space can be established through a proportionality between the curvature and the squared mass.

The expansion of the string scattering amplitudes is given by an integration over moduli space followed by a sum over the genus. The domain of string perturbation theory has been postulated to consist of surfaces confined to a finite interaction region and characterized by an arbitrary number of handles including those effectively closed infinite-genus surfaces with accumulating handles of decreasing size [3]. The specification of a size of a handle that may be arbitrarily small and even less than the Planck length shall be reviewed in the context on the uncertainty principle in string theory, which sets a lower bound on observable distances.

It will be concluded that the virtual nature of the process has a role in determining whether the minimum defined by the uncertainty principle is applicable. The relation with other finite-size effects such as the addition of boundary states to the string amplitudes is discussed.

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2 The Uncertainty Principle in String Theory

Several derivations of the uncertainty principle in string theory have been given. First, it has been been shown that it can be derived through discretization of the string worldsheet independently of the fine graining [9]. Secondly, the introduction of curvature yields a curvature-dependent contribution to the commutator of the position and momentum operators that implies a minimal uncertainty in the distance [8]. Thirdly, there is a version of the string action which, upon canonical quantization, yields a non-zero commutator for the string coordinates [10].

Consider a discretization of the worldsheet and let the partition function be defined by

$$Z = lim_{N,M\to\infty} \prod_{i,j} \int dx_{i,j} \sqrt{g(x_{i,j})} exp\left(-\frac{1}{2\lambda^2}\sum[\sigma(x_{i+1j};x_{i,j})+\sigma(x_{i,j+1},x_{i,j})]\right)$$

$$\lambda = \left(\frac{2\alpha'\hbar}{c^2}\right) \tag{2.1}$$

where

$$\sigma(x,x') = g_{\mu\nu}(x)\xi^\mu(x,x')\xi^\nu(x,x') \tag{2.2}$$

and

$$g_{\mu\nu} = \delta_{\mu\nu} + \frac{1}{3}x^\lambda x^\kappa R_{\mu\lambda\kappa\nu}(0) + ... \qquad \xi^\mu(x,x') = \Delta x^\mu + \frac{1}{3}R^\mu{}_{\nu_1\lambda\nu_2}(\Delta x)^{\nu_1}(\Delta x)^{\nu_2}x^\lambda \tag{2.3}$$

as an approximation to

$$\int \mathcal{D}X(\sigma,\tau) exp\left(-\frac{1}{2\lambda^2}\int d\tau d\sigma\sqrt{h}h^{\alpha\beta}\partial_\sigma X^\mu \partial_\alpha X^\nu g_{\mu\nu}\right). \tag{2.4}$$

By the Migdal-Kadanoff approximation, the sums over the vertical and horizontal sides of the plaquettes may be replaced by sums without intermediate edges with a compensating numerical factor. For example,

$$\sum_{vert}\sigma \to \frac{1}{2}\sum_{vert}{}''(\sigma - \frac{1}{6}\lambda^2 R_{\mu\nu}\xi^\mu\xi^\nu + \frac{1}{3}\lambda^4 R) \tag{2.5}$$

in substitution of a square grid by a grid with the twice the spacing. To reproduce the same model, $R_{\mu\nu}$ must be set equal to zero, which can be achieved through a field redefinition $g_{\mu\nu} \to g_{\mu\nu} - \frac{1}{6}\epsilon\hbar R_{\mu\nu}$, $\lambda = \epsilon\hbar$. After this redefinition, it is found that the expectation of the square of the distance to the next neighbouring point on the lattice is equal to the original space

$$\langle\sigma(x_{i+2,j};x_{i,j})\rangle \simeq \lambda^2. \tag{2.6}$$

Regardless of the discretization of the worldsheet, the proper distance between adjacent points is found to be bounded below [9]. There is an additional contribution to the uncertainty relation

$$\Delta x \geq \lambda\left(\frac{\kappa}{2\Delta p} + \frac{\Delta p}{\kappa}\right) = \frac{\hbar}{2\Delta p} + const.\alpha'\Delta p \tag{2.7}$$

and a minimum value at the Planck length ℓ_P.

Another derivation of the result [8] follows from the Poisson brackets derived from an action that is classically equivalent to the Nambu-Goto action and equivalent to the Polyakov action at the quantum level

$$S_n = \int d^2\xi e\left\{\frac{1}{e^n}\left[\frac{1}{2\lambda^2}(\epsilon^{ab}\partial_a X^\mu \partial_b X_\mu)^2\right]^{\frac{n}{2}} + n - 1\right\}. \tag{2.8}$$

For $n = 2$, the Poisson bracket of the coordinate fields has the form

$$\{X^\mu, X^\nu\} = \frac{1}{e}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu. \tag{2.9}$$

With the Lagrange multiplier condition

$$\frac{1}{2}(\{X^\mu, X^\nu\}_{P.B})^2 = \lambda^2 \tag{2.10}$$

implying a minimal uncertainty in the position $\Delta x \geq \lambda$ at the quantum level.

