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Online Particle Filtering of Stochastic Volatility

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Abstract

A method for online estimation of the volatility when observing a stock price is proposed. This is based on modeling the volatility dynamics as a stochastic differential equation that is constructed using a technique from the control theory. Identification of the model parameters using the observations is proposed afterwards. It is based on some stochastic calculus. Volatility estimation is then reformulated as a filtering problem. An alternative filter instead of the optimal one is proposed since the latter is not computationally feasible. It is based on samples (or particles) drawn by discretization of the stochastic volatility model. Besides, the main feature that makes online particle filtering possible is analytic resolution of the Fokker-Planck equation for the current return. To the best of our knowledge, these techniques for modeling and estimating the volatility are quiet novel. The method is implemented on real data: the Heng Seng index price; this shows a period of relatively high volatility that corresponds obviously to the Asiatic crisis of October 1997.

Keywords: stochastic volatility, stochastic differential equations, Fokker-Planck equation, particle filtering

1 Introduction

Let $S = (S_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}_+$-valued semimartingale based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ which is assumed to be continuous. The process S is interpreted to model the price of a stock. A basic problem arising in Mathematical Finance is to estimate the price volatility, i.e. the square of the parameter σ in the following stochastic differential equation

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$$

where $W = (W_t)_{t\in\mathbb{R}_+}$ is a Wiener process. It turns out that the assumption of a constant volatility does not hold in practice. Even to the most casual observer of the market, it should be clear that volatility is a random function of time which we denote σ_t^2. Itô's formula for the return $y_t = \log(S_t/S_0)$ yields

$$dy_t = \left(\mu - \frac{\sigma_t^2}{2}\right) dt + \sigma_t\,dW_t \quad t > 0 \tag{1}$$

together with the initial condition $y_0 = 0$.

The main objective is to estimate in discrete real-time one particular sample path of the volatility process using one observed sample path of the return. As regards the drift μ, it is constant but unknown. Under the so-called risk-neutral measure, the drift is a riskless rate which is well known; actually one finds that μ does not cancel out, for instance, when calculating conditional expectations in a filtering problem. For this argument no change of measure is required, we work directly in the original measure $\mathbb{P}$, and μ has to be estimated from the observed sample path of the return as well.

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2 A Model for the Stochastic Volatility

Let $(z_t)_{t\in\mathbb{R}_+}$ be an arbitrary $\mathbb{R}$-valued process; at the moment, this is not the unknown process σ_t^2 of instantaneous volatility. Let us assume prior information about the process z_t: wide sense stationarity and a parametric model for its covariance function

$$\gamma(\tau) = D\exp(-\alpha|\tau|) \quad \tau \in \mathbb{R}$$

for some constants $D, \alpha > 0$. Then the spectral density of z_t is given by the formula

$$\Gamma(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(\tau)\exp(-j\omega\tau)d\tau = \frac{1}{2\pi}\frac{2D\alpha}{\omega^2+\alpha^2}$$

where $j = \sqrt{-1}$. The spectral density $\Gamma(\omega)$ may be rewritten as

$$\Gamma(\omega) = \frac{1}{2\pi}\left|\frac{H(j\omega)}{F(j\omega)}\right|^2 \quad \omega \in \mathbb{R}$$

where $H(j\omega) = \sqrt{2D\alpha}$ and $F(j\omega) = j\omega + \alpha$. Notice now that

$$\Phi(s) = \frac{H(s)}{F(s)} \quad s \in \mathbb{C}$$

represents the transfer function of some temporally homogeneous linear filter; this filter is furthermore stable as the root of $F(s)$ is in the left half-plane of the complex variable s. Recalling that $1/2\pi$ is the spectral density of a white noise with unit intensity, we come to the conclusion that

$$z_t - \mathbb{E}[z_t]$$

may be considered as the response of the filter whose transfer function is $\Phi(s)$, to a zero-mean white noise with unit intensity. The differential equation describing such a filter is

