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A New Method for Multi-Objective Linear Programming Models with Fuzzy Random Variables

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Abstract

This paper proposes a new method for solving a multi-objective linear programming model with fuzzy random variables. In this model, a multi-objective linear programming problem with real variables and fuzzy random coefficients is introduced. Then, a new algorithm is developed to solve the model based on the concepts of mean value of fuzzy random variables, chance-constrained programming and piecewise linear approximation method. Furthermore, an illustrative numerical example is also given to clarify the method discussed in this paper.

Keywords: multi-objective programming, fuzzy random variable, chance-constrained programming, integer programming

1 Introduction

In a real world, we must often make a decision on the basis of uncertain data or information. For such decision making problems involving uncertainty, there exist two typical approaches: stochastic programming and fuzzy programming.

Stochastic programming model can be applied when probabilistic properties of unknown elements are at hand [12, 30]. By having several objective functions, multi-objective stochastic linear programming model is appropriate. Stancu-Minasian [28, 29] used the minimum risk approach, while Leclercq [17] and Teghem Jr. et al. [31] proposed interactive methods for solving multi-objective stochastic linear programming problems.

Fuzzy optimization [4, 18, 19, 21] has to be chosen when we encounter inherent imprecision or vagueness [35, 36]. Therefore, fuzzy mathematical programming representing the uncertainty or ambiguity in decision making situations by fuzzy concepts has attracted attention of many researchers [15, 16, 26, 27]. Fuzzy multi-objective linear programming, first proposed by Zimmermann [37] has been rapidly developed by numerous researchers and it is most frequently applied to the increasing number of real problems.

As a mixture of the stochastic approach and fuzzy approach, Wang and Zhong [33] considered mathematical programming problems with fuzzy random variables. A Fuzzy Random Variable (FRV) is a random variable whose actual value is a fuzzy number. The concept of fuzzy random variables was first introduced by Kwakernaak [14] and then developed by Puri and Ralescu [25].

In addition, Chakraborty et al. [5] presented an interactive method for multi-objective linear programming problems with fuzzy number coefficients and normal random variable in objective functions and/or constraints, and Katagiri et al. [13] suggested an interactive method to solve the fuzzy random multi-objective 0-1 programming problem. Eshghi et al. [10] discussed special classes of mathematical programming models with FRVs and fuzzy random quadratic minimum spanning tree problem in which Er-expected value of FRVs was applied. They also used fuzzy stochastic optimization in the redundancy optimization problem and in the resource-constraint project scheduling problem [22, 23, 24].

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This paper considers multi-objective LP problems whose parameters are FRVs but the decision variables are crisp. The aim of this paper is to introduce a new method for this type of problems by using fuzzy stochastic optimization and integer programming.

Indeed, a new model is introduced for a multi-objective linear programming in which the parameters have both fuzzy and random properties, simultaneously. We consider both of them and FRVs can be useful to illustrate this hybrid uncertainty. Then, a new method is developed to solve this new model. Firstly, the mean value of FRVs and Chance-Constrained Programming (CCP) method are used to convert fuzzy stochastic programming model to mathematical programming model. Subsequently, Piecewise Linear Approximation (PLA) approach is applied to obtain a linear programming model.

This paper is organized as follows: in Section 2 some basic concepts on fuzzy theory and fuzzy random theory are presented. In Section 3, our model is introduced as a multi-objective linear programming with FRVs. Then the proposed problem is converted to a new model by using the concepts of mean value of FRVs, CCP and PLA method (integer programming). Furthermore, Er-expected value model [10, 11] will be used to solve the proposed problem. Finally, appropriate algorithms for solving the proposed problem are presented and an example is also solved and analyzed to clarify the algorithm.

2 Preliminaries

This section reviews some technical terms presented by Puri and Ralescu [25]. In the following definitions, we assume that $(\Omega, \Im, \mathrm{P})$ is a probability space and $(\Theta, \mathrm{P}(\Theta), \mathrm{Pos})$ is a possibility space where Θ is universe, $\mathrm{P}(\Theta)$ is the power set of Θ and Pos is a possibility measure defined on fuzzy sets. Furthermore, $\mathrm{F_c}(\Re)$ is a collection of all normalized fuzzy numbers whose α-level sets are convex subsets of $\Re$.

A fuzzy set on $\Re$ is called a fuzzy number if it is normal, convex and upper semi-continuous and its support set is compact. LR fuzzy number [7, 9] is a special fuzzy number used frequently. We will use standard fuzzy arithmetic, from the extension principle, to perform sums, products, etc. of fuzzy numbers [8].

A FRV is a random variable and a Borel measurable function whose actual value is a fuzzy number [25]. It is frequently used in uncertain systems.

Lemma 1 [32] If X is a FRV, then an α-cut $X_\alpha(\omega) = \{t \in \Re \mid \mu_{X(\omega)}(t) \geq \alpha\} = [X_\alpha^-(\omega), X_\alpha^+(\omega)]$ is a random interval for every $\alpha \in (0,1]$.