3 Curvature Contribution to Uncertainty Principle

It has been shown also that any curvature in space-time induces additional terms in the commutation relations between the position and momentum operators

$$[x^\mu, p_\nu] = i\hbar\delta^\mu{}_\nu - \frac{i\hbar}{6}(R^\mu{}_{\alpha\nu\beta} + R^\mu{}_{\beta\nu\alpha})x^\alpha x^\beta + ... \tag{3.1}$$

as a result of the Lie transport of the position vector and the expansion of the metric in normal coordinates [10].

The modified momentum operator yields the following inequality

$$\Delta p^A \Delta x^A - \frac{1}{6}(R^A{}_{BCD} + R^A{}_{CBD})\Delta x^B \Delta x^C \geq \frac{\hbar}{2}\delta_D{}^A. \tag{3.2}$$

The trace is

$$\frac{2}{3}R_{BC}\Delta x^B \Delta x^C + \Delta p_A \Delta x^A - \frac{D}{2}\hbar \geq 0. \tag{3.3}$$

Averaging over coordinate directions, $\langle \Delta x_A \Delta x^A \rangle \sim D\langle \Delta x^A \rangle^2$, where $\langle \Delta X^A \rangle$ refers to a single coordinate direction. Similarly, $\langle \Delta p_A \Delta x^A \rangle \sim D\langle \Delta p^A \rangle \langle \Delta x^A \rangle$, with $\langle \Delta p^A \rangle$ being expectation value of the uncertainty in the momentum in a single coordinate direction.

Since $\Delta x^A \Delta x_A = g_{BC}\Delta x^B \Delta x^C$ and $\langle \Delta x^B \Delta x^C \rangle \sim \frac{1}{D} g^{BC} \langle \Delta x_A \Delta x^A \rangle$,

$$\frac{2}{3}R_{BC}\frac{1}{D}g^{BC} \cdot D\langle \Delta x^A \rangle^2 + D\langle \Delta p^A \Delta x^A \rangle - \frac{D}{2}\hbar \geq 0 \tag{3.4}$$

and

$$\frac{2}{3}R\langle \Delta x^A \rangle^2 + D\langle \Delta p^A \rangle \langle \Delta x^A \rangle - \frac{D}{2}\hbar \geq 0. \tag{3.5}$$

The solutions are

$$\langle \Delta x^A \rangle \geq -\frac{3}{4}DR^{-1}\langle \Delta p^A \rangle \pm \frac{3}{4}R^{-1}D\langle \Delta p^A \rangle \left[1 + \frac{4}{3}\frac{R\hbar}{D\langle \Delta p^A \rangle^2} \right]^{\frac{1}{2}}. \tag{3.6}$$

Expanding

$$\left[1 + \frac{4}{3}\frac{R\hbar}{D\langle \Delta p^A \rangle^2}\right]^{\frac{1}{2}} = 1 + \frac{2}{3}\frac{R\hbar}{D\langle \Delta p^A \rangle^2} - \frac{2}{9}\frac{R^2\hbar^2}{D^2\langle \Delta p^A \rangle^4} + ... \tag{3.7}$$

the two solutions are

$$\begin{aligned} \langle \Delta x^A \rangle &\geq \frac{3}{4}R^{-1}D\langle \Delta p^A \rangle \cdot \left[\frac{2}{3}\frac{R\hbar}{D\langle \Delta p^A \rangle^2} - \frac{2}{9}\frac{R^2\hbar^2}{D^2\langle \Delta p^A \rangle^4} + ...\right] \\ &= \frac{\hbar}{2}\frac{1}{\langle \Delta p^A \rangle} - \frac{R\hbar^2}{6D\langle \Delta p^A \rangle^3} + ... \end{aligned} \tag{3.8}$$

and

$$\langle \Delta x^A \rangle \geq -\frac{3}{2}DR^{-1}\langle \Delta p^A \rangle - \frac{\hbar}{2}\frac{1}{\langle \Delta p^A \rangle} + ... \tag{3.9}$$

In the first inequality, the curvature correction appears with decreasing powers of $\langle \Delta p^A \rangle$, and it increases with R. While the flat-space limit follows immediately, there is no direct comparison with the string uncertainty relation. The second uncertainty relation has a form similar to that of string theory. While $\frac{\hbar}{2\langle \Delta p^A \rangle}$ is a flat-space term that would be unchanged by curvature corrections, $|R^{-1}|$ should be considerably larger than α'.