$$\dot{u}(t) + \alpha u(t) = \sqrt{2D\alpha}\, w(t)$$

where $w(t)$ and $u(t)$ are respectively the input and the output of the filter. Setting $m = \mathbb{E}[z_t]$ and $z_t - m = u(t)$ yields an unbounded diffusion process on $\mathbb{R}$:

$$dz_t = -\alpha(z_t - m)\,dt + \sqrt{2D\alpha}\,dW_t \tag{2}$$

Right now it is easy to see that (2) is linear and thus z_t is Gaussian. For finding the first and second order moments of the process solving a linear stochastic differential equation (SDE) the reference [5] is our key source. The ordinary differential equation (ODE) for the mean gives

$$\mathbb{E}_x[z_t] = (x - m)\exp(-\alpha t) + m \quad x \in \mathbb{R}.$$

Thus for any $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}_x[z_t] = m.$$

The ODE for the variance gives

$$\mathbb{E}_x\left[z_t^2\right] - \mathbb{E}_x[z_t]^2 = D\left(1 - \exp(-\alpha t)\right) \quad x \in \mathbb{R}.$$

Thus for any $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}_x\left[z_t^2\right] - \mathbb{E}_x[z_t]^2 = D.$$

The ODE for the covariance function

$$K_z(t+\tau, t) = \mathbb{E}_x[z(t+\tau)z(t)] - \mathbb{E}_x[z(t+\tau)]\,\mathbb{E}_x[z(t)] \quad x \in \mathbb{R}$$

yields

$$K_z(t+\tau,t) = \begin{cases} \exp(-\alpha\tau)D\left(1-\exp(-\alpha t)\right) & \tau \geq 0 \\ \exp(\alpha\tau)D\left(1-\exp(-\alpha\tau)\exp(-\alpha t)\right) & \tau \leq 0 \end{cases} \qquad t \in \mathbb{R}_+.$$

Then

$$\lim_{t\to\infty} K_z(t+\tau,t) = D\exp(-\alpha|\tau|) \quad \tau \in \mathbb{R}.$$

Since z_t is Gaussian, the convergence of the its mean and variance as $t \to \infty$ means the convergence in law (or weak convergence) of z_t as $t \to \infty$ to a Gaussian distribution with mean m and variance D. In other words z_t is ergodic with ergodic distribution density

$$p_z(x) = \frac{1}{\sqrt{2\pi D}} \exp\left\{-\frac{x^2}{2D}\right\} \quad x \in \mathbb{R}$$

and asymptotic covariance function

$$k_z(\tau) = D\exp(-\alpha|\tau|) \quad \tau \in \mathbb{R}.$$

In case where $m = 0$ the asymptotic covariance function coincides with the asymptotic correlation function. The asymptotic variance is also the asymptotic second order moment, and thus

$$\lim_{t\to\infty} \mathbb{E}_x\left[z_t^2\right] = D.$$

In this case, with given initial condition z_0, consider the process $|z_t|$, i.e., the reflected diffusion process, written x_t. x_t is a solution of

$$dx_t = -\alpha x_t\,dt + \sqrt{2D\alpha}\,dW_t \quad t > 0 \tag{3}$$

with reflection on the boundary $\{0\}$ of its state space $\mathbb{R}_+$. Starting from any fixed point strictly greater than zero, x_t reaches this boundary by a predictable stopping time with finite expectation because of the negative sign of the drift. The initial condition x_0 is a random variable with known distribution since $x_0 = |z_0|$. We claim that x_t is ergodic with ergodic distribution

$$\pi(A) = \frac{\int_A p_z(x)\,dx}{\int_{\mathbb{R}_+} p_z(x)\,dx} \quad A \in \mathcal{B}(\mathbb{R}_+)$$

because an invariant measure for z_t is an invariant measure for $|z_t|$ too. But

$$\int_{\mathbb{R}_+} p_z(x)\,dx = \frac{1}{2}.$$

Therefore the ergodic distribution density of x_t is given by

$$p(x) = \frac{2}{\sqrt{2\pi D}} \exp\left\{-\frac{x^2}{2D}\right\} \quad x \in \mathbb{R}_+.$$

