Journal of Uncertain Systems
Vol.6, No.1, pp.30-37, 2012
Online at: www.jus.org.uk

Averaging of the Fuzzy Differential Equations

Andrej V. Plotnikov*, Tatyana A. Komleva
Odessa State Academy of Civil Engineering and Architecture, 4 Didrihsona St., 65029 Odessa, Ukraine

Received 24 March 2010; Revised 21 October 2011

Abstract
In this article we prove the substantiation of the method of full averaging for the fuzzy differential equations with small parameter on the metric space $(E^n; D)$. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

Keywords: fuzzy differential equation, differential equations with small parameter, averaging scheme

1 Instroduction

The absence of exact universal research methods for nonlinear systems has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

Many important problems of analytical dynamics are described by the nonlinear mathematical models that as a rule are presented by the nonlinear differential or the integro - differential equations. The absence of exact universal research methods for nonlinear systems has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

The averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. It became possible due to the works of N.N. Bogolyubov, Yu.A. Mitropolskij, A.M. Samojlenko, V.M. Volosov, M.A. Krasnoselskiy, N.N. Moiseev, S.G. Krein, F.L. Chernousko, L.D. Akulenko, V.A. Plotnikov, A.N. Filatov, etc. (see [2, 3, 5, 15, 17, 23, 26, 27, 29]).

In recent years, the fuzzy set theory introduced by Zadeh [36] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of science as physical, mathematical, differential equations and engineering sciences. Recently there have been new advances in the theory of fuzzy differential equations.

The theory of fuzzy differential equations has been studied by many authors [1, 8, 9, 10, 11, 12, 14, 16, 19, 24, 25, 26, 32, 33] by using the H-differentiability for the fuzzy valued mappings of a real variable whose values are normal, convex, upper semicontinuous and compactly supported fuzzy sets in $\mathbb{R}^n$. S. Seikkala [30, 31] defined the fuzzy derivative which is generalization of the Hukuhara derivative in [7]. The local existence theorems are given in [34], and the existence theorems under compactness-type conditions are investigated in [35], for the Cauchy problem $x' = f(t, x), x(t_0) = x_0$ when the fuzzy valued mapping f satisfies the generalized Lipschitz condition. Park et al [20, 21, 22] studied the fuzzy differential equation with nonlocal condition. J.J. Nieto [18] proved an existence theorem for fuzzy differential equations on the metric space $(\mathbb{E}^n; D)$.

In this article we prove the substantiation of the method of full averaging for the fuzzy differential equations with small parameter on the metric space $(E^n; D)$. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

2 Preliminaries

Let $(conv(\mathbb{R}^n), h)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$h(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\},$$

*Corresponding author. Email: a-plotnikov@ukr.net (A.V. Plotnikov).

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is clear that $(conv(\mathbb{R}^n), h)$ becomes a metric space.

Denote $\mathbb{E}^n = \{\, x : \mathbb{R}^n \to [0,1] \mid u \text{ satisfies (i)–(iv) below} \,\}$, where
(i) x is normal, i.e., there exists an $\xi_0 \in \mathbb{R}^n$ such that $x(\xi_0) = 1$,
(ii) x is fuzzy convex; that is, for $\xi, \gamma \in \mathbb{R}^n$ and $0 \leq \beta \leq 1$, $x(\beta\xi + (1-\beta)\gamma) \geq \min\{x(\xi), x(\gamma)\}$,
(iii) x is upper semicontinuous,
(iv) $[x]^0 = cl\{\xi \in \mathbb{R}^n \mid x(\xi) > 0\,\}$ is compact.

For $0 < \alpha \leq 0$ denote $[x]^\alpha = \{\xi \in \mathbb{R}^n \mid x(\xi) \geq \alpha\,\}$. Then from (i)–(iv), it follows that the α-level sets $[x]^\alpha \in conv(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

Let $\widehat{o}$ be the fuzzy set defined by $\widehat{o}(\xi) = 1$ if $\xi = 0$ and $\widehat{o}(\xi) = 0$ if $\xi \neq 0$.