From Eq.(3.3), a similarity transformation which diagonalizes the matrix represented by the Ricci tensor, $\Lambda(R)_{BC}\Lambda^{-1} = (R')_{BC}$, yields

$$\frac{2}{3}(R')_{AA}(\Delta x'^A)^2 + (\Delta\vec{p}\Lambda^{-1})_A(\Delta x'^A) - c^A = 0$$

$$\sum_A c^A = \frac{D}{2}\hbar, \tag{3.10}$$

where $\Delta\vec{x}' = \Lambda\Delta\vec{x}$. The two solutions are

$$\Delta x'^A = \frac{c^A}{(\Delta\vec{p}\Lambda^{-1})_A} - \frac{2}{3}\frac{(R')_{AA}c^A}{(\Delta\vec{p}\Lambda^{-1})^3_A} + ... \tag{3.11}$$

and

$$\Delta x'^A = -\frac{3}{2}(R')^{-1}_{AA}(\Delta p'\Lambda^{-1})_A - \frac{c^A}{(\Delta\vec{p}\Lambda^{-1})_A} + ... \,. \tag{3.12}$$

Again, the difference in sign of the terms proportional to $\frac{1}{\Delta p}$ may be noted and consistency is achieved only if the sum of the terms in Eq.(3.12) is positive.

4 Fermions and Nonrelativistic Strings

Consider the expansion of a fermion field in terms of positive-energy states [1].

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} c_r(\vec{p})u_r(\vec{p})e^{ip\cdot x} \tag{4.1}$$

such that

$$\begin{aligned} z\psi(x) &= \frac{1}{i}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} c_r(\vec{p})u_r(\vec{p})\frac{\partial}{\partial p_z}e^{ip\cdot x} \\ &= i\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{\partial}{\partial p_z}[c_r(\vec{p})u_r(\vec{p})]e^{ip\cdot x} \end{aligned} \tag{4.2}$$

after integration by parts. Let the average position and be $\bar{z} = 0$ and momentum be $\overline{p_z}$. Then

$$\begin{aligned} \overline{\Delta z^2} &= \int\int d^3x \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{d^3k}{(2\pi)^{\frac{3}{2}}}\frac{\partial}{\partial p_z}[c_r(\vec{p})u_r(\vec{p})]\frac{\partial}{\partial k_z}[c_r(\vec{k})u_r^*(\vec{k})]e^{ip\cdot x}e^{-ik\cdot x} \\ &= \frac{1}{(2\pi)^3}\int d^3p\ d^3k \frac{\partial}{\partial\ p_z}[c_r(\vec{p})u_r(\vec{p}]\frac{\partial}{\partial k_z}[c_r^*(\vec{k})u_r^*(\vec{k})]\ \cdot (2\pi)^3\delta(\vec{p}-\vec{k}) \\ &= \int d^3p \frac{\partial}{\partial p_z}[c_r(\vec{p})u_r(\vec{p})]\frac{\partial}{\partial p_z}[c_r^*(\vec{p})u_r^*(\vec{p})]. \end{aligned} \tag{4.3}$$

It follows that

$$\begin{aligned} \overline{\Delta z^2} = &\int d^3p \left|\frac{\partial c_r(\vec{p})}{\partial p_z}\right|^2 |u|^2 + \int d^3p\ |c(\vec{p})|^2 \left|\frac{\partial u_r(\vec{p})}{\partial p_z}\right|^2 \\ &+ \int d^3p \left\{\frac{\partial c_r^*(\vec{p})}{\partial p_z}c_r(\vec{p})\frac{\partial u_r(\vec{p})}{\partial p_z}u_r^*(\vec{p}) + c_r^*(\vec{p})\frac{\partial c_r(\vec{p})}{\partial p_z}\frac{\partial u_r^*(\vec{p})}{\partial p_z}u_r(\vec{p})\right\}. \end{aligned} \tag{4.4}$$

The coefficients $c_r(\vec{p}) = f\left(\frac{|\vec{p}|}{p_0}\right)\frac{1}{p_0^{\frac{3}{2}}}$ can be chosen to be real [1] and

$$\begin{aligned} &\int d^3p \left\{\frac{\partial c_r^*(\vec{p})}{\partial p_z}c_r(\vec{p})\frac{\partial u_r(\vec{p})}{\partial p_z}u_r^*(\vec{p}) + c_r^*(\vec{p})\frac{\partial c_r(\vec{p})}{\partial p_z}\frac{\partial u_r^*(\vec{p})}{\partial p_z}u_r(\vec{p})\right\} \\ &\int d^3p c_r(\vec{p})\frac{\partial c_r(\vec{p})}{\partial p_z}\frac{\partial}{\partial p_z}(u_r^*(\vec{p})u_r(\vec{p})). \end{aligned} \tag{4.5}$$