This is beyond our expectation in view of the required wide sense stationarity. It follows that for each $x \in \mathbb{R}_+$, for any bounded continuous function f on $\mathbb{R}_+$,

$$\lim_{t\to\infty} \mathbb{E}_x\left[f(x_t)\right] = \int_{\mathbb{R}_+} f(x)p(x)\,dx$$

and in particular

$$\lim_{t\to\infty} \mathbb{E}_x\left[x_t\right] = \frac{2D}{\sqrt{2\pi D}}.$$

Obviously, the asymptotic second order moment for x_t coincides with that of z_t, namely

$$\lim_{t\to\infty} \mathbb{E}\left[x_t^2\right] = \lim_{t\to\infty} \mathbb{E}\left[z_t^2\right] = D.$$

Let us now compute $r(\tau)$, the asymptotic correlation function of x_t, for $\tau \neq 0$.

$$\begin{aligned} \mathbb{E}\left[x(t+\tau)x(t)\right] &= \mathbb{E}\left[\,|z(t+\tau)z(t)|\,\right] \\ &= \mathbb{E}\left[z(t+\tau)z(t)\right]\mathbb{P}\left\{\textit{even passage number by 0}\right\} \\ &\quad -\mathbb{E}\left[z(t+\tau)z(t)\right]\mathbb{P}\left\{\textit{odd passage number by 0}\right\} \\ &\to C\exp(-\alpha|\tau|) \text{ as } t\to\infty \end{aligned}$$

for some constant $0 < C < D$. Note the discontinuity of $r(\tau)$ at $\tau = 0$:

$$r(\tau) = \begin{cases} C\exp(-\alpha|\tau|) & \tau \neq 0 \\ D & \tau = 0. \end{cases}$$

It should be noted that this type of correlation function may include short-term or long-term memory in the volatility. We shall freely call the process x_t or equivalently the SDE (3) our stochastic volatility model. We just have to denote the Wiener process in (3) differently, say $\tilde{W}$, since it is independent of the Wiener process in (1).

3 Filtering

Now we consider the filtering problem associated to the couple (x_t, y_t): we have noisy nonlinear observations of x_t, the $\mathbb{R}$-valued discrete-time process of returns $(y_n)_{n=1,2,...}$ indexed at irregularly spaced instants $t_1, t_2, ...$ The observation times are assumed to be rigourously determined. The observations process is related to the state process $(x_t)_{t\in\mathbb{R}_+}$ via the conditional distribution

$$\mathbb{P}\left\{y_n \in \Gamma | y_1, ..., y_{n-1}, (x_t : 0 \le t \le t_n)\right\} \quad n \ge 1$$

for Γ a Borel-measurable set from $\mathbb{R}$. For homogeneity of notation we set $t_0 = 0$ so that $y_{n=0} = y_{t=t_0} = 0$. Now look at the distribution above and recall that $y_n = y(t_n)$ and that the process y_t solves the SDE

$$dy_t = \left(\mu - \frac{x_t}{2}\right)dt + \sqrt{x_t}\,dW_t \quad y_0 = 0. \tag{4}$$

This is (1) where σ_t is denoted $\sqrt{x_t}$. For $t \ge t_{n-1}$

$$y_t = y_{n-1} + \int_{t_{n-1}}^{t}\left(\mu - \frac{x_s}{2}\right)ds + \int_{t_{n-1}}^{t}\sqrt{x_s}\,dW_s \tag{5}$$

and thus

$$\mathbb{P}\left\{y_n \in \Gamma | y_1, ..., y_{n-1}, (x_t : 0 \le t \le t_n)\right\} = \mathbb{P}\left\{y_n \in \Gamma | y_{n-1}, (x_t : t_{n-1} \le t \le t_n)\right\}.$$

