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CDS Pricing Formula in the Fuzzy Credit Risk Market*

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Abstract

In this paper, credit default swap (CDS) pricing formula is obtained in the fuzzy credit risk market. The formula solution is given by the method of fuzzy expect. In addition, some illustrative examples are also documented.

Keywords: credit risk, CDS, fuzzy process, geometric Liu process

1 Introduction

Recently, the credit risk of firms has been one of the most active areas in finance. In the early 1970s, Scholes [1] and Merton [7] began the credit risk research with stochastic process. In the next few decades, this field of research, which is based on randomness as one type of uncertainty, develops rapidly.

Different from randomness, fuzziness is another type of uncertainty in the real world, which always is used to describe the phenomena with vague and subjective information, especially those lack of or even without historical records. The concept of fuzzy set was initially proposed by Zadeh [11] in 1965. To measure a fuzzy event, Liu [6] presented the concept of credibility measure. Credibility theory was founded by Liu [4] in 2004 and refined by Liu [5] in 2007 as a branch of mathematics for studying the behavior of fuzzy phenomena. In order to deal with the evolution of fuzzy phenomena with time, Liu [3] proposed a fuzzy process, a differential formula and a fuzzy integral. Later, the community renamed them Liu process, Liu formula and Liu integral. Some more researches about the subject have been made. You [10] studied differential and integral of multi-dimensional Liu process. some properties of analytic functions of complex Liu process were considered by Qin [9]. Dai [2] proposed a reflection principle. Peng [8] studied credibilistic stopping problems for fuzzy stock market.

Credit default swap (for short CDS) have been proven to be one of the most successful financial innovations of the 1990s. They are instruments that provide insurance against the default of a particular company (or sovereign entity) on its debt. The company is known as the reference entity, and a default is known as a credit event. The buyer of protection pays periodic payments to the seller of protection at a predetermined fixed rate per year. The payments continue until the maturity of the contract or until occurrence of the default, whichever is earlier. If default event occurs, the buyer of protection has the right to deliver to the seller of protection a bond issued by the reference entity in exchange for its face value.

In this paper, we study the credit default swap (CDS) pricing formula for fuzzy credit risk market. The pricing formula is obtained with geometric Liu process [3]. Examples are given by the method of Quasi-Monte Carlo numerical approach at last.

2 Preliminaries

Definition 1 [6] The set function Cr is called a credibility measure if it satisfies the following four axioms:
Axiom 1. (Normality) $Cr\{\Theta\} = 1$.

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Axiom 2. (Monotonicity) $Cr\{A\} \leq Cr\{B\}$ whenever $A \subset B$.
Axiom 3. (self-Duality) $Cr\{A\} + Cr\{A^C\} = 1$ for any event A.
Axiom 4. (Maximality) $Cr\{\cup_i A_i\} = sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $sup_i Cr\{A_i\} < 0.5$. Above Θ is a nonempty set and P is the power set of Θ. Each element A in P is called an event.

Definition 2 [5] Let ξ be a fuzzy variable defined on the Medibility space $(\Theta,\ P,\ Cr)$, Then its membership function is derived from the credibility measure by

$$\phi(x) = (2Cr\{\xi = x\}) \wedge 1,\ x \in \mathbf{R}.$$

Definition 3 [6] Let ξ be a fuzzy variable. Then the expected value of the ξ is defined by

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq \eta\}d\eta - \int_{-\infty}^0 Cr\{\xi \leq \eta\}d\eta.$$

Definition 4 [3] Geometric Liu process is defined by a lognormal membership function

$$\phi(z) = 2(1 + \exp(\frac{\pi\,|\ln z - \mu t|}{\sqrt{6}\sigma t}))^{-1},$$

denoted by $L(\mu, \sigma^2)$.

Lemma 1 [5] Let ξ be a fuzzy variable with membership function ϕ. Then for any set A of real numbers, we have

$$Cr\{\xi \in A\} = \frac{1}{2}(\sup_{\xi \in A} \phi(\xi) + 1 - \sup_{\xi \in A^C} \phi(\xi)).$$

Definition 5 [2] The first passage time of fuzzy process G_t to level x is defined as follows,

$$\tau_x = inf\{t > 0 | G_t = x\}.$$

3 CDS Pricing Formula

Theorem 1 Assume that G_t is a Geometric Liu process with drift μ and diffusion σ. Let τ_B be the first passage time of G_t to down-barrier V_B. Then the credibility distribution of τ_B is given as follows,

$$\begin{aligned} Cr\{\tau_B \leq t\} &= (1 + \exp(\frac{\pi(\ln V_B - \mu t)}{\sqrt{6}\sigma t}))^{-1}, & \mu t < \ln V_B, \\ Cr\{\tau_B \leq t\} &= 1 - (1 + \exp(\frac{\pi(\mu t - \ln V_B)}{\sqrt{6}\sigma t}))^{-1}, & \mu t \geq \ln V_B. \end{aligned}$$