The spinor u_r satisfies the relation $u_r^*(\vec{p})u_r(\vec{p}) = \sum_{\alpha=1}^4 u_{r\alpha}^*(\vec{p})u_{r\alpha}(\vec{p}) = 1$. For example, when

$$u_1 = \frac{1}{\sqrt{2}}\left(1+\frac{m}{E}\right)^{\frac{1}{2}}, \quad u_2 = 0,$$

$$u_3 = \frac{1}{\sqrt{2}} \frac{p_z}{m+E} \left(1+\frac{m}{E}\right)^{\frac{1}{2}}, \quad u_4 = \frac{1}{\sqrt{2}} \frac{p_x + ip_y}{m+E} \left(1+\frac{m}{E}\right)^{\frac{1}{2}}, \tag{4.6}$$

$$\begin{aligned} \sum_{\alpha=1}^{4} u^*_{r\alpha}(\vec{p}) u_{r\alpha}(\vec{p}) &= \frac{1}{2}\left(1+\frac{m}{E}\right) + \frac{1}{2} \frac{p_x^2+p_y^2+p_z^2}{(m+E)^2}\left(1+\frac{m}{E}\right) \\ &= \frac{1}{2} \frac{[(E+m)^2 + p_x^2 + p_y^2 + p_z^2]}{E^2\left(1+\frac{m}{E}\right)} \\ &= \frac{1}{2} \cdot \frac{2E(E+m)}{E(E+m)} = 1. \end{aligned} \tag{4.7}$$

Let $\xi = \frac{|\vec{p}|}{p_0}$ such that $p_z = p_0 \xi \, cos\, \theta$. Since

$$\frac{\partial c_r(\vec{p})}{\partial p_z} = \frac{\partial \xi}{\partial p_z} \frac{\partial c_r(\vec{p})}{\partial \xi} + \frac{\partial \theta}{\partial p_z} \frac{\partial c_r(\vec{p})}{\partial \theta} = \frac{1}{p_0^{\frac{5}{2}} cos\, \theta} \frac{df}{d\theta} \tag{4.8}$$

and

$$\begin{aligned} I_1 &= \int d^3p \left(\frac{\partial c_r(\vec{p})}{\partial p_z}\right) = \int |\vec{p}|^2 d|\vec{p}| d\Omega \left(\frac{\partial c_r(\vec{p})}{\partial p_z}\right)^2 \\ &= \int p_0^3 \xi^2 d\xi d\Omega \, \frac{1}{p_0^5 \, cos^2\theta} \left(\frac{df}{d\xi}\right)^2 \\ &= \frac{1}{p_0^2} \int \frac{d\Omega}{cos^2\theta} \int_0^\infty \left(\frac{df}{d\xi}\right)^2 \xi^2 d\xi \\ &= \frac{4\pi}{p_0^2} \int \left(\frac{df}{d\xi}\right)^2 \xi^2 d\xi = \frac{\alpha''}{p_0^2}. \end{aligned} \tag{4.9}$$

The second integral in Eq.(4.4) is

$$\begin{aligned} I_2 &= \int d^3p \; c_r^2(\vec{p}) \frac{\partial u_r^*(\vec{p})}{\partial p_z} \frac{\partial u_r(\vec{p})}{\partial p_z} \\ &= \int p_0^3 \xi^2 d\xi \frac{f^2(\xi)}{p_0^3} \int d\Omega \; \frac{\partial u_r^*(\vec{p})}{\partial p_z} \frac{\partial u_r(\vec{p})}{\partial p_z} \\ &= 4\pi \int f^2(\xi) \xi^2 M\left(\xi, \frac{m}{p_0}\right) d\xi. \end{aligned} \tag{4.10}$$