Given a sample path of $(x_t)_{t_{n-1}\le t\le t_n}$ and the observation y_{n-1}, $(y_t)_{t_{n-1}\le t\le t_n}$ is a Markov process with state space $\mathbb{R}$ satisfying (5). This leads to the central concept of this section: the Fokker-Planck equation [2, 3, 4, 6]. The domain of the Fokker-Planck operator:

$$\mathcal{L}_{FP}p(y,t) = \left(\frac{x_t}{2} - \mu\right)\frac{\partial p}{\partial y}(y,t) + \frac{x_t}{2}\frac{\partial^2 p}{\partial y^2}(y,t)$$

is the set of distribution densities on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ under $\mathbb{P}$. Given a sample path of $(x_t)_{t_{n-1}\le t\le t_n}$ and the observation y_{n-1}, the distribution density $p(y,t)$ of y_t solves the Fokker-Planck equation

$$\frac{\partial p}{\partial t}(y,t) = \mathcal{L}_{FP}p(y,t) \quad t_{n-1} < t \le t_n \tag{6}$$

with the initial condition $p(y, t_{n-1}) = \delta(y - y_{n-1})$. The formal solution of the above partial differential equation is

$$p(y,t) = \exp\{(t - t_{n-1})\mathcal{L}_{FP}\}\, p(y, t_{n-1}).$$

Since $\mathcal{L}_{FP}$ is a sum of two non commuting operators, the exponential operator $\exp\{(t - t_{n-1})\mathcal{L}_{FP}\}$ cannot be expressed as simple products of terms involving each of these. Nevertheless, the solution of the Fokker-Planck equation is obtained using the Trotter product formula [7]. For two arbitrary operators A and B

$$\exp\{t(A+B)\} = \lim_{n\to\infty} \left(\exp\left\{\frac{t}{n}A\right\} \exp\left\{\frac{t}{n}B\right\}\right)^n.$$

Then the solution of (6) is the limit as $n \to \infty$ of

$$\left(\exp\left\{\frac{\rho(t-t_{n-1})}{n}\frac{d}{dy}\right\}\exp\left\{\frac{\varrho(t-t_{n-1})}{n}\frac{d^2}{dy^2}\right\}\right)^n \delta(y - y_{n-1})$$

where

$$\rho = \frac{x_t}{2} - \mu, \quad \varrho = \frac{x_t}{2}.$$

For algebraic manipulations we use the integral representation of the delta function and write the solution of (6) as

$$p(y,t) = \lim_{n\to\infty} \Theta^n \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\{-jzy\}\exp\{jzy_{n-1}\}\, dz$$

where

$$\Theta = \exp\left\{\frac{\rho(t-t_{n-1})}{n}\frac{d}{dy}\right\}\exp\left\{\frac{\varrho(t-t_{n-1})}{n}\frac{d^2}{dy^2}\right\}.$$

We claim that

$$\exp\left\{\frac{\varrho(t-t_{n-1})}{n}\frac{d^2}{dy^2}\right\}\exp\{-jzy\} = \exp\left\{-\frac{\varrho(t-t_{n-1})}{n}z^2 - jzy\right\},$$

$$\exp\left\{\frac{\rho(t-t_{n-1})}{n}\frac{d}{dy}\right\}\exp\{-jzy\} = \exp\left\{-\frac{\rho(t-t_{n-1})}{n}jz - jzy\right\}.$$

Therefore

$$\Theta\exp\{-jzy\} = \exp\left\{-\frac{\varrho(t-t_{n-1})}{n}z^2 - \frac{\rho(t-t_{n-1})}{n}jz - jzy\right\},$$

$$\Theta^n\exp\{-jzy\} = \exp\left\{-\varrho(t-t_{n-1})z^2 - \rho(t-t_{n-1})jz - jzy\right\}$$

and thus

$$p(y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\{-\varrho(t-t_{n-1})z^2 + jz\left[-y + y_{n-1} - \rho(t-t_{n-1})\right]\}\, dz.$$

