Journal of Uncertain Systems
Vol.6, No.1, pp.51-55, 2012
Online at: www.jus.org.uk

Fuzzy Automata based on Lattice-Ordered Monoid and Associated Topology*

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Received 13 March 2011; Revised 22 August 2011

Abstract

The concepts of fuzzy source and fuzzy successor operators for an L-fuzzy automaton (L is a lattice-ordered monoid) are introduced, which turn out to be L-fuzzy closure operators. When L is a quantale, these operators introduce L-fuzzy topologies. These observations are then used to give topological characterization of the separatedness and connectedness properties of an L-fuzzy automaton.

Keywords: L-fuzzy automaton, fuzzy source, fuzzy successor, L-fuzzy subautomaton, L-fuzzy topology

1 Introduction

The concept of separatedness and connectedness properties of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) were explored and studied by Malik, Mordeson and Sen [3] (cf. [4], for more details). Srivastava and Tiwari [8] shown that certain topological concepts can be used in fuzzy automata theory to obtain certain results pertaining to their separation and connectivity properties. In [5, 6], the concept of automata with membership values in complete residuated lattice and its topological characterization were introduced. Recently, the concept of fuzzy finite automata and grammar theory based on lattice ordered-monoid were introduced and it was studied that up to which extent the properties of such automata and their languages depend on associated lattice (cf., [1, 2]). In [10], Tiwari and Sharan introduced the concept of fuzzy automata based on lattice-orderd monoid (called L-fuzzy automata) as a generalization of fuzzy automata introduced in [4, 8] and shown that several results of such L-fuzzy automata depend on the fact that whether the associated monoid is with or without zero divisors. In this paper, chiefly motivated from [3, 8], we introduce the separatedness and connectedness properties of an L-fuzzy automaton and study their L-fuzzy topological characterization. Specifically, we introduce the concept of fuzzy source and fuzzy successor for an L-fuzzy automaton as a generalization of the concepts of source and successor introduced in [10] for an L-fuzzy automaton and discuss the separatedness and connectedness properties of such L-fuzzy automaton. Finally, we try to characterize these properties in terms of L-fuzzy topological concepts.

2 Preliminaries

In this section, we recall some basic concepts related to a lattice ordered monoid. We collect the followings from [1].

Definition 2.1 *An algebra* $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ *is called a* **lattice-ordered monoid** *if*

1. $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ *is a lattice with the least element 0 and the greatest element 1,*
2. $(L, \bullet, e)$ *is a monoid with identity* $e \in L$ *such that for all* $a, b, c \in L$,

*The authors acknowledge with thanks the support received through a research grant, provided by the Council of Scientific and Industrial Research, New Delhi, under which this work has been carried out.
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(i) $a \bullet 0 = 0 \bullet a = 0$,

(ii) $a \leq b \Rightarrow \forall x \in L, a \bullet x \leq b \bullet x$ *and* $x \bullet a \leq x \bullet b$,

(iii) $a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c)$ *and* $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.

Definition 2.2 *A* **quantale** *is a complete lattice ordered monoid* $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ *satisfying infinite distributive law, i.e.,* $\forall a, b_i \in L (i \in I)$, $a \bullet \vee\{b_i : i \in I\} = \vee\{a \bullet b_i : i \in I\}$ *and* $\vee\{b_i : i \in I\} \bullet a = \vee\{b_i \bullet a : i \in I\}$.

Definition 2.3 *A monoid* $(L, \bullet, e)$ *is called* **monoid without zero divisors** *if for all* $a, b \in L, a \neq 0, b \neq 0 \Rightarrow a \bullet b \neq 0$.

3 L-Fuzzy Automaton

In this section, we introduce the concepts of fuzzy source and fuzzy successor for L-fuzzy automata and study the separatedness and connectedness properties of such automata. We begin with the following.