Expected value is a fundamental concept for FRV and. In order to define the expected value of an FRV, several operators were introduced in literature [33]. The expectation of a FRV is a fuzzy number [32, 33].

Definition 1 [10, 11] Let X be a FRV then we can define the scalar expected value of X, denoted by $Er(X)$ and called it Er-expected value of X, as follows:

$$Er(X) = \frac{1}{2}\int_0^1 \{E(X_\alpha^-) + E(X_\alpha^+)\}\, d\alpha$$

where $E(X_\alpha^-)$ and $E(X_\alpha^+)$ are expected values of X_α^- and X_α^+ respectively.

Definition 2 If X is a FRV, then for any $\omega \in \Omega$, $X(\omega)$ is fuzzy number. We define the mean value of fuzzy number $X(\omega)$, denoted by $M(X(\omega))$, as follows:

$$M\big(X(\omega)\big) = \frac{1}{2}\int_0^1 [X_\alpha^-(\omega) + X_\alpha^+(\omega)]d\alpha \qquad \forall \omega \in \Omega.$$

It is clear that M(X) is random variable.

Corollary 1 Let X be a FRV. $E\big(M(X)\big) = Er(X)$.

Proof: $M(X)$ is a random variable. Therefore, $E\big(M(X)\big) = \int_\Omega M\big(X(\omega)\big)P(d\omega)$ and from definition 1 it is equal to $Er(X)$.

Definition 3 Let X be a FRV. We define the scalar variance of X, denoted by $Vr(X)$, as follows:

$$Vr(X) = Var\big(M(X)\big) = \int_{\Omega} \Big(M\big(X(\omega)\big)\Big)^2 P(d\omega) - (Er(X))^2.$$

Corollary 2 Let X and Y be FRV and $\lambda \in \Re$ then $M(X + \lambda Y) = M(X) + \lambda M(Y)$.

Proof: Since $(X + \lambda Y)_\alpha = X_\alpha + \lambda Y_\alpha$ then $\mathrm{M}(\mathrm{X} + \lambda\mathrm{Y}) = \frac{1}{2}\int_0^1 \{(\mathrm{X} + \lambda\mathrm{Y})_\alpha^- + (\mathrm{X} + \lambda\mathrm{Y})_\alpha^+\} d\alpha = \frac{1}{2}\int_0^1 (X_\alpha^- + X_\alpha^+) d\alpha + \lambda\frac{1}{2}\int_0^1 (Y_\alpha^- + Y_\alpha^+) d\alpha = M(X) + \lambda M(Y)$. □

Now we discuss a method to evaluate the fuzzy random inequality $\mathrm{X} \lesseqqgtr \mathrm{Y}$ or $\mathrm{X} \gtreqqless \mathrm{Y}$ where X and Y are FRVs. It can be easily compared by Er-expected value of FRVs. Furthermore, it is obvious that $\mathrm{M(X)}$ and $\mathrm{M(Y)}$ in this case are random variable according to definition 2 and can be compared based on CCP method.

Definition 4 Let X and Y be FRVs. Then the relations " $\cong$ " and " $\lesseqqgtr$" are defined respectively as follows:

i) $X \cong Y$ Iff $M(X) = M(Y)$;

ii) $X \lesseqqgtr Y$ Iff $M(X) \le M(Y)$.

3 Problem Formulation

In this section, Fuzzy Multi-Objective Linear Programming (FMOLP) and Fuzzy Random Multi-Objective Linear Programming (FRMOLP) are introduced and solved. Er-expected value model [10] and Mean & CCP Model which is a new method will be used to FRMOLP.

3.1 Fuzzy Multi-Objective LP

The fuzzy multi-objective linear programming is proposed as follows:

(FMOLP)

$$\begin{aligned} \text{Max} \quad & [\tilde{c}_1^t x, \tilde{c}_2^t x, \dots, \tilde{c}_r^t x] \\ \text{s.t.} \quad & \tilde{a}_i x \le \tilde{b}_i, \quad i = 1, \dots, m \\ & x \ge 0 \end{aligned}$$

where all the parameters are fuzzy variables. Furthermore, $x = (x_1, x_2, \dots, x_n)$ is the crisp decision variable vector. If a FRV is degenerated to a fuzzy variable, its mean value is a real number. The mean value model of FMOLP, denoted by M-FMOLP, is defined as follows:

(M-FMOLP)

$$\begin{aligned} \text{Max} \quad & [M(\tilde{c}_1^t x), M(\tilde{c}_2^t x), \dots, M(\tilde{c}_r^t x)] \\ \text{s.t.} \quad & M(\tilde{a}_i x) \le M(\tilde{b}_i), \quad i = 1, \dots, m \\ & x \ge 0. \end{aligned}$$

By this model whose all parameters are real, solutions with optimal expected return subject to expected constraints will be obtained. Therefore, we can design the following algorithm to solve FMOLP:

Algorithm 3.1 (Expected Value Algorithm)

Step 1. Define fuzzy parameters of FMOLP by using information of experts or decision makers and determine their mean values.

Step 2. Convert FMOLP to M-FMOLP by using mean value of fuzzy variables and Corollary (2).