For the spinor (4.6),

$$\begin{aligned} \frac{\partial u_1}{\partial p_z} &= -\frac{1}{2\sqrt{2}}\left(1+\frac{m}{E}\right)^{-\frac{1}{2}} \frac{mp_z}{E^3}, \quad \frac{\partial u_2}{\partial p_z} = 0, \\ \frac{\partial u_3}{\partial p_z} &= -\frac{1}{\sqrt{2}}\left(1+\frac{m}{E}\right)^{-\frac{1}{2}} \frac{p_z^2}{E^3} + \frac{1}{\sqrt{2}E}\left(1+\frac{m}{E}\right)^{-\frac{1}{2}} + \frac{m}{2\sqrt{2}}\left(1+\frac{m}{E}\right)^{-\frac{3}{2}} \frac{p_z^2}{E^4}, \\ \frac{\partial u_4}{\partial p_z} &= \frac{1}{\sqrt{2}}(p_x + ip_y)\left(-\left(1+\frac{m}{E}\right)^{-\frac{1}{2}} \frac{p_z}{E^3} + \frac{1}{2}\left(1+\frac{m}{E}\right)^{-\frac{3}{2}} \frac{mp_z}{E^4}\right) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} M\left(\xi, \frac{m}{p_0}\right) = \frac{1}{4\pi} \int d\Omega \; \Bigg\{ &\frac{1}{8}\left(1+\frac{m}{E}\right)^{-1} \frac{m^2p_z^2}{E^6} + \frac{1}{2}\left(1+\frac{m}{E}\right)^{-1} \frac{1}{E^2}\Bigg[1 - \frac{p_z^4}{E^4} \\ &+ \frac{1}{4} \frac{m^2p_z^2}{\left(1+\frac{m}{E}\right)^2 E^6} - \frac{2p_z^2}{\left(1+\frac{m}{E}\right) E^2} - \frac{mp_z^2}{\left(1+\frac{m}{E}\right) E^3} + \frac{mp_z^4}{\left(1+\frac{m}{E}\right)^2 E^5}\Bigg]\Bigg\} \end{aligned}$$

$$+\frac{1}{4\pi}\int d\Omega\ (p_x^2+p_y^2)\left[\frac{1}{2}\left(1+\frac{m}{E}\right)^{-1}\frac{p_z^2}{E^6}-\frac{1}{2}\left(1+\frac{m}{E}\right)^{-2}\frac{mp_z^2}{E^7}\right.$$
$$\left.+\frac{1}{8}\left(1+\frac{m}{E}\right)^{-3}\Big\}\frac{m^2p_z^2}{E^8}\right]. \qquad (4.12)$$

With $p_x=|\vec{p}|\ sin\ \theta\ cos\ \phi$,

$$\int_0^{2\pi}\int_0^{\pi}|\vec{p}|^2 sin^2\theta cos^2\phi\ |\vec{p}|^2 cos^2\theta sin\ \theta d\theta d\phi$$
$$=\pi\left\{\frac{cos^3\theta}{3}\Bigg|_{cos\ \theta=-1}^{cos\ \theta=1}-\frac{cos^5\theta}{5}\Bigg|_{cos\ \theta=-1}^{cos\ \theta=1}\right\}|\vec{p}|^4=\frac{4\pi}{15}|\vec{p}|^4. \qquad (4.13)$$

From Eq.(4.13),

$$M\left(\xi,\frac{m}{p_0}\right)\approx\frac{1}{4m^2c^2} \qquad |\vec{p}|\ll mc,$$
$$M\left(\xi,\frac{m}{p_0}\right)\approx\frac{1}{4\pi}\cdot\frac{1}{2}\left(1+\frac{m}{E}\right)^{-1}\frac{1}{E^2}\left[4\pi-\frac{4\pi}{5}-2\cdot\frac{4\pi}{3}\right]+\frac{1}{4\pi}\frac{1}{2E^2}\cdot 2\cdot\frac{4\pi}{15}\simeq\frac{2}{15|\vec{p}|^2} \qquad |\vec{p}|\gg mc. \qquad (4.14)$$

Suppose that $\int_0^\infty f^2(\xi)d\xi=\beta'$, $4\pi\int f^2(\xi)\xi^2 d\xi=\alpha'''$, and inequalities are used for the uncertainty relations.

$$\overline{\Delta z^2}\geq\frac{\alpha'\alpha''}{\overline{\Delta p_z^2}}+\frac{\alpha'''}{4m^2c^2} \qquad \overline{\Delta p_z^2}\ll m^2c^2,$$
$$\overline{\Delta z^2}\geq\frac{\alpha'\alpha''}{\overline{\Delta p_z^2}}+\frac{2}{15}\frac{\alpha'\beta'}{\overline{\Delta z^2}} \qquad \overline{\Delta p_z^2}\gg m^2c^2. \qquad (4.15)$$

Let $\alpha'\alpha''=\alpha\hbar^2$, $\alpha'''=4\beta\hbar^2$ and $\alpha'\alpha''+\frac{2}{15}\alpha'\beta'=\gamma\hbar^2$. The uncertainty relations have the form [1]

$$\overline{\Delta z^2}\geq\frac{\alpha\hbar^2}{\overline{\Delta p_z^2}}+\frac{\beta\hbar^2}{m^2c^2} \qquad \overline{\Delta p_z^2}\ll m^2c^2,$$
$$\overline{\Delta z^2}\geq\gamma\hbar^2 \qquad \overline{\Delta p_z^2}\gg m^2c^2. \qquad (4.16)$$