Define $D : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}^+ \bigcup\{0\}$ by the equation $D(x,y) = \sup\limits_{0\leq\alpha\leq 1} h([x]^\alpha, [y]^\alpha)$. It is easy to show that D is a metric in $\mathbb{E}^n$. Using the results of [28], we know that
(1) $(\mathbb{E}^n, D)$ is a complete metric space,
(2) $D(x+z, y+z) = D(x,y)$ for all $x, y, z \in \mathbb{E}^n$,
(3) $D(kx, ky) = |k|D(x,y)$ for all $x, y \in \mathbb{E}^n$, $k \in \mathbb{R}$.

Definition 1 [20] A mapping $f : [0,T] \times \mathbb{E}^n \to \mathbb{E}^n$ is called continuous at point $(t_0, x_0) \in (0,T) \times \mathbb{E}^n$ provided for any $\varepsilon > 0$, there exists an $\delta > 0$ such that $D(f(t,x), f(t_0,x_0)) < \varepsilon$ whenever $|t - t_0| < \delta$ and $D(x, x_0) < \delta$.

The fact that $f : [0,T] \times \mathbb{E}^n \to \mathbb{E}^n$ is continuous, we shall denote by $f \in C([0,T] \times \mathbb{E}^n, \mathbb{E}^n)$.

Definition 2 [20] A mapping $f : [0,T] \times \mathbb{E}^n \to \mathbb{E}^n$ is called bounded if there exists a constant $\lambda > 0$ such that $\|y\| \leq \lambda$ for all $y \in [f(t,x)]^0$.

Definition 3 [9] Let $f : [0,T] \to \mathbb{E}^n$. The integral of $f(\cdot)$ over $[0,T]$, denote by $\int_0^T f(t)dt$, is defined by the equation

$$\left[\int_0^T f(t)dt\right]^\alpha = \int_0^T [f(t)]^\alpha dt$$

$$= \left\{ \int_0^T \phi(t)dt \mid \phi : [0,T] \to \mathbb{R}^n \text{ is a measurable selection for } [f(\cdot)]^\alpha \right\}$$

for all $0 < \alpha \leq 1$.

Definition 4 [9] A mapping $f : [0,T] \to \mathbb{E}^n$ is called differentiable at $t \in [0,T]$ if for any $\alpha \in [0,1]$ the set-valued mapping $f_\alpha(t) = [f(t)]^\alpha$ is Hukuhara differentiable at point t [7] and the family $\{D_H f_\alpha(t) \mid \alpha \in [0,1]\}$ defines a fuzzy number $f'(t) \in \mathbb{E}^n$.

If $f : [0,T] \to \mathbb{E}^n$ is differentiable at $t \in [0,T]$, we say that $f'(t)$ is the fuzzy derivative of $f(t)$ at the point t.

3 Main Result

Consider the fuzzy differential equation with small parameter

$$x' = \varepsilon f(t,x), \tag{1}$$

where t – time, $x \in \mathbb{E}^n$, $\varepsilon > 0$ – small parameter, $f : \mathbb{R} \times \mathbb{E}^n \to \mathbb{E}^n$.

Definition 5 [9, 20] A mapping $x : \mathbb{R} \to \mathbb{E}^n$ is a solution of the problem (1) with initial value $x(0) = x_0$, if it is continuous and satisfies the integral equation

$$x(t) = x_0 + \int_0^t \varepsilon f(s, x(s))ds. \tag{2}$$

Consider also the following averaged system

$$y' = \varepsilon \overline{f}(y), \tag{3}$$

where

$$\lim_{T\to 0} D\left(\frac{1}{T}\int_0^T f(t,x)dt, \overline{f}(x)\right) = 0. \qquad (4)$$

Theorem Let in the domain $Q = \{t \geq 0,\ x \in P \subset \mathbb{E}^n\}$ the following hold:

1) the mapping $f(t,x)$ is continuous, uniformly bounded with constant M, satisfies the Lipschitz condition in x with constant λ, i.e.

$$D(f(t,x),\hat{0}) \leq M, \quad D(f(t,x_1), f(t,x_2)) \leq \lambda D(x_1,x_2);$$

2) uniformly with respect to x in the domain P the limit (4)exists;

3) for any $x_0 \in P' \subset P$ and $t \geq 0$ the solution of the differential equation (3) together with a $\rho-$neighborhood belong to the domain P.