Step 3. Solve M-FMOLP which is a multi-objective optimization problem by Zimmermann approach [37]. The obtained optimal solution is called M-optimal solution of the original problem.

We will explain Zimmermann approach at the end of this section.

3.2 Fuzzy Random Multi-Objective LP

Consider the following multi-objective linear programming with FRVs:

(FRMOLP)

$$\begin{aligned} \text{Max} \quad & \left[\tilde{\bar{c}}_1^t x, \tilde{\bar{c}}_2^t x, \dots, \tilde{\bar{c}}_r^t x\right] \\ \text{s.t.} \quad & \tilde{\bar{a}}_i x \leq \tilde{\bar{b}}_i, \quad i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

where $\tilde{\bar{c}}_i = (\tilde{\bar{c}}_{k1}, \tilde{\bar{c}}_{k2}, \dots, \tilde{\bar{c}}_{kn}), k = 1, \dots, l,$ $\tilde{\bar{A}} = [\tilde{\bar{a}}_{ij}]_{m \times n},$ $\tilde{\bar{b}} = \left(\tilde{\bar{b}}_1, \tilde{\bar{b}}_2, \dots, \tilde{\bar{b}}_m\right)^t$ represent FRVs involved in the objective functions and constraints respectively.

3.2.1 Er-expected Value Model

Er-expected value model of FRMOLP, represented by Er-FRMOLP, is defined as follows:

(Er-FRMOLP)

$$\begin{aligned} \text{Max} \quad & \left[Er(\tilde{\bar{c}}_1^t x), Er(\tilde{\bar{c}}_2^t x), \dots, Er(\tilde{\bar{c}}_r^t x)\right] \\ \text{s.t.} \quad & Er(\tilde{\bar{a}}_i x) \leq Er(\tilde{\bar{b}}_i), \quad i = 1, \dots, m \\ & x \geq 0. \end{aligned}$$

By this model whose all parameters are real, solutions with optimal Er-expected return subject to Er-expected constraints will be obtained. We design the following algorithm to solve FRMOLP:

Algorithm 3.2 (Er-expected Value Algorithm)
Step 1. Define fuzzy random parameters of FRMOLP by using information of experts or decision makers and determine their Er-expected values.
Step 2. Convert FRMOLP to Er-FRMOLP by using Er-expected value of fuzzy random variables and Corollary (2).
Step 3. Solve Er-FRMOLP which is a multi-objective optimization problem by Zimmermann approach. The obtained optimal solution is called Er-optimal solution of the original problem.

3.2.2 Mean & CCP Model

In this model, mean value of FRVs and CCP method are used. The mean value of FRMOLP, denoted by M-FRMOLP, is defined as follows:

(M-FRMOLP)

$$\begin{aligned} \text{Max} \quad & \left[M(\tilde{\bar{c}}_1^t x), M(\tilde{\bar{c}}_2^t x), \dots, M(\tilde{\bar{c}}_r^t x)\right] \\ \text{s.t.} \quad & M(\tilde{\bar{a}}_i x) \leq M(\tilde{\bar{b}}_i), \quad i = 1, \dots, m \\ & x \geq 0. \end{aligned}$$

Now, we have the following stochastic linear programming model:

$$\begin{aligned} \text{Max} \quad & [\bar{z}_1, \bar{z}_2, \dots, \bar{z}_r] = \left[\bar{c}_1^t x, \bar{c}_2^t x, \dots, \bar{c}_r^t x\right] \\ \text{s.t.} \quad & \bar{a}_i x \leq \bar{b}_i, \quad i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

where $\bar{z}_k = M(\tilde{\bar{z}}_k)$, $\bar{c}_k = (\bar{c}_{k1}, \bar{c}_{k2}, \dots, \bar{c}_{kn}) = M(\tilde{\bar{c}}_k)$, $k = 1, \dots, r$, $\bar{A} = [\bar{a}_{ij}]_{m\times n} = M(\tilde{\bar{A}})$, $\bar{b} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_m)^t = M(\tilde{\bar{b}})$ are random variables.

By using the chance-constrained multi-objective programming model [20], M-FRMOLP can be written as follows and denoted by M&CCP-FRMOLP:

(M&CCP-FRMOLP)

$$\begin{aligned} \text{Max } & [z_1, z_2, \dots, z_r] \\ \text{s.t. } & Pr\{\bar{c}_k^t x \geq z_k\} \geq \alpha, \quad k = 1, \dots, r \\ & Pr\{\bar{a}_i x \leq \bar{b}_i\} \geq \beta, \quad i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

where α and β are the predetermined confidence levels defined by a decision maker. In this model, the random variables can have different probability distributions. Therefore, several models will be found from M&CCP-FRMOLP. Based on probability properties and expert opinions, a proper model will be selected and solved by Zimmermann approach. Anyway, the following algorithm is designed.