Definition 3.1 [10] *Let L be an lattice-ordered monoid. An* L**-fuzzy automaton** *is a triple* $M = (Q, X, \delta)$, *where Q is a nonempty set (of* **states** *of M), X is a monoid (the* **input monoid** *of M), whose identity shall be denoted as e_X, and* $\delta : Q \times X \times Q \to L$ *is a map, such that* $\forall q, p \in Q, \forall x, y \in X$,

$$\delta(q, e_X, p) = \begin{cases} e & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\text{and} \quad \delta(q, xy, p) = \vee\{\delta(q, x, r) \bullet \delta(r, y, p) : r \in Q\}.$$

Definition 3.2 [10] *Let* (Q, X, δ) *be an L-fuzzy automaton and* $A \subseteq Q$. *The* **source** *and the* **successor** *of A are respectively the sets*

$$\sigma(A) = \{q \in Q : \delta(q, x, p) > 0, \text{ for some } (x, p) \in X \times A\}, \text{ and}$$
$$s(A) = \{p \in Q : \delta(q, x, p) > 0, \text{ for some } (x, q) \in X \times A\}.$$

Definition 3.3 *Let* (Q, X, δ) *be an L-fuzzy automaton and* $A \in L^Q$. *The* **fuzzy source** *and the* **fuzzy successor** *of A are respectively defined as:*

$$so(A)(q) = \vee\{\delta(q, x, p) \bullet A(p) : (x, p) \in X \times Q\}, \text{ and}$$
$$su(A)(q) = \vee\{A(p) \bullet \delta(p, x, q) : (x, p) \in X \times Q\}, \ q \in Q.$$

Proposition 3.1 *Let L be a lattice-ordered monoid without zero divisors and* $A \subseteq Q$. *Then for all* $p \in Q$,

(i) $so(1_A)(p) > 0$ *iff* $p \in \sigma(A)$,

(ii) $su(1_A)(p) > 0$ *iff* $p \in s(A)$.

Proof: To prove (i), let $A \subseteq Q$ and $p \in Q$. Then $so(1_A)(p) > 0 \Leftrightarrow \vee\{\delta(p, x, q) \bullet 1_A(q) : q \in Q, x \in X\} > 0 \Leftrightarrow \delta(p, x, q) \bullet 1_A(q) > 0$, for some $q \in Q, x \in X \Leftrightarrow \delta(p, x, q) > 0$ and $1_A(q) > 0$ (as L is a monoid without zero divisors) $\Leftrightarrow \delta(p, x, q) > 0$, for some $q \in A$ and $x \in X \Leftrightarrow p \in \sigma(A)$. The proof of (ii) follows analogously.

Proposition 3.2 *Let L be a quantale and* (Q, X, δ) *be an L-fuzzy automaton. Then* $\forall A, B, A_i \in L^Q$ *and* $i \in I$,

(i) *if* $A \subseteq B$, *then* $so(A) \subseteq so(B)$ *and* $su(A) \subseteq su(B)$,

(ii) $A \subseteq so(A)$ *and* $A \subseteq su(A)$,

(iii) $so(\cup\{A_i : i \in I\}) = \cup\{so(A_i : i \in I)\}$ *and* $su(\cup\{A_i : i \in I\}) = \cup\{su(A_i : i \in I)\}$,

(iv) $so(so(A)) = so(A)$ *and* $su(su(A)) = su(A)$.

Proof: We are giving here the proofs for the properties involving the operator so. The proof for the operator su can be given analogously.

(i) Let $A \subseteq B$. Then $A(p) \leq B(p), \forall p \in Q$. Now, $so(A)(p) = \vee\{\delta(p,x,q) \bullet A(q) : q \in Q, x \in X\} \leq \vee\{\delta(p,x,q) \bullet B(q) : q \in Q, x \in X\} = so(B)(p)$. Thus $so(A) \subseteq so(B)$.

(ii) Let $p \in Q$. Then $so(A)(p) = \vee\{\delta(p,x,q) \bullet A(q) : q \in Q, x \in X\} \geq \delta(p, e_X, p) \bullet A(p) \geq e \bullet A(p) = A(p)$. Thus $A \subseteq so(A)$.

(iii) Let $p \in Q$. Then $so(\cup\{A_i : i \in I\})(p) = \vee\{\delta(p,x,q) \bullet (\vee\{A_i(q) : i \in I\}) : q \in Q, x \in X\} = \vee\{\delta(p,x,q) \bullet A_i(q) : q \in Q, x \in X, i \in I\} = \cup\{so(A_i(p)) : i \in I\} = \cup\{so(A_i) : i \in I\}(p)$.