Then for any $\eta > 0$ and $L > 0$ there exists $\varepsilon_0(\eta, L) > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq t \leq L\varepsilon^{-1}$ the following condition is satisfied

$$D(x(t), y(t)) \leq \eta. \qquad (5)$$

Proof: From the conditions 1), 2) follows that the mapping $\overline{f}(x)$ is uniformly bounded and satisfies the Lipschitz condition with constant λ. Really in view of the condition 2) of the theorem for any $\delta > 0$ it is possible to find $T(\delta) > 0$ such that for all $T > T(\delta)$ the estimate is fair:

$$d\left(\overline{f}(x), \frac{1}{T}\int_0^T f(t,x)dt\right) < \delta.$$

Then

$$D(\overline{f}(x),\widehat{o}) \leq D\left(\overline{f}(x), \frac{1}{T}\int_0^T f(t,x)dt\right) + D\left(\frac{1}{T}\int_0^T f(t,x)dt, \widehat{o}\right)$$

$$\leq D\left(\overline{f}(x), \frac{1}{T}\int_0^T f(t,x)dt\right) + M \leq \delta + M,$$

$$D(\overline{f}(y_1),\overline{f}(y_2)) \leq D\left(\overline{f}(y_1), \frac{1}{T}\int_0^T f(t,y_1)dt\right) + D\left(\frac{1}{T}\int_0^T f(t,y_1)dt, \frac{1}{T}\int_0^T f(t,y_2)dt\right)$$

$$+D\left(\frac{1}{T}\int_0^T f(t,y_2)dt, \overline{f}(y_2)\right) \leq 2\delta + \frac{1}{T}\int_0^T D(f(t,y_1), f(t,y_2))dt$$

$$\leq 2\delta + \frac{1}{T}\int_0^T \lambda D(y_1,y_2)dt = 2\delta + \lambda D(y_1,y_2).$$

As the value δ is chosen arbitrarily, in a limit we will receive:

$$|\overline{f}(x)| \leq M, \quad D(\overline{f}(y_1),\overline{f}(y_2)) \leq \lambda D(y_1,y_2).$$

From Definition 5, we have

$$x(t) = x_0 + \varepsilon\int_0^t f(s,x(s))ds, \qquad (6)$$

$$y(t) = x_0 + \varepsilon\int_0^t \overline{f}(y(s))ds. \qquad (7)$$

Then, we obtain

$$D(x(t),y(t)) \leq \varepsilon\lambda \int_0^t D(x(s),y(s))ds + \varepsilon D\left(\int_0^t f(s,y(s))ds, \int_0^t \overline{f}(y(s))ds\right). \quad (8)$$

Let us consider the division $0 = t_0 < t_1 < \ldots < t_m = \frac{L}{\varepsilon}$ of $[0, \frac{L}{\varepsilon}]$ such that $t_{i+1} - t_i = \frac{L}{\varepsilon m}$, $i = 0, 1, \ldots, m$ and let $y_i = y(t_i)$, $i = 0, 1, \ldots, m$.

Let $t \in (t_k, t_{k+1})$ for any $k \in \{0, 1, \ldots, m-1\}$. Then

$$\varepsilon D\left(\int_0^t f(s,y(s))ds, \int_0^t \overline{f}(y(s))ds\right) \leq \varepsilon \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} D(f(\tau,y(\tau)), f(\tau,y_i))d\tau$$

$$+\varepsilon \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} D(\overline{f}(y_i), \overline{f}(y(\tau)))d\tau + \varepsilon \sum_{i=0}^{k-1} D\left(\int_0^{t_{i+1}} f(\tau,y_i)d\tau, \int_0^{t_{i+1}} \overline{f}(y_i)d\tau\right)$$