Algorithm 3.3 (M&CCP Algorithm)
Step 1. Define fuzzy random parameters of FRMOLP by using information of experts or decision makers and determine their mean values.
Step 2. Convert FRMOLP to M-FRMOLP by using the mean value of fuzzy random variables and Corollary (2).
Step 3. Convert M-FRMOLP to M&CCP-FRMOLP by using chance-constrained multi-objective programming method.
Step 4. Solve M&CCP-FRMOLP which is a multi-objective optimization problem by Zimmermann approach.

3.2.3 FRMOLP Model with Normal Distribution

Let the random aspect of FRVs has a normal distribution with a definite expectation and an exact variance. Therefore, we have the following mathematical programming:

(Problem 1)

$$\begin{aligned} \text{Max } & [z_1, z_2, \dots, z_r] \\ \text{s.t. } & z_k - E(\bar{c}_k^t)x \leq k_\alpha \sqrt{\sum_{j=1}^{n} Var(\bar{c}_{kj}) x_j^2}, \quad k = 1, \dots, r \\ & E(\bar{a}_i)x - k_\beta \sqrt{\sum_{j=1}^{n} Var(\bar{a}_{ij}) x_j^2 + Var(\bar{b}_i)} \leq E(\bar{b}_i), \quad i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

where k_α is a point that $Pr\{U \sim N[0,1] \geq k_\alpha\} = \alpha$. Problem (1) is a nonlinear programming model and we use PLA approach [6, 34] to solve it and obtain an approximated global optimal solution.

Let $f(x_1, x_2, \dots, x_n)$ be a nonlinear function with n variables. We say that f is a separable function if $f(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} f_j(x_j)$ where $f_j(x_j), j = 1, \dots, n$, are functions of a single variables. A nonlinear programming model is a separable programming if all nonlinear functions can be converted to separable functions.

Separable programming has an important role in PLA approach. Fortunately, all nonlinear functions in our model are convertible to separable functions. New, PLA approach is applied to obtain an approximated linear programming model. The following problem is a separable programming form of problem (1):

(Problem 2)

$$\begin{aligned}
&\text{Max}\quad [z_1, z_2, \dots, z_r]\\
&\text{s.t.}\quad z_k - \sum_{j=1}^{n} c_{kj}x_j \le k_\alpha\, t_k, \quad k = 1, \dots, r\\
&t_k^{\,2} = \sum_{j=1}^{n} Var(\bar{c}_{kj})x_j^2, \quad k = 1, \dots, r\\
&\sum_{j=1}^{n} a_{ij}x_j - k_\beta w_i \le b_i, \quad i = 1, \dots, m\\
&w_i^{\,2} = \sum_{j=1}^{n} Var(\bar{a}_{ij})x_j^2 + Var(\bar{b}_i), \quad i = 1, \dots, m\\
&t_k \ge 0, \quad k = 1, \dots, r\\
&w_i \ge 0, \quad i = 1, \dots, m\\
&x \ge 0
\end{aligned}$$

where $c_k = (c_{k1}, c_{k2}, \dots, c_{kn}) = E(\bar{c}_k), k = 1, \dots, r$, $[a_{ij}]_{m\times n} = E(\bar{A})$, $(b_1, b_2, \dots, b_m)^t = E(\bar{b})$.

Now, interval and break points for each decision variable are considered as described in Table 1.

Table 1: Break points and new decision variables of PLA programming

Decision variables	Interval	Break points	New decision variables
x_j	$[a_j, b_j]$	$p_{j1}, p_{j2}, \dots, p_{js}$ $\quad a_j = p_{j1} \le p_{j2} \le \cdots \le p_{js} = b_j$	λ_{jl}
t_k	$[c_k, d_k]$	$q_{k1}, q_{k2}, \dots, q_{ks}$ $\quad c_k = q_{k1} \le q_{k2} \le \cdots \le q_{ks} = d_k$	η_{kl}
w_i	$[e_i, f_i]$	$r_{i1}, r_{i2}, \dots, r_{is}$ $\quad e_i = r_{i1} \le r_{i2} \le \cdots \le r_{is} = f_i$	κ_{il}

By using new decision variables and PLA programming approach, Problem (2) which is a separable programming model can be converted as follows:

(Problem 3)

$$\begin{aligned}
&\text{Max}\quad [z_1, z_2, \dots, z_r]\\
&\text{s.t.}\quad z_k - \sum_{j=1}^{n}\sum_{l=1}^{s} c_{kj}p_{jl}\lambda_{jl} \le k_\alpha \sum_{l=1}^{s} q_{kl}\eta_{kl}, \quad k = 1, \dots, r\\
&\sum_{j=1}^{n}\sum_{l=1}^{s} a_{ij}\, p_{jl}\lambda_{jl} - k_\beta \sum_{l=1}^{s} r_{il}\kappa_{il} \le b_i, \quad i = 1, \dots, m\\
&\sum_{l=1}^{s} q_{kl}^{\,2}\eta_{kl} = \sum_{j=1}^{n}\sum_{l=1}^{s} Var(\bar{c}_{kj})p_{jl}^{\,2}\lambda_{jl}, \quad k = 1, \dots, r\\
&\sum_{l=1}^{s} r_{il}^{\,2}\kappa_{il} = \sum_{i=1}^{n}\sum_{l=1}^{s} Var(\bar{a}_{ij})p_{jl}^{\,2}\lambda_{jl} + Var(\bar{b}_i), \quad i = 1, \dots, m\\
&\sum_{l=1}^{s} \lambda_{jl} = 1, \quad j = 1, \dots, n\\
&\sum_{l=1}^{s} \eta_{kl} = 1, \quad k = 1, \dots, r
\end{aligned}$$