At nonrelativistic velocities,

$$(\overline{\Delta z^2})^{\frac{1}{2}}\geq\left(\frac{\alpha\hbar^2}{\overline{\Delta p_z^2}}\right)^{\frac{1}{2}}\left[1+\frac{\beta\hbar}{\alpha\hbar^2}\frac{(\Delta p_z^2)}{m^2c^2}\right]^{\frac{1}{2}} \qquad (4.17)$$

and

$$\langle\Delta z\rangle_{rms}\geq\frac{\alpha^{\frac{1}{2}}\hbar}{\langle\Delta p_z\rangle_{rms}}+\frac{\beta\hbar}{2\alpha^{\frac{1}{2}}}\frac{\langle\Delta p_z\rangle_{rms}}{m^2c^2}+... \qquad (4.18)$$

which is compatible with the uncertainty relation in curved space because the coefficient of $\langle\Delta p_z\rangle_{rms}$ decreases as m^{-2}. Consistency is achieved if $R\propto m^2$, with more massive fermions curving the space to a greater extent.

It has been verified, that if the covariant constancy of Dirac matrices is relaxed, such that $\nabla_\mu\gamma_\nu=\phi\gamma_{\mu\nu}$, with ϕ being a field with dimensions of mass, the commutation of the covariant derivatives yields $(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu)\gamma_\rho=-R^\sigma{}_{\rho\mu\nu}\gamma_\sigma+i[\Omega_{\mu\nu},\gamma_\rho]+T^\sigma_{\mu\nu}\nabla_\sigma\gamma_\rho$, which requires $i[\Omega_{\mu\nu},\gamma_\rho]=-\Omega^\sigma{}_{\rho\mu\nu}\gamma_\sigma+\Omega^\sigma{}_{\mu\nu}\gamma_{\sigma\rho}$, where $\Omega_{\mu\nu}$ represents the spin-connection contribution to the covariant derivative of the metric [2]. The coefficients $\Omega^\sigma{}_{\rho\mu\nu}\gamma_\sigma$ and $\Omega^\sigma{}_{\mu\nu}$ may be identified as the curvature and torsion for the spin connection, and the curvature must satisfy

$$\Omega^\sigma{}_{\rho\mu\nu}=-R^\sigma{}_{\rho\mu\nu}-\phi^2(g_{\rho\mu}\delta^\sigma{}_\nu-g_{\rho\nu}\delta^\sigma_\mu). \qquad (4.19)$$

The trace of the equation gives

$$R=12\phi^2-\Omega. \qquad (4.20)$$

Suppose that the spin curvature is constant. Then $R\sim 12\phi^2$ for large ϕ^2, which is consistent with the proportionality to m^2.

The effect of observation of the motion of a string through the use of $N-1$ strings would be reduction in the uncertainty. If each measurement x_i of a variable x has a standard deviation $\sigma_i = \sigma$, then the estimator after n measurements is $x^* = \frac{1}{n}(x_1 + ... + x_n)$ and the standard deviation is $\sigma_{x^*} = \frac{\sigma}{\sqrt{n}}$, Therefore, the effect of $N-1$ strings on the uncertainty in a string momentum would be multiplication by $\frac{1}{\sqrt{N-1}}$. Given that the flat-space term is unchanged, the uncertainty relation for a string in an N-string configuration is

$$\Delta x \geq const \cdot \alpha' \frac{\Delta p}{\sqrt{N-1}} + \frac{\hbar}{2\Delta p}. \tag{4.21}$$

Identifying the N-string background with that of a curved space of Ricci scalar R,

$$const. \cdot \alpha' \cdot \frac{\Delta p}{\sqrt{N-1}} = -\frac{3}{2} D R^{-1} \Delta p \tag{4.22}$$

and

$$\sqrt{N-1} = -const. \cdot \alpha' \cdot \frac{2R}{3D}, \qquad N = const. \cdot \alpha'^2 \cdot \frac{4R^2}{9D} + 1. \tag{4.23}$$

The interaction of a probe and a string is the combination of two processes consisting of the propagation of the string and the scattering by a probe in observation of the motion of the string. The propagation of the string, with the inclusion of higher-genus contributions, is described by a two-point amplitude that generates an R^2 term in the effective action. These R^2 terms will be dominant in the high-curvature region of the string. For an N-string background, the integral over the entire time interval can be split into a sum of integrals over the subintervals corresponding to the interaction of each probe with the string.