$$+\varepsilon \sum_{i=1}^{k} D\left(\int_0^{t_i} f(\tau,y_i)d\tau, \int_0^{t_i} \overline{f}(y_i)d\tau\right) + D\left(\int_0^t f(\tau,y_k)d\tau, \int_0^t \overline{f}(y_k)d\tau\right),$$

$$\varepsilon \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} D(\overline{f}(y_i), \overline{f}(y(\tau)))d\tau \leq \frac{ML^2\lambda}{m},$$

$$\varepsilon \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} D(f(\tau,y(\tau)), f(\tau,y_i))d\tau \leq \frac{ML^2\lambda}{m}.$$

From the condition 2) of the theorem there exists an increasing function $\varphi(t)$, such that

a) $\lim\limits_{t\to\infty} \varphi(t) = 0$;

b) $D\left(\int_0^t f(\tau,x)d\tau, \int_0^t \overline{f}(x)d\tau\right) < t\varphi(t)$ for all $x \in P$.

Therefore,

$$\varepsilon D\left(\int_0^t f(\tau,x)d\tau, \int_0^t \overline{f}(x)d\tau\right) < \varepsilon t\varphi(t) \leq F(\varepsilon),$$

where $F(\varepsilon) = \sup\limits_{\tau\in[0,L]} (\tau\varphi(\tau/\varepsilon))$, $\tau = \varepsilon t$ and so, $\lim\limits_{\varepsilon\to 0} F(\varepsilon) = 0$.

Hence, we obtain

$$\varepsilon \sum_{i=0}^{k-1} D\left(\int_0^{t_{i+1}} f(\tau,y_i)d\tau, \int_0^{t_{i+1}} \overline{f}(y_i)d\tau\right) + \varepsilon \sum_{i=1}^{k} D\left(\int_0^{t_i} f(\tau,y_i)d\tau, \int_0^{t_i} \overline{f}(y_i)d\tau\right)$$

$$+D\left(\int_0^t f(\tau,y_k)d\tau, \int_0^t \overline{f}(y_k)d\tau\right) \leq 2mF(\varepsilon)$$

and therefore,

$$\varepsilon D\left(\int_0^t f(s,y(s))ds, \int_0^t \overline{f}(y(s))ds\right) \leq \frac{2ML^2\lambda}{m} + 2mF(\varepsilon). \quad (9)$$

From the estimates (8), (9) and Gronwall-Bellman's lemma follows that

$$D(x(t),y(t)) \leq \varepsilon \exp(\lambda L) D\left(\int_0^t f(\tau,y(\tau))d\tau, \int_0^t \overline{f}(y(\tau))d\tau\right)$$

$$\leq \exp(\lambda L)\frac{2ML^2\lambda}{m} + \exp(\lambda L)2mF(\varepsilon).$$

We take m such that

$$\exp(\lambda L)\frac{\lambda ML^2}{m} \leq \eta/2. \tag{10}$$

After that, we choose ε_0 such that for all $\varepsilon \in (0, \varepsilon_0]$

$$\exp(\lambda L)2mF(\varepsilon) \leq \eta/2. \tag{11}$$

From (8),(10) and (11), we obtain (5). The proof is completed. □
Remark If the condition 3) doesn't hold it can be replaced by the following condition:

3') for any $x_0 \in P' \subset P$ the solutions of the differential equation (4) together with a $\rho-$neighborhood belong to the domain P for $\tau \in [0, T_1]$, where $\tau = \varepsilon t$.

Then for any $\eta \in (0, \rho]$ and $L \in [0, T_1]$ there exists $\varepsilon_0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, L\varepsilon^{-1}]$ the condition (5) of the theorem fulfill. Let's show outcomes of the given paper on several modelling examples:
Example We will consider now the averaging scheme for a linear Cauchy problem of the second order with small parameter:

$$x' = \varepsilon(A(t)x + B(t)), \quad x(0) = x_0, \tag{12}$$

where $u : I \to \mathbb{E}^2, \quad A : I \to \mathbb{R}^{2\times 2}, B : I \to \mathbb{E}^2$.