$$\sum_{l=1}^{s} \kappa_{il} = 1, \quad i = 1, \dots, m$$
$$\left(\lambda_{j1}, \lambda_{j2}, \dots, \lambda_{js}\right) \in SOS2, \quad j = 1, \dots, n$$
$$\left(\eta_{k1}, \eta_{k2}, \dots, \eta_{ks}\right) \in SOS2, \quad k = 1, \dots, r$$
$$\left(\kappa_{i1}, \kappa_{i2}, \dots, \kappa_{is}\right) \in SOS2, \quad i = 1, \dots, m$$
$$0 \le \lambda_{jl} \le 1, \quad j = 1, \dots, n, l = 1, \dots, s$$
$$0 \le \eta_{kl} \le 1, \quad k = 1, \dots, r, l = 1, \dots, s$$
$$0 \le \kappa_{il} \le 1, \quad i = 1, \dots, m, l = 1, \dots, s$$

where $SOS2$ constraints refer to special ordered sets of type 2 introduced by Beale and Tomlin [2] and developed by Beale and Forrest [1]. The above problem is a multi-objective Mixed Integer Programming (MIP) problem. Now, Zimmermann fuzzy approach [37] will be used to solve multi-objective optimization problem. In this approach, Bellman and Zadeh's max-min operator [3] has been used.

Let $z^{(k)}$ be upper bounds of $z_k, k = 1, \dots, r$, and $z^{(k)} - p^{(k)}$ be their initial values, $k = 1, \dots, r$. By considering the membership function of fuzzy objective function and using Bellman and Zadeh's max-min operator, Problem (3) can be converted to the following problem:

(Problem 4)

$$\text{Max} \quad \lambda$$
$$\text{s.t.} \quad z_k \ge z^{(k)} - (1-\lambda)p^{(k)}, \quad k = 1, \dots, r$$
$$z_k - \sum_{j=1}^{n}\sum_{l=1}^{s} c_{kj} p_{jl} \lambda_{jl} \le k_\alpha \sum_{l=1}^{s} q_{kl}\eta_{kl}, \quad k = 1, \dots, r$$
$$\sum_{j=1}^{n}\sum_{l=1}^{s} a_{ij}\, p_{jl}\lambda_{jl} - k_\beta \sum_{l=1}^{s} r_{il}\kappa_{il} \le b_i, \quad i = 1, \dots, m$$
$$\sum_{l=1}^{s} {q_{kl}}^2 \eta_{kl} = \sum_{j=1}^{n}\sum_{l=1}^{s} Var\left(\overline{c}_{kj}\right) {p_{jl}}^2 \lambda_{jl}, \quad k = 1, \dots, r$$
$$\sum_{l=1}^{s} {r_{il}}^2 \kappa_{il} = \sum_{i=1}^{n}\sum_{l=1}^{s} Var\left(\overline{a}_{ij}\right) {p_{jl}}^2 \lambda_{jl} + Var\left(\overline{b}_i\right), \quad i = 1, \dots, m$$
$$\sum_{l=1}^{s} \lambda_{jl} = 1, \quad j = 1, \dots, n$$
$$\sum_{l=1}^{s} \eta_{kl} = 1, \quad k = 1, \dots, r$$
$$\sum_{l=1}^{s} \kappa_{il} = 1, \quad i = 1, \dots, m$$
$$\left(\lambda_{j1}, \lambda_{j2}, \dots, \lambda_{js}\right) \in SOS2, \quad j = 1, \dots, n$$
$$\left(\eta_{k1}, \eta_{k2}, \dots, \eta_{ks}\right) \in SOS2, \quad k = 1, \dots, r$$
$$\left(\kappa_{i1}, \kappa_{i2}, \dots, \kappa_{is}\right) \in SOS2, \quad i = 1, \dots, m$$
$$0 \le \lambda_{jl} \le 1, \quad j = 1, \dots, n, l = 1, \dots, s$$
$$0 \le \eta_{kl} \le 1, \quad k = 1, \dots, r, l = 1, \dots, s$$
$$0 \le \kappa_{il} \le 1, \quad i = 1, \dots, m, l = 1, \dots, s.$$

The above problem is a MIP problem which can be solved by one of the MIP solvers. If we suppose that $\lambda_{jl}^{\star}, \eta_{kl}^{\star}, \kappa_{il}^{\star},\ i = 1, \ldots, m,\ j = 1, \ldots, n, k = 1, \ldots, r, l = 1, \ldots, s$ are the optimal solution of Problem 4 then a PLA-optimal solution of the original problem can be obtained by $x_j^{\star}(PLA) = \sum_{l=1}^{s} P_{jl}\lambda_{jl}^{\star},\ j = 1, \ldots, n$.