Proof: Because

$$Cr\{\tau_B \leq t\} = Cr\{\inf_{0 \leq s \leq t} G_s \leq V_B\} = 1 - Cr\{\sup_{0 \leq s \leq t} G_s \geq V_B\},$$

the follows can be obtained from Reflected Principle that

$$Cr\{\tau_B \leq t\} = 1 - Cr\{G_t \geq V_B\} = 1 - \frac{1}{2}(\sup_{\xi \geq V_B} \phi(\xi) + 1 - \sup_{\xi \leq V_B} \phi(\xi)).$$

We have

$$\phi(\xi) = 2(1 + \exp(\frac{\pi\,|\ln \xi - \mu t|}{\sqrt{6}\sigma t}))^{-1}.$$

Therefore, it can be obtained that If $\mu t < \ln V_B$,

$$Cr\{\tau_B \leq t\} = 1 - (1 + \exp(\frac{\pi(\ln V_B - \mu t)}{\sqrt{6}\sigma t}))^{-1}.$$

If $\mu t \geq \ln V_B$,

$$Cr\{\tau_B \leq t\} = (1 + \exp(\frac{\pi(\mu t - \ln V_B)}{\sqrt{6}\sigma t}))^{-1}.$$

The proof is complete.

In the following, we consider a CDS, which's target bond value is 1. Maturity, rate and recovery are denoted by T, r and R. ω is used to represent the annual fee of CDS. Let t_i $(i = 1, 2, \ldots, N)$ denote paying day, where $t_N = T$. Δt is constant as the paying interval, so the holders should pay $\omega \Delta t$ at every pay day.

If the firm defaults at time τ, there must be natural number k $(k \leq N)$ which satisfies $t_{k-1} \leq \tau < t_k$. Thereby, the present value of premiums which should be paid from 0 to T is

$$\sum_{i=1}^{k-1} \omega \Delta t \cdot e^{-rt_i} + \omega(\tau - t_{k-1})e^{-rt_k} = \omega U(\tau),$$

where $t_0 = 0$.

Furthermore, $U(\tau)$ can be represented as follows

$$U(\tau) = \begin{cases} U_1 = (\tau - t_0)e^{-rt_1}, & 0 \leq \tau < t_1, \\ U_2 = \sum\limits_{i=1}^{1} \Delta t \cdot e^{-rt_i} + (\tau - t_1)e^{-rt_2}, & t_1 \leq \tau < t_2, \\ \cdots\cdots & \\ U_k = \sum\limits_{i=1}^{k-1} \Delta t \cdot e^{-rt_i} + (\tau - t_{k-1})e^{-rt_k}, & t_{k-1} \leq \tau < t_k, \\ \cdots\cdots & \\ U_N = \sum\limits_{i=1}^{N-1} \Delta t \cdot e^{-rt_i} + (\tau - t_{N-1})e^{-rt_N}, & t_{N-1} \leq \tau \leq t_N, \\ U(t_N), & t_N < \tau. \end{cases}$$

On the other hand, if the default event occurs at τ, the present value of compensation is $(1 - R)e^{-r\tau}$. Because the initial value of CDS should be zero, the following equation can be given as

$$E[(1 - R)e^{-r\tau}] = \omega E[U(\tau)],$$

$$\omega = \frac{E[(1 - R)e^{-r\tau}]}{E[U(\tau)]}.$$

Theorem 2 Assume that the CDS's target G_t is a Geometric Liu process with drift μ and diffusion σ, τ is the default time with barrier V_B. All the other conditions have been mentioned above. Then the annual fee of CDS can be priced as follow

$$\omega = \frac{\int_0^{(1-R)} (1 + \exp(\frac{\pi(-\frac{\mu}{r}\ln\frac{\eta}{1-R} - \ln V_B)}{-\sqrt{6}\frac{\sigma}{r}\ln\frac{\eta}{1-R}}))^{-1} d\eta}{\sum\limits_{k=1}^{N} \int_{U(t_{k-1})}^{U(t_k)} 1 - (1 + \exp\frac{\pi(\mu U_k^{-1}(\eta) - \ln V_B)}{\sqrt{6}\sigma U_k^{-1}(\eta)})^{-1} d\eta},$$

where

$$U_k^{-1}(\eta) = (\eta - \sum_{i=1}^{k-1} \Delta t e^{-rt_i})e^{rt_k} + t_{k-1},$$

$$m_k = (\frac{1}{\mu}\ln V_B - t_{k-1})e^{rt_k} + \sum_{i=1}^{k-1} \Delta t e^{-rt_i}. \quad (k = 1, 2, \ldots, N)$$

Proof: On the one hand, we obtain by *Definition* 3 that

$$\begin{aligned} E[(1 - R)e^{-r\tau}] &= \int_0^{+\infty} Cr\{(1 - R)e^{-r\tau} \geq \eta\} d\eta - \int_{-\infty}^{0} Cr\{(1 - R)e^{-r\tau} \leq \eta\} d\eta \\ &= \int_0^{(1-R)} Cr\{\tau \leq -\frac{1}{r}\ln\frac{\eta}{1 - R}\} d\eta. \end{aligned}$$