Let $A(t) = \begin{pmatrix} \cos(t) & 1 \\ -1 & \sin(t) \end{pmatrix}$, $B(t)$ such that $B_\alpha(t) = K_{40(1-\alpha)|\sin(t)|}\begin{pmatrix} 10\sin(t) \\ 10\cos(t) \end{pmatrix}$,

$$x_0(\xi) = \begin{cases} \sqrt{1-(\xi_1-1)^2-(\xi_2-1)^2},\ \xi \in S_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ 0,\ \xi \notin S_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{cases}$$

Along with the differential equation (12) we will consider the averaged differential equation

$$\bar{x}' = \varepsilon(\bar{A}\bar{x} + \bar{B}), \quad x(0) = x_0, \tag{13}$$

where $\bar{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\bar{B}$ such that $\bar{B}_\alpha = K_{40(1-\alpha)/\pi}\begin{pmatrix} 10 \\ 10 \end{pmatrix}$.

Further in drawings graphs of solutions of the initial (12) and averaged (13) systems and also as "floating" character of a constant term of an initial problem is exhibited on dynamics drawing the $\alpha-$level sets are reduced.

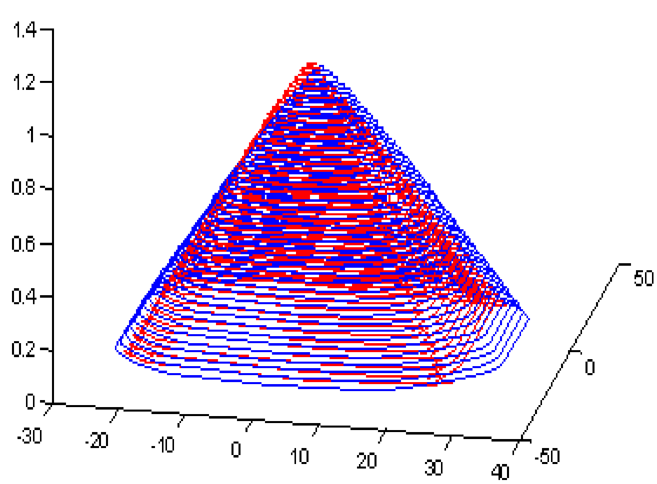

Figure 1: Fuzzy solutions $x(t)$ (blue) and $\bar{x}(t)$ (red) for $\varepsilon = 0.1,\ T = 10$

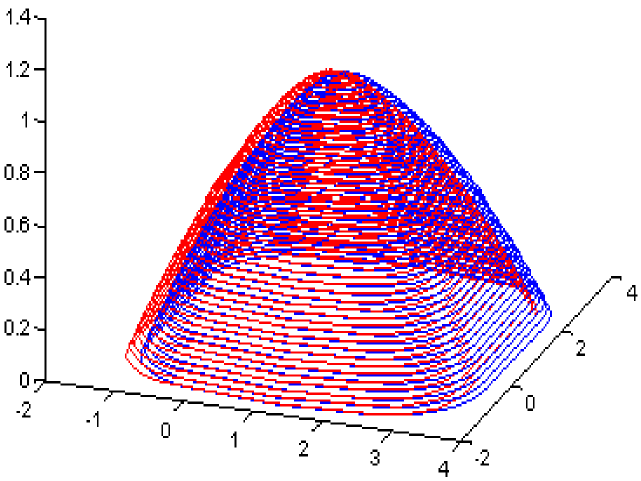

Figure 2: Fuzzy solutions $x(t)$ (blue) and $\bar{x}(t)$ (red) for $\varepsilon = 0.01,\ T = 10$

4 Conclusions

This result generalize the results of M. Kisielewicz [13], A.V. Plotnikov [26, 27] for the differential equations with the Hukuhara's derivative and M. M. Hapaev [6] for the ordinary differential equations with small parameter. Also, in this article, we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