Let Z be a Gaussian random variable and $\psi(u)$, $u \in \mathbb{R}$, be its characteristic function:

$$\begin{aligned}\psi(u) &= \mathbb{E}[\exp\{juZ\}]\\ &= (2\pi\,\mathbb{V}\text{ar}[Z])^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\exp\{juz\}\exp\left\{-\frac{(z-\mathbb{E}[Z])^2}{2\mathbb{V}\text{ar}[Z]}\right\}dz\\ &= \exp\left\{ju\,\mathbb{E}[Z] - \frac{u^2}{2}\mathbb{V}\text{ar}[Z]\right\}.\end{aligned}$$

Then

$$p(y,t) = \frac{1}{2\sqrt{\pi\varrho(t-t_{n-1})}}\,\psi\left(-y + y_{n-1} - \rho(t-t_{n-1})\right)$$

with

$$\mathbb{E}[Z] = 0 \quad \mathbb{V}\text{ar}[Z] = \frac{1}{2\varrho(t - t_{n-1})}$$

and hence we obtain for $t_{n-1} \le t \le t_n$

$$p(y,t) \qquad = \frac{1}{\sqrt{2\pi x_t(t - t_{n-1})}} \exp\left\{-\frac{[-y + y_{n-1} + \left(\mu - \frac{x_t}{2}\right)(t - t_{n-1})]^2}{2x_t(t - t_{n-1})}\right\}.$$

Conditional Density Characterization: The Optimal Filter

The optimal estimate – in the sense of a least mean square error – of $f(x_t)$ given the observations y_1,...,y_{n-1} up to time t is the conditional expectation

$$\mathbb{E}\left[f(x_t)|y_1, ..., y_{n-1}\right] \quad t_{n-1} \le t < t_n \quad n \ge 1$$

for all reasonable functions f on $\mathbb{R}_+$. We assume that $\mathbb{P}\left\{x_t \le x|y_1, ..., y_{n-1}\right\}$ possesses a density with respect to the Lebesgue measure λ on $\mathbb{R}_+$:

$$\Pi_{x_t|y_1,...,y_{n-1}}(x) = \frac{d\mathbb{P}\left\{x_t \le x|y_1, ..., y_{n-1}\right\}}{\lambda(dx)}.$$

Now look at the SDE (3), the Fokker-Planck operator for x_t is

$$\mathcal{L}_{FP}p(x) = \alpha p(x) + \alpha(x - m)p'(x) + D\alpha p''(x).$$

The domain of this operator is the set of distribution densities $p(x)$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, under $\mathbb{P}$, satisfying

$$mp(0) - Dp'(0) = 0.$$

This is due to the reflection of the process x_t on the boundary $\{0\}$ of its state space $\mathbb{R}_+$.

It follows that the posterior distribution density $\Pi_{x_t|y_1,...,y_{n-1}}(x)$ for $t_{n-1} \le t < t_n$, $n \ge 1$, solves the Fokker-Planck equation

$$\frac{\partial p}{\partial t}(x,t) = \mathcal{L}_{FP}p(x,t) \quad t_{n-1} < t < t_n$$

i.e.

$$\frac{\partial p}{\partial t}(x,t) = \alpha p(x,t) + \alpha(x - m)\frac{\partial p}{\partial x}(x,t) + D\alpha\frac{\partial^2 p}{\partial x^2}(x,t) \tag{7}$$

with the initial condition

$$p(x, t_{n-1}) = \Pi_{x(t_{n-1})|y_1,...,y_{n-1}}(x) \tag{8}$$

and the boundary condition

$$mp(0,t) - D\frac{\partial p}{\partial x}(0,t) = 0. \tag{9}$$

This is a static relation for $x = 0$, i.e., it holds for any $t \in [t_{n-1}, t_n[$.