In the following steps, we summarize the necessary steps to solve the problem discussed in this section.

Algorithm 3.4: (PLA Algorithm)
Step 1. Convert M&CCP-FRMOLP to Problem (1) by using the concept of normal probability distribution of the random aspect of FRVs.
Step 2. Convert Problem (1) to Problem (3) by applying Piecewise Linear Approximation method.
Step 3. Convert Problem (3) to Problem (4) by Zimmermann approach (Bellman and Zadeh's Max-min operator). Problem (4) is a mixed integer programming model.
Step 4. Solve Problem (4) as a mixed integer programming model by one of the MIP solvers. Then, $\lambda_{jl}^{\star}$ is an optimal solution of PLA method and an optimal solution of the original problem is obtained by $x_j^{\star}(PLA) = \sum_{l=1}^{s} p_{jl}\lambda_{jl}^{\star},\ j = 1, \ldots, n.$

As this section shows, FMOLP and FRMOLP are used to modeling a multi-objective linear programming problem whose parameters have vague properties. This vagueness can be possibilistic imprecision, probability uncertainty or both of them. Fuzzy, random, and fuzzy random variables s can be applied for it respectively. Furthermore, Expected value model is used for FMOLP and Er-expected value model and Mean & CCP model are used for FRMOLP. Mean & CCP model is a quite complex approach but its results are more veracious because of using the probability distribution function of the random aspect of fuzzy random parameters instead of using the expect value of them.

In our model, parameters are assumed to be fuzzy random variables which are more suitable to real-world problems. However, the model is not well defined theoretically in this case due to fuzzy randomness of parameters. Therefore, we used the concept of mean value of fuzzy random variables, CCP and PLA approaches to overcome this problem. This replacement enabled us to convert the original complex model to a mixed integer programming model. Another advantage of our method is using the probability distribution function and the variance effect of parameter which have a direct effect on the optimal solutions. However, using the expectation value is tantamount to focus on the center of the distribution while neglecting other parameters of the distribution.

Unfortunately, obtaining the optimal solution is not an easy task due to the complexity of the final nonlinear programming model if some other parameters of the fuzzy random variables are taken into consideration and the generalization of the model in this case can be an interesting idea for future researches.

Now, we will formulate a problem with these different assumptions. Then, our proposed Algorithms will be applied to obtain optimal solutions.

4 A Numerical Example

In this section, we will design a Production Planning problem with an uncertain framework and two objective functions. One of the objective functions is to maximize the total benefit of the production process and the other one is also to maximize the total quality rate. This problem has two resource constraints. Furthermore, parameters of this problem have uncertain properties in both random and fuzzy aspects. Anyway, the problem is described as follows:

In a factory, a manager plans to manufacture two new products A and B. He has two major resources, resource1 and resource 2. The estimated usage rates of resources for producing a batch of product A are the following: about $\tilde{\bar{a}}_1$ for resource 1 and about $\tilde{\bar{a}}_3$ for resource 2. On the other hand, the usage rates of resources for producing a batch of product B are as follows: about $\tilde{\bar{a}}_2$ for resource 1 and about $\tilde{\bar{a}}_4$ for resource 2. The availability of resource 1 and 2 are about $\tilde{\bar{d}}_1$ and $\tilde{\bar{d}}_2$ respectively and the profits of product A and B are about $\tilde{\bar{c}}_1$ and $\tilde{\bar{c}}_2$ respectively. Furthermore, the quality performance rats of product A and B are about $\tilde{\bar{c}}_3$ and $\tilde{\bar{c}}_4$.

For simplicity we have assumed that all uncertain parameters are FRVs and their values are given in Table 2.

Table 2: Uncertain parameters of the example

$\tilde{\bar{A}}$	$\tilde{\bar{c}}_1$	$\tilde{\bar{c}}_2$	$\tilde{\bar{c}}_3$	$\tilde{\bar{c}}_4$	$\tilde{\bar{a}}_1$	$\tilde{\bar{a}}_2$	$\tilde{\bar{a}}_3$	$\tilde{\bar{a}}_4$	$\tilde{\bar{d}}_1$	$\tilde{\bar{d}}_2$
$r \sim N[\mu, \sigma^2]$	$N(19, 5^2)$	$N(29, 4^2)$	$N(32, 4^2)$	$N(21, 5^2)$	$N(4, 2^2)$	$N(2, 2^2)$	$N(3, 1.5^2)$	$N(6, 3^2)$	$N(34, 4^2)$	$N(40, 5^2)$
β	2.3	2.7	3.8	2.5	1.8	1.5	2.3	3.7	1.8	4.2
γ	4.3	3.7	1.8	0.5	0.8	2.5	4.3	1.7	3.8	2.2
$Er(\tilde{\bar{A}}) = E(\bar{A})$	19.5	29.25	31.5	20.5	3.75	2.25	3.5	5.5	34.5	39.5

Now, we want to determine how many Product A and B should be manufactured in order to maximize the total production benefit and maximize the total quality performance.