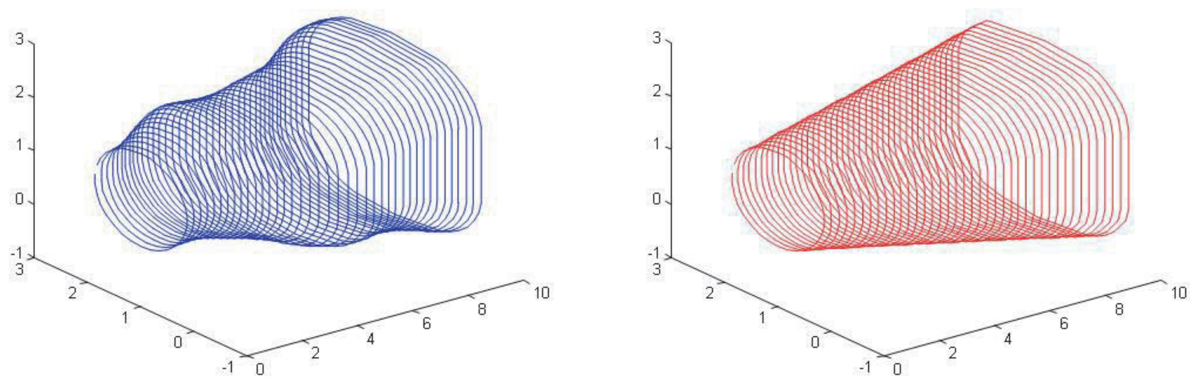

Figure 3: Dynamics drawing the $\alpha-$level sets of the fuzzy solutions $x(t)$ (blue) and $\bar{x}(t)$ (red) for $\alpha = 0.5,\ \varepsilon = 0.01$

References

[1] Bede, B., and T. Gnana Bhaskar, Perspectives of fuzzy initial value problems, The University of Texas at El Paso, Department of Math, Sciences, Research Reports Series.

[2] Bogolyubov, N.N., and J.A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillations*, Gordon and Breach, New York, 1961.

[3] Burd, V., *Method of Averaging for Differential Equations on an Infinite Interval: Theory and Applications*, no.255, 2007.

[4] Chen, M.H., C. Wu, X. Xue, and G. Liu, On fuzzy boundary value problems, *Information Sciences*, no.178, pp.1877–1892, 2008.

[5] Filatov, A.N., *Asymptotic Methods in the Theory of the Differential and Integro-Differential Equations*, Fan, Tashkent, 1974.

[6] Hapaev, M.M., On the averaging method in certain problems connected with averaging, *Differencialnye Uravnenija*, no.2, pp.600–608, 1966.

[7] Hukuhara, M., Integration des applications mesurables dont la valeur est un compact convexe, *Funkcialaj Ekvacioj*, no.10, pp.205–223, 1967.

[8] Jeong, J.U., Fuzzy differential equations with nonlocal condition, *Journal of Applied Mathematics and Computing*, vol.17, nos. 1-3, pp.509–517, 2005.

[9] Kaleva, O., Fuzzy differential equations, *Fuzzy Sets and Systems*, vol.24, no.3, pp.301–317, 1987.

[10] Kaleva, O., The Cauchy problem for fuzzy differential equations, *Fuzzy Sets and Systems*, vol.35, pp.389–396, 1990.

[11] Kaleva, O., The Peano theorem for fuzzy differential equations revisited, *Fuzzy Sets and Systems*, vol.98, pp.147–148, 1998.

[12] Kaleva, O., A notes on fuzzy differential equations, *Nonlinear Analysis*, vol.64, pp.895–900, 2006.

[13] Kisielewicz, M., Method of averaging for differential equations with compact convex valued solutions, *Rendiconti di Matematica*, vol.9, no.3, pp.397–408, 1976.

[14] Komleva, T.A., A.V. Plotnikov, and N.V. Scripnik, Differential equations with set-valued solutions, *Ukrainian Mathematical Journal*, vol.60, no.10, pp.1540–1556, 2008.

[15] Krylov, N.M., and N.N. Bogolyubov, *Introduction in the Nonlinear Mechanics*, Publishing House ANUSSR, Kiev, 1937.