(iv) Let $p \in Q$. Then $so(so(A))(p) = \vee\{\delta(p,x,q) \bullet so(A(q)) : q \in Q, x \in X\} = \vee\{\delta(p,x,q) \bullet (\vee\{\delta(q,y,r) \bullet A(r) : r \in Q, y \in X\} : q \in Q, x \in X\} = \vee\{\delta(p,x,q) \bullet \delta(q,y,r) \bullet A(r) : r, q \in Q, y, x \in X\} = \vee\{(\vee\{\delta(p,x,q) \bullet \delta(q,y,r) : q \in Q\}) \bullet A(r) : r \in Q, y, x \in X\} = \vee\{\delta(p,xy,r) \bullet A(r) : r \in Q, y, x \in X\} = \vee\{\delta(p,z,r) \bullet A(r) : r \in Q, z \in X\} = so(A)(p)$. Hence $so(so(A)) = so(A)$.

Definition 3.4 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton. $R \in L^Q$ is called an* L**-fuzzy subautomaton** *of M if $su(R)(q) \leq R(q)$, $\forall q \in Q$. Further, this L-fuzzy subautomaton is called* **separated** *if $su(1-R)(q) \leq (1-R)(q), \forall q \in Q$.*

Remark 3.1 *For $L = [0,1]$ and $\bullet = \wedge$, L-fuzzy subautomaton of an L-fuzzy automaton defined above reduces to the fuzzy subsystem of a fuzzy automaton introduced in* [3].

Remark 3.2 *From Proposition 3.2, it follows that for an L-fuzzy automaton $M = (Q, X, \delta)$ and $R \in L^Q$, $su(R)$ is always an L-fuzzy subautomaton of M.*

Before stating next proposition, recall the following from [12].

Definition 3.5 *Let X be a nonempty set. An* L**-fuzzy point** *x_α of X is an L-fuzzy set defined by*

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases} \quad \alpha \in L.$$

Proposition 3.3 *Let R be an L-fuzzy subautomaton of L-fuzzy automaton $M = (Q, X, \delta)$. Then R is separated iff for all L-fuzzy points $p_t \in R, s_r \in (1-R), su(p_t)(q) + su(s_r)(q) \leq 1, \forall q \in Q$.*

Proof: Let $R \in L^Q$ be separated L-fuzzy subautomaton. Then $su(R) = R$ and $su(1-R) = 1-R$. Also, let $p_t \in R$, $s_r \in 1-R$ and $q \in Q$. Then $su(p_t)(q) + su(s_r)(q) \leq su(R)(q) + su(1-R)(q) = R(q) + (1-R)(q) = 1$. Conversely, let R be an L-fuzzy subautomaton of L-fuzzy automaton M such that for all L-fuzzy points $p_t \in R, s_r \in (1-R)$ and for all $q \in Q, su(p_t)(q) + su(s_r)(q) \leq 1$. Then we have to show that R is separated. Let $su(R)(q) = t = R(q)$, and $su(1-R)(q) = s$. If $t = 0$ or $s = 0$, it is trivial. If $t > 0$ and $s > 0$, then $1 = R(q) + (1-R)(q) \leq R(q) + su(1-R)(q) = q_t(q) + q_s(q) \leq su(q_t)(q) + su(q_s)(q) \leq 1$. Hence $R(q) + su(1-R)(q) = 1$, or $su(1-R)(q) = (1-R)(q)$, showing that R is separated.

Definition 3.6 *An L-fuzzy automaton M is called* **connected** *if M has no non-constant separated L-fuzzy subautomaton.*

Proposition 3.4 *An L-fuzzy automaton $M = (Q, X, \delta)$ is not connected iff there exist L-fuzzy subsets R, S of Q such that $su(R)$ and $su(S)$ are non-constant and for all $q \in Q$, $su(R)(q) + su(S)(q) = 1$.*

Proof: Let M be not connected. Then there exists a non-constant separated L-fuzzy subautomaton, say, R of M. Let $S = 1-R$. Then S is non-constant and for all $q \in Q$, $su(R)(q) + su(S)(q) = 1$.