Also, we can obtaine from $Theorem$ 2 that

$$Cr\{\tau \le t\} = 1 - Cr\{G_t \ge V_B\} = 1 - \frac{1}{2}(\sup_{\xi \ge V_B} \phi(\xi) + 1 - \sup_{\xi \le V_B} \phi(\xi)),$$

where

$$\phi(\xi) = 2(1 + \exp(\frac{\pi\,|\ln \xi - \mu t|}{\sqrt{6}\sigma t}))^{-1}.$$

Generally, $V_B < 1$, because the face value of the target bond is 1. Otherwise the default will occur at initial time. Therefore, when $\eta \le (1-R)$, $\mu \cdot (-\frac{1}{r}\ln\frac{\eta}{1-R}) > \ln V_B$. then

$$Cr\{\tau \le -\frac{1}{r}\ln\frac{\eta}{1-R}\} = (1 + \exp(\frac{\pi(-\frac{\mu}{r}\ln\frac{\eta}{1-R} - \ln V_B)}{-\sqrt{6}\frac{\sigma}{r}\ln\frac{\eta}{1-R}}))^{-1}.$$

Furthermore,

$$E[(1-R)e^{-r\tau}] = \int_0^{(1-R)} (1 + \exp(\frac{\pi(-\frac{\mu}{r}\ln\frac{\eta}{1-R} - \ln V_B)}{-\sqrt{6}\frac{\sigma}{r}\ln\frac{\eta}{1-R}}))^{-1} d\eta.$$

On the other hand, we have

$$\begin{aligned} E[U(\tau)] &= \int_0^{+\infty} Cr\{U(\tau) \ge \eta\} d\eta \\ &= \int_0^{U(T)} Cr\{U(\tau) \ge \eta\} d\eta \\ &= \sum_{k=1}^{N} \int_{U(t_{k-1})}^{U(t_k)} Cr\{\tau \ge U_k^{-1}(\eta)\} d\eta \\ &= \sum_{k=1}^{N} \int_{U(t_{k-1})}^{U(t_k)} 1 - Cr\{\tau \le U_k^{-1}(\eta)\} d\eta, \end{aligned}$$

where

$$U_k^{-1}(\eta) = (\eta - \sum_{i=1}^{k-1} \Delta t e^{-rt_i}) e^{rt_k} + t_{k-1}, \quad (k = 1, 2, \ldots, N).$$

Let $m_k = (\frac{1}{\mu}\ln V_B - t_{k-1})e^{rt_k} + \sum_{i=1}^{k-1} \Delta t e^{-rt_i}$. If $\mu U_k^{-1}(\eta) < \ln V_B$, we will have $\eta < m_k$. However, $m_k < U_k^{-1}$, which means $\mu U_k^{-1}(\eta) > \ln V_B$. The follows can be given from from $Theorem$ 2 that

$$\begin{aligned} E[U(\tau)] &= \int_0^{+\infty} Cr\{U(\tau) \ge \eta\} d\eta \\ &= \sum_{k=1}^{N} [\int_{U(t_{k-1})}^{U(t_k)} 1 - (1 + \exp\frac{\pi(\mu U_k^{-1}(\eta) - \ln V_B)}{\sqrt{6}\sigma U_k^{-1}(\eta)})^{-1} d\eta]. \end{aligned}$$

Because $E[(1-R)e^{-r\tau}] = \omega E[U(\tau)]$, we have

$$\omega = \frac{E[(1-R)e^{-r\tau}]}{E[U(\tau)]}.$$

The proof is complete.

Theorem 3 CDS formula $\omega = \omega(R, \mu, r, \sigma, V_B)$ has the following properties.

(a). ω is a decreasing function R;

(b). ω is an decreasing function of μ;

(c). ω is an decreasing function of r;
(d). ω is an increasing function of σ;
(e). ω is a increasing function of V_B.

They are easy to be proved from the monotonicity of the integrand.

Example Suppose that the recovery R is 0.8, the drift μ is 0.2, the volatility σ is 0.3, the down-barrier V_B=0.8 and the rate r is 0.1. The following MATLAB codes by the method of Quasi-Monte Carlo, may be employed to calculate CDS price:

$for \quad i = 1 : ex - num$
$hal(i) = haltonBaseb(b, n);$
$cpsa(i) = (1 - R) * 1/((1 + exp((pi * ((mu/r) * log(hal(i)) + log(vb)))/(sqrt(6) * sigma * log(hal(i))/r))));$
end
$Ecpsa = mean(cpsa);$
$for \quad k = 2 : pay - num$
$for \quad i = 1 : ex - num$
$pay(i) = (ut(k) - ut(k - 1)) * (1 - 1/(1 + exp((mu * u_r - log(vb))/(ad = sqrt(6) * sigma * u_r))));$
end
$U(k - 1) = mean(pay);$
end
$EU = sum(U);$
$omega = Ecpsa/EU;$

The result shows that $\omega = 0.0687$. This means the premium is about 6.9%.

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