The multi-objective linear programming model with FRVs of the problem is formulated as follows:

$$\begin{aligned} \text{Max} \quad & [\tilde{\bar{z}}_1, \tilde{\bar{z}}_2] = [\tilde{\bar{c}}_1 x_1 + \tilde{\bar{c}}_2 x_2, \tilde{\bar{c}}_3 x_1 + \tilde{\bar{c}}_4 x_2] \\ s.t. \quad & \tilde{\bar{a}}_1 x_1 + \tilde{\bar{a}}_2 x_2 \le \tilde{\bar{d}}_1, \\ & \tilde{\bar{a}}_3 x_1 + \tilde{\bar{a}}_4 x_2 \le \tilde{\bar{d}}_2, \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

Let $\tilde{\bar{A}} = (r, \beta, \gamma)$ be FRV where r is random variable, $E(r) = \mu$, $Var(r) = \sigma^2$. The mean value of $\tilde{\bar{A}}$ is calculated as follows which is random variable:

$$\tilde{\bar{A}}_\alpha^- = r + \beta(\alpha - 1), \quad \tilde{\bar{A}}_\alpha^+ = r + \gamma(1 - \alpha),$$

$$M(\tilde{\bar{A}}) = \frac{1}{2}\int_0^1 (X_\alpha^- + X_\alpha^+) d\alpha = r + \frac{1}{4}(\gamma - \beta) = \bar{A}.$$

Apply Er-expected Value Algorithm to the above fuzzy random multi-objective linear programming problem and solve the obtained LP problem by the one of the LP solver. The following LP model is generated:

$$\begin{aligned} \text{Max} \quad & \lambda \\ s.t. \quad & 19.5x_1 + 29.25x_2 \ge 217.1 - 7(1 - \lambda), \\ & 31.5x_1 + 20.5x_2 \ge 293.2 - 3.4(1 - \lambda), \\ & 3.75x_1 + 2.25x_2 \le 34.5, \\ & 3.5x_1 + 5.5x_2 \le 39.5, \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

The above problem can be easily solved by LINGO 8.0, which is one of the commercial ILP solvers, and the Er-optimal solution of the problem is reported in the second column of Table 3.

Table 3: Numerical results of our example

	Fuzzy random parameters			Fuzzy parameters (optimal solution)
	Er-expected value model (Er-optimal solution)	Mean & CCP model		
		(PLA-optimal solution)	Optimal solution by LINGO Global Solver	
$x^\star = (x_1^\star, x_2^\star)$	$x_{Er}^\star(FRMOLP) = (7.91, 2.15)$	$x_{PLA}^\star(FRMOLP) = (4.20, 2.07)$	$x_{Lingo}^\star(FRMOLP) = (4.37, 2.02)$	$x_M^\star(FMOLP) = (6.13, 2.71)$
$(z_1^\star, z_2^\star)$	$([\tilde{\bar{z}}_1, \tilde{\bar{z}}_2])_{Er}^\star$ =(217.08,293.24)	$([\tilde{\bar{z}}_1, \tilde{\bar{z}}_2])_{PLA}^\star$ =(113.66,149.35)	$([\tilde{\bar{z}}_1, \tilde{\bar{z}}_2])_{Lingo}^\star$ =(114.47,153.26)	$([\tilde{\bar{z}}_1, \tilde{\bar{z}}_2])_M^\star$ =(211.82,255.48)

Now apply M&CCP Algorithm to the above fuzzy random multi-objective linear programming problem. The following nonlinear programming model is generated by considering the predetermined confidence levels:

$$
\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & z_k \geq z^{(k)} - (1-\lambda)p^{(k)}, \quad k = 1,2 \\
& z_1 - 19.5x_1 - 29.25x_2 \leq k_\alpha \sqrt{25x_1^2 + 16x_2^2}, \\
& z_2 - 31.5x_1 - 20.5x_2 \leq k_\alpha \sqrt{16x_1^2 + 25x_2^2}, \\
& 3.75x_1 + 2.25x_2 - k_\beta \sqrt{4x_1^2 + 4x_2^2 + 16} \leq 34.5, \\
& 3.5x_1 + 5.5x_2 - k_\beta \sqrt{2.25x_1^2 + 9x_2^2 + 25} \leq 39.5, \\
& x_1 \geq 0, x_2 \geq 0.
\end{aligned}
$$

Use the proposed PLA Algorithm to the nonlinear programming problem. The following separable programming problem can be obtained:

$$
\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & z_k \geq z^{(k)} - (1-\lambda)p^{(k)}, \quad k = 1,2 \\
& z_1 - 19.5x_1 - 29.5x_2 \leq k_\alpha t_1, \\
& t_1^2 = 25x_1^2 + 16x_2^2, \\
& z_2 - 31.5x_1 - 20.5x_2 \leq k_\alpha t_2, \\
& t_2^2 = 16x_1^2 + 25x_2^2, \\
& 3.75 + 2.25x_2 - k_\beta w_1 \leq 34.5, \\
& w_1^2 = 4x_1^2 + 4x_2^2 + 16, \\
& 3.5x_1 + 5.5x_2 - k_\beta w_2 \leq 39.5, \\
& w_2^2 = 2.25x_1^2 + 9x_2^2 + 25, \\
& x_1 \geq 0, x_2 \geq 0.
\end{aligned}
$$

The estimated interval of x_1, x_2, t_1, t_2, w_1 and w_2 and their break points are given in Table 4 for $\alpha = \beta = 0.9$.