At each observation instant t_n, $n \ge 1$, $\Pi_{x(t_n)|y_1,...,y_n}(x)$ solves the Bayes rule

$$\Pi_{x(t_n)|y_1,...,y_n}(x) \propto \Pi_{x(t_n^-)|y_1,...,y_{n-1}}(x)\Pi_{y_n|y_1,...,y_{n-1},x(t_n)=x}(y_n) \tag{10}$$

where

$$\Pi_{y_n|y_1,...,y_{n-1},x(t_n)=x}(y_n) = \frac{1}{\sqrt{2\pi x(t_n - t_{n-1})}} \exp\left\{-\frac{\left[-y_n + y_{n-1} + (\mu - \frac{x}{2})(t_n - t_{n-1})\right]^2}{2x(t_n - t_{n-1})}\right\}$$

and $\Pi_{x(t_n^-)|y_1,...,y_{n-1}}(x)$ is the solution of (7-9) as $t \uparrow t_n$.

4 Identification

It follows from (4) that the variation process $[y]_t$ of y_t is given by

$$[y]_t = \int_0^t x_s\, ds,$$

thus

$$[y]_{t_n} - [y]_{t_{n-1}} = \int_{t_{n-1}}^{t_n} x_s\, ds \quad n = 1, 2, \ldots.$$

On the other hand, so long as every duration between two successive observations is small, the following approximation holds

$$[y]_{t_n} \approx \sum_{i=1}^{n} (y_i - y_{i-1})^2.$$

Thus

$$\int_{t_{n-1}}^{t_n} x_s\, ds \approx (y_n - y_{n-1})^2$$

i.e., the couple of series below coincide approximatively

$$S = \left\{ \int_{t_{n-1}}^{t_n} x_s\, ds \right\}_{n=1,2,\ldots} \qquad S' = \left\{ (y_n - y_{n-1})^2 \right\}_{n=1,2,\ldots}$$

and so do their first and second order moments. The following is the computation of the mean and the correlation function for the series S of aggregations of the instantaneous volatility on the observation intervals. To do this we need to have $t_n - t_{n-1} = \delta$ for each $n = 1, 2, \ldots$ and as mentioned above δ must be small (we set $\delta = 1$ time unit).

$$\mathbb{E}\left[\int_{t_{n-1}}^{t_n} x_s\, ds \right] = \frac{2D\,\delta}{\sqrt{2\pi D}}$$

and for $k = 1, 2, \ldots$

$$\mathbb{E}\left[\int_{t_{n-1}}^{t_n} x_u\, du\,, \int_{t_{n-k-1}}^{t_{n-k}} x_v\, dv \right] = \int_{t_{n-1}}^{t_n} \int_{t_{n-k-1}}^{t_{n-k}} r_x(u - v)\, du\, dv.$$

If we replace $r_x(u - v)$ by its expression, we obtain the following formula for $k = 1, 2, \ldots$

$$\mathbb{E}\left[\int_{t_{n-1}}^{t_n} x_u\, du\,, \int_{t_{n-k-1}}^{t_{n-k}} x_v\, dv \right] = \frac{C}{\alpha^2} \left(\exp\{-\alpha\delta(k-1)\} - 2\exp\{-\alpha\delta k\} + \exp\{-\alpha\delta(k+1)\} \right). \tag{11}$$

It follows that C and α may be obtained by least squares of the difference between the correlation function of S', estimated from the observations, and the correlation function given by formula (11).