If the observations are made at $t_1, t_2, ..., t_{N-1}$, the amplitude could be described by the propagation of the string together with the insertion of external states at these times. These states can be regarded as additional source terms such that the partition function for the process is $Z = Z_{12} Z_{13} ... Z_{1N}$ where the Euclidean expressions are

$$\begin{aligned} Z_{12} &= e^{-\int_{t_0}^{t_1} \int \mathcal{L}_{eff} + J_1[X,\psi,...] \ d^3x \ dt} \\ Z_{13} &= e^{-\int_{t_1}^{t_2} \int \mathcal{L}_{eff} + J_2[X,\psi,...] \ d^3x \ dt} \\ &\vdots \\ Z_{1N} &= e^{-\int_{t_{N-2}}^{t_{N-1}} \int \mathcal{L}_{eff} + J_N[X,\psi,...] \ d^3x \ dt}. \end{aligned} \tag{4.24}$$

Then

$$Z = e^{-\int_{t_0}^{t_{N-1}} \int \mathcal{L}_{eff} \ d^3x \ dt - \int_{t_0}^{t_1} J_1[X,\psi,...] \ d^3x \ dt - ... - \int_{t_{N-2}}^{t_{N-1}} J_N[X,\psi,...] \ d^3x \ dt}. \tag{4.25}$$

Therefore, the effective Lagrangian would be given by that of the propagator of the string in each process. Instead of an $(N+1)$-point amplitude, only a two-point amplitude is necessary., and the effective Lagrangian remains quadratic in the curvature. The proportionality between $N-1$ and R^2 follows.

5 Minimal Length of Closed Geodesics on Riemann Surfaces

These results on the minimal uncertainty in position raise the problem of the size of the handles and the lengths of closed geodesics on Riemann surfaces of arbitrary genus g. It may be recalled that the regularization of the bosonic string partition function requires a genus-independent cut-off in moduli space. This procedure has been implemented explicitly in the Schottky parameterization through integration of the measure over the uniformizing parameters with the auxiliary condition of the radii of the isometric circles decreasing as $\frac{1}{\sqrt{g}}$. Since the area in the intrinsic metric satisfies

$$A = \int d^2\xi \sqrt{-g} = -\int d^2\xi \sqrt{-g} R \propto (2g-2), \tag{5.1}$$

it increases linearly with the genus. Because areas are proportional to the squares of lengths, the lengths of closed geodesics around the handles would be increased by a factor of $\sqrt{g}$ relative to the lengths in the target space. The lower bound on the radii of the isometric circles in the Schottky covering surface implies that the lengths of closed geodesics around the handles are $\mathcal{O}(1)$, and the cut-off in genus-independent in the intrinsic metric [4].

As the genus increases to infinity, other surfaces may be included in the perturbative expansion of the scattering amplitudes. In addition to closed surfaces of finite genus, effectively closed infinite-genus surfaces may be added to the sum over histories. One class of infinite-genus surfaces is uniformized by Schottky groups with isometric circles having radii that decrease as $r_n \sim \frac{1}{n^q}$, $q > \frac{1}{2}$. This depdendence is necessary for the convergence of the Poincare series $\sum_{\alpha \neq I} |\gamma|^{-2}$, where $\frac{V_\alpha z - \xi_{1\alpha}}{V_\alpha z - \xi_{2\alpha}} = K_\alpha \frac{z-\xi_{1\alpha}}{z-\xi_{2\alpha}}$, where $V_\alpha = T_1^{\pm 1} ... T_\ell^{\pm 1}$, $V_\alpha z = \frac{\alpha_\alpha z + \beta_\alpha}{\gamma_\alpha z + \delta_\alpha}$, and the N-point amplitude based on a series for the Green function.

However, it is evident that the dependence of r_n on n with $q > \frac{1}{2}$ produces handles of arbitrarily infinitesimal size even in the intrinsic metric [5]. The genus-indpendent cut-off in moduli space is not satisfied. There is no requirement for this cut-off in superstring theory where the amplitudes at each genus are finite. Therefore, these surfaces can be included in the domain of superstring perturbation theory. Nevertheless, the existence of handles of arbitrarily infinitesimal thickness might be considered in the context of the generalized uncertainty principle. There is a minimal uncertainty in Δx for measurements from the target space. If the Riemann surface represents a virtual process, this lower bound should have no effect on the size of the handles which are not observed experimentally in the target space. The possibility of the splitting and joining of strings in a real process has been suggested for electron-positron creation and annihilation [6]. There, the times for each splitting and joining of the strings are chosen to be identical for each loop. The average time for the pair annihilation reaction is computed to be slightly larger for the real process in comparison with the virtual process. If the surfaces, as the genus increase to infinity, are embedded in the target space as ladder diagrams, with the mapping between distances in the worldsheet and the target space satisfying a boundedness condition, $k_1 d_{targ.} < d_{intr.} < k_2 d_{targ.}$, a minimal length condition on the size of the handles, compatible with the uncertainty relation, may be imposed, However, if the infinite-genus surface is confined to a finite region, it will have the form of a sphere with an infinite number of handles having radii satisfying $r_n \sim \frac{1}{n^2}$, $q > \frac{1}{2}$, is necessary, and the lower bound does not exist. For the real process, this might be interpreted as an indication that there is a large-genus cut-off to satisfy a bound of this kind. This would preserve the property of lengths less than the uncertainty bound, or the Planck length, being unobservable from the target space.