Table 4: The estimated intervals and the break points

Variables	Estimated intervals	Break points
x_1	[0, 6]	0, 2, 4, 6
x_2	[0,4]	0, 2, 4
t_1	[0,25]	0, 5, 10, 15, 20, 25
t_2	[0,20]	0, 5, 10, 15, 20
w_1	[4,12]	4, 8, 12
w_2	[5,15]	5, 10, 15

PLA method is used to convert the above separable programming problem to a Mixed Integer programming Model. The PLA-optimal solution of the problem is reported in the second column of Table 5.

My new approach used mean value of fuzzy random variable to handle fuzzy properties of fuzzy stochastic programming, chance-constrained programming to manage random properties of fuzzy stochastic programming and PLA method to solve the obtained nonlinear programming problem. By comparing the results of Er-expected value model, we realize that the variance effect of parameters is important and has a direct effect on the optimal solutions.

Table 5: Numerical results of PLA method

	Mean & CCP model (PLA-optimal solution)
$(\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14})$	(0,0,0.899,0.101)
$(\lambda_{21}, \lambda_{22}, \lambda_{23})$	(0,0.961,0.039)
$(\lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{34}, \lambda_{35}, \lambda_{36})$	(0,0,0,0,0.459,0.541)
$(\lambda_{41}, \lambda_{42}, \lambda_{43}, \lambda_{44}, \lambda_{45})$	(0,0,0,0,1)
$(\lambda_{51}, \lambda_{52}, \lambda_{53})$	(0,0.476,0.524)
$(\lambda_{61}, \lambda_{62}, \lambda_{63})$	(0,0.954,0.046)
$x^\star = (x_1^\star, x_2^\star)$	$x_{PLA}^\star(FRMOLP) = (4.20,2.07)$
$(z_1^\star, z_2^\star)$	$([\tilde{\bar{z}}_1, \tilde{\bar{z}}_2])_{PLA}^\star$=(113.66,149.35)

We also solve the nonlinear programming model, obtained by M&CCP Algorithm, by Global solver of LINGO 8.0 and the results are collected in the forth column of Table 3. By comparing the results, we can see that the results of PLA method are extremely near to the optimal solution of LINGO Global Solver.

Now, the above production planning problem is also solved by considering the assumption that the parameters are fuzzy numbers obtained by experts or decision makers and their values are given in Table 6. In this case, information of experts is more effective to define and determine uncertain parameters.

Table 6: Fuzzy parameters of the example

$\tilde{A} = (r, \beta, \gamma)$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{c}_3$	$\tilde{c}_4$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{d}_1$	$\tilde{d}_2$
r	22	27	34	19	5	1	4	6	32	43
β	2.3	2.7	3.8	2.5	1.8	1.5	2.3	3.7	1.8	4.2
γ	4.3	3.7	1.8	0.5	0.8	2.5	4.3	1.7	3.8	2.2
$M(\tilde{A})$	22.5	27.25	33.5	18.5	4.75	1.25	4.5	5.5	32.5	42.5

The fuzzy multi-objective linear programming model can be used to formulate this production planning problem with fuzzy number parameters.

We apply the expected value Algorithm and solve the obtained LP problem by LINGO 8.0. Its optimal solution is reported in the fifth column of Table 3.

As this result shows, PLA -optimal solution has not been improved because of considering probability distribution function and the variance effect. Therefore, PLA-optimal solution is more confident than optimal solution of the other models for a decision maker.

Because this paper is an early step in the study of developing a fuzzy stochastic optimization, we compared the results of our two methods and compared them to the results of fuzzy programming model. But, additional research is also needed to see the efficiency of our methods in more practical cases.

5 Concluding Remarks

In this paper, a new method for multi-objective linear programming models with fuzzy random variables has been discussed. Then a new algorithm based on mean value of fuzzy random variables, chance-constrained programming and integer programming developed to solve the model. Because of using variance effect, our optimal solution is more confident. Furthermore, a nonlinear programming problem which is obtained by Charnes and Cooper's chance-constrained approach has been converted to a mixed integer programming problem by using the piecewise linear approximation method. We also find that the global optimal solution of this nonlinear programming problem is incredibly near to the optimal solution of PLA approach.

As it was mentioned before, considering probability distribution function and the variance effect have direct effect on optimal solutions and its optimal solution is more confident than other optimal solutions.

This paper is also necessarily restricted to simple assumptions because of its early step in the study of developing a fuzzy random approach. To improve the method discussed in this paper, the reader can generalize the model by adding the effect of other factors of a fuzzy random variable to model. For example, in some circumstances the selection of fuzzy membership function might be better described by new rules or the other probability distribution can be discussed for random property of fuzzy random variables.

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