[16] Lakshmikantham, V., T. Granna Bhaskar, and V. Devi, *Theory of Set Differential Equations in Metric Spaces*, Cambridge Scientific Publishers, 2006.

[17] Mitropolsky, J.A., *Method of an Average in the Nonlinear Mechanics*, Naukova Dumka, Kiev, 1971.

[18] Nieto, J.J., The Cauchy problem for continuous fuzzy differential equations, *Fuzzy Sets and Systems,* vol.102, pp.259–262, 1999.

[20] Park, J.Y., and H.K. Han, Existence and uniqueness theorem for a solution of fuzzy differential equations, *International Journal of Mathematics and Mathematical Sciences*, vol.22, no.2, pp.271–279, 1999.

[21] Park, J.Y., and H.K. Han, Fuzzy differential equations, *Fuzzy Sets and Systems*, vol.110, pp.69–77, 2000.

[22] Park, J.Y., H.K. Han, and K.H. Son, Fuzzy differential equation with nonlocal condition, *Fuzzy Sets and Systems*, vol.115, pp.365–369,2000.

[19] Park, J.Y., I.H. Jung, and M.J. Lee, Almost periodic solutions of fuzzy systems, *Fuzzy Sets and Systems*, no.119, pp.367–373, 2001.

[23] Perestyuk, N.A., V.A. Plotnikov, A.M. Samoilenko, and N.V. Skripnik, *Differential Equations with Impulse Effects: Multivalued Right-hand Sides with Discontinuities (De Gruyter Studies in Mathematics: 40)*, Walter De Gruyter GmbH& Co., Berlin/Boston, 2011.

[24] Plotnikov, A.V., T.A. Komleva, and A.V. Arsiry, Necessary and sufficient optimality conditions for a control fuzzy linear problem, *International Journal of Industrial Mathematics*, vol.1, no.3, pp.197–207, 2009.

[27] Plotnikov, V.A., A.V. Plotnikov, and A.N. Vityuk, Differential equations with a multivalued right-hand side, *Asymptotic Methods*, AstroPrint, Odessa, 1999.

[25] Plotnikov, A.V., and N.V. Skripnik, The generalized solutions of the fuzzy differential inclusions, *International Journal of Pure and Applied Mathematics*, vol.56, no.2, pp.165–172, 2009.

[26] Plotnikov, A.V., and N.V. Skripnik, Differential equations with "clear" and fuzzy multivalued right-hand sides, *Asymptotics Methods*, AstroPrint, Odessa, 2009.

[28] Puri, M.L., and D.A. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, vol.114, pp.409–422, 1986.

[31] Seikkala, S., On the fuzzy initial value problem, *Fuzzy Sets and Systems,* vol.24, pp.319–330, 1987.

[32] Song, S.J., and C.X. Wu, Existence and uniqueness of solutions to Cauchy problem of fuzzy differential equations, *Fuzzy Sets and Systems,* vol.110, pp.55–67, 2000.

[33] Song, S.J., C.X. Wu, and X. Xue, Existence and uniqueness of cauchy problem for fuzzy differential equations under dissipative conditions, *Computers & Mathematics with Applications*, vol.51, nos.9-10, pp.1483–1492, 2006.

[29] Volosov, V.M., and B.I. Morgunov, *Metod of Average in the Theory of Nonlinear Oscillations*, Moscow State University Publishing house, Moscow, 1971.

[30] Vorobiev, D., and S. Seikkala, Towards the theory of fuzzy differential equations, *Fuzzy Sets and Systems,* vol.125, pp.231–237, 2002.

[35] Wu, C., and S.J. Song, Existence theorem to the Cauchy problem of fuzzy differential equations under compactness type conditions, *Journal of Informations Sciences*, vol.108, pp.123–134, 1998.

[34] Wu, C., S.J. Song, and E. Lee, Approximate solutions, existence and uniqueness of the Cauchy problem of fuzzy differential equations, *Journal of Mathematical Analysis and Applications*, vol.202, pp.629–644, 1996.

[36] Zadeh L.A., Fuzzy sets, *Information and Control*, vol.8, pp.338–353, 1965.