6 Boundary States on the String Worldsheet

The other possibility, that arises, is the addition of Dirichlet boundaries to the closed string surface. This nonperturbative effect is interpreted as an open-closed string interaction. Since the open string can be prepared at microscopic distances from the interaction region, it could be used also as a probe of the closed string propagation. In this circumstance, the size of the disk, or the open string, would have a lower bound of Planck scale dimensions. The embedding coordinates of open-string theories satisfy Neumann conditions. Neumann conditions are necessary also for there to be no momentum loss through the boundary of a bordered surface. For the string amplitudes, momentum would be conserved because the open-string boundary state can be identified with a zero-momentum physical state. As a probe, however, the open string would have momentum which could be imparted to the closed string. This process is sufficient to introduce a minimal uncertainty in the location of the string coordinates, and thereby, allows the uncertainty bound in Δx to be satisfied.

It is known that either Neumann or Dirichlet boundary conditions can be imposed at the borders of P_G surfaces. While conservation of momentum is valid for Neumann conditions, there can be a flow of momentum from the surface if Dirichlet boundary conditions are imposed. The flow of momentum is given by

$$g \oint d\sigma_B \zeta \cdot \partial_n X(\sigma_B, \tau_B) \tag{6.1}$$

where τ_B is the worldsheet position of the boundary and ζ is the open-string state [7]. This momentum may curve the surrounding local region or the open-string state might propagate freely as a physical state.

When the Neumann condition is satisfied, the closed-string worldsheet and open-string boundary form a closed system. At the point of the creation of the open-string state, the combined open-closed string state can be regarded as a mixed state in the Hilbert space. Since the time evolution is unitary in a closed system, it cannot be transformed into a pure state and the open and closed strings must continue to propagate.

With the Dirichlet boundary condition, it is an open system and unitarity is not necessarily preserved. Momentum can flow through the open-string channel without being returned or the open string may interact again with the closed string changing the mixed state to a pure state. In the first instance, if unitarity is not required, this would remove the necessity for introducing additional types of surfaces besides the closed finite-genus surfaces to ensure that the norm of the vacuum state is one.

Then the closed string amplitude can be factored from the final physical state leaving the propagation of an open string with momentum. If the initial theory is that of the Type II superstring, the open-string state would belong to the Hilbert space of a Type I superstring model. One of the effects of the border with Dirichlet boundary conditions, therefore, has been the partial breaking of supersymmetry.

When the supersymmetry algebra has a subalgebra with bosonic generators that preserve the boundary, there exists a Nicolai map which can transform the supersymmetric sigma-model to a bosonic sigma model. Since the propagating state then must be bosonic, or a fermion which can be bosonized, it would be consistent with the mediation of the elementary particle forces by vector bosons.

7 Conclusion

The two solutions to the quadratic inequality for the uncertainty in the position of the worldline of a particle in curved space-time correspond to different physical configurations. The existence of a flat-space limit of the first relation is consistent with a weak gravitational field. The qualitatively distinct second relation is similar to that of string theory, yields a minimum length uncertainty and contains a duality between momentum modes and winding modes to leading order. The latter inequality, therefore, reflects a string-theoretic model of space-time curvature in a gravitational field. This result is confirmed through an identification of the coefficients of terms in the uncertainty relation in an N-string background and the curvature. The proportionalty between $N-1$ and R^2 is attained through a discussion of the string scattering processes necessary for a determination of the position coordinates.

This string scattering process can include higher-genus surfaces. However, the effect on the intraction between the probe and a string is given by a quadratic curvature term in the effective action. The curvature of the target space-time rather than the Riemann surface occurs in the uncertainty relations. Since the curvature of an arbitrary-genus surface in the string scattering amplitude does not determine the uncertainty in the position coordinates, it could be concluded that there are no initial constraints on the cross-sections of the handles on these surfaces. This conclusion is corroborated by the presence of infinitesimal handles on surfaces of infinite genus, which could be included in the sum of string worldsheets without any restrictions on minimal lengths in the two-dimensional manifold. The existence of infinitesimal handles would be consistent with the uncertainty principle for virtual diagrams that are not observable from target space. By contrast, the inverse process of the emission of a nonperturbative open-string boundary state could be viewed as the interaction of a probe with the closed string and may be used to determine the uncertainty in the length coordinates in the target space. However, unitarity of the combined system would require that the closed string amplitude and propagation of the open string state are considered separately with Dirichlet boundary conditions. The existence of handles of infinitesimal size on the closed string worldsheet would remain consistent with the independence of the two processes and the differentiation between the length of closed geodesics and the minimal length uncertainty.

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