Conversely, let there exist L-fuzzy subsets R and S of Q such that $su(R)$ and $su(S)$ be non-constant and for all $q \in Q$, $su(R)(q) + su(S)(q) = 1$. Then $su(1 - su(R))(q) = su(su(S))(q) = su(S)(q) = (1 - su(R))(q)$, whereby $su(R)$ is separated. Hence M is not connected.

4 L-Fuzzy Topologies for L-Fuzzy Automata

In this section, we see that the concepts of fuzzy source and fuzzy successor give rise to two L-fuzzy topologies on the state-set of an L-fuzzy automaton. Further, we show that the separatedness and connectedness properties of an L-fuzzy automaton can be in terms of these topologies as in [8].

Proposition 4.1 *Let (Q, X, δ) be an L-fuzzy automaton.*

(a) The fuzzy source and the fuzzy successor, viewed as functions so : $L^Q \to L^Q$ and su : $L^Q \to L^Q$, turn out to be Kuratowski L-fuzzy closure operators on Q, inducing two L-fuzzy topologies on Q (which we shall respectively denote as $\tau(Q)$ and $\tau^*(Q)$).

(b) Both the L-fuzzy topologies $\tau(Q)$ and $\tau^*(Q)$ are 'saturated' (in the sense that they are closed under arbitrary intersections also).

(c) The L-fuzzy topologies $\tau(Q)$ and $\tau^*(Q)$ are dual in the sense that each $A \in L^Q$ is L-fuzzy $\tau(Q)$-open iff A is L-fuzzy $\tau^*(Q)$-closed.

Proof: The proof for (a) and (b) follow from the Proposition 3.2 and that for (c) is straightforward.

Remark 4.1 *Note that the operators so and su are also upper L-fuzzy approximation operators on an approximation space Q (i.e., a set with an L-fuzzy reflexive and L-fuzzy transitive relation on it, (cf.,* [7]*), which is a concept used in the study of rough set theory.*

Proposition 4.2 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $R \in L^Q$. Then*
(i) R is L-fuzzy subautomaton of M iff R is L-fuzzy $\tau(Q)$-open.
(ii) R is a separated L-fuzzy subautomaton of M iff R is L-fuzzy $\tau(Q)$-clopen.
(iii) M is connected iff L-fuzzy $\tau(Q)$ is connected.

Proof: (i) This follows from the Definition 3.4 of L-fuzzy subautomaton and Proposition 4.1.
(ii) R is a separated L-fuzzy subautomaton of M iff $su(R) = R$ and $su(1 - R) = 1 - R$ iff R and $1 - R$ are L-fuzzy $\tau(Q)$-open iff R is L-fuzzy $\tau(Q)$-open and L-fuzzy $\tau(Q)$-closed iff R is L-fuzzy $\tau(Q)$-clopen.
(iii) Follows from (ii) and definition of connectedness.

Proposition 4.3 *An L-fuzzy subautomaton $R \in L^Q$ of an L-fuzzy automaton $M = (Q, X, \delta)$ is separated iff $1 - R$ is a separated L-fuzzy subautomaton of M.*

Proof: R is separated L-fuzzy subautomaton of M iff R is L-fuzzy $\tau(Q)$-clopen iff $1-R$ is L-fuzzy $\tau(Q)$-clopen iff $1 - R$ is separated L-fuzzy subautomaton of M.

Proposition 4.4 *The arbitrary union and arbitrary intersection of separated L-fuzzy subautomatons of L-fuzzy automaton M is separated L-fuzzy subautomaton of M.*

Proof: Follows from the Proposition 4.1 and the fact that the L-fuzzy topology $\tau(Q)$ is saturated.

5 Conclusion

We tried to introduce here the separatedness and connectedness properties of an L-fuzzy automaton and characterized these properties in terms of well known L-fuzzy topological concepts, which are then used to show some separatedness properties of an L-fuzzy automaton. Even, it may be interesting to introduce the concept of primaries of such automata and their decompositions as in [8, 9, 11]. Also, it may be worthwhile to see if some standard L-fuzzy topological concepts have some meaningful L-fuzzy automata theoretic interpretations, when applied to the L-fuzzy topology associated with such automata.

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