The following gives an approximation for the drift parameter μ in (1). We have

$$y_n - y_{n-1} = \int_{t_{n-1}}^{t_n} \left(\mu - \frac{x_s}{2} \right) ds + \int_{t_{n-1}}^{t_n} \sqrt{x_s}\, dW_s.$$

Then

$$\mathbb{E}\left[y_n - y_{n-1} \right] = \mu\,\delta - \frac{1}{2}\,\mathbb{E}\left[\int_{t_{n-1}}^{t_n} x_s\, ds \right] = \mu\,\delta - \frac{D\,\delta}{\sqrt{2\pi D}},$$

and thus

$$\mu = \frac{1}{\delta}\,\mathbb{E}\left[y_n - y_{n-1} \right] + \frac{D}{\sqrt{2\pi D}}.$$

The Hang Seng index price of the market of Hong Kong is observed during 3191 successive trading days from 1995 to 2007. This is plotted in Figure 1. Figure 2 shows the daily returns

$$y_n - y_{n-1} = \log\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right) \quad n = 1, ..., 3190.$$

The empirical mean of S' yields an approximation for the second order moment D of $1.0977e-007$. This approximation together with the empirical mean of the daily returns yield an approximation for the drift μ of $5.4008e-004$. The constant C and the rate α that give a good fitting between the correlation function of S and its approximation are $3.5926e-007$ and 0.0857 respectively. The model for the stochastic volatility of the stock is thus calibrated, and we now go back to filtering.

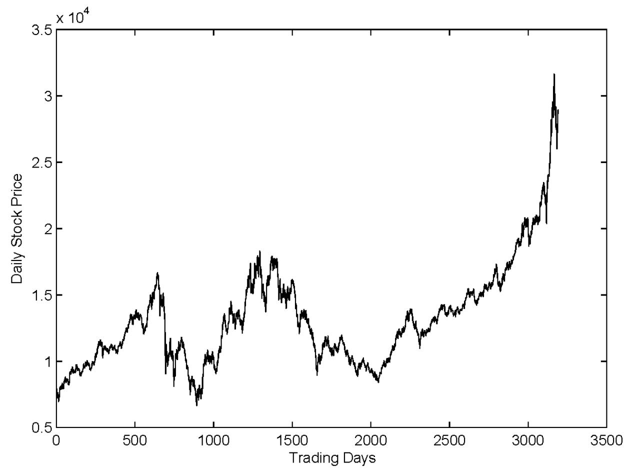

Figure 1: The observed sample path for the daily price of the Hang Seng index

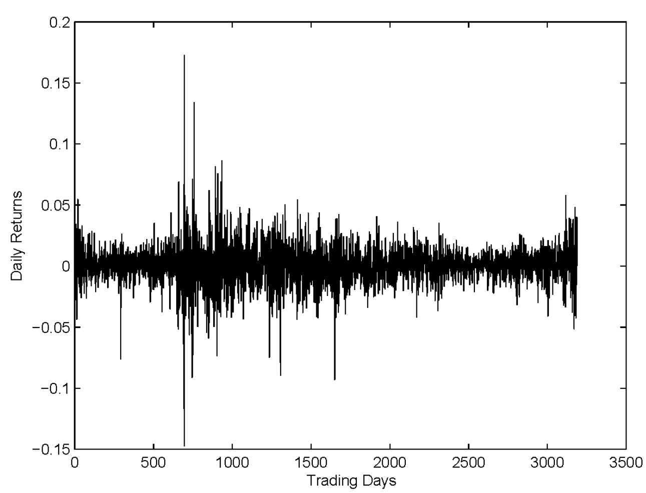

Figure 2: The observed sample path for the daily return

5 A Monte-Carlo Particle Filter

The true filter (7-10) which is optimal in a mean square sense involves a resolution of the Fokker-Planck equation. Both analytic and numerical solutions for this partial differential equation are computationally intractable. This drives us to an alternative Monte-Carlo filter [1]. We wish to approximate the posterior distribution as a weighted sum of random Dirac measures: for Γ a Borel-measurable set from $\mathbb{R}_+$,

$$\mathbb{P}\{x_t \in \Gamma | y_1, ..., y_{n-1}\} \approx \sum_{k=1}^{K} w_k \, \epsilon_{\xi_k}(\Gamma) \quad t_{n-1} \le t < t_n \quad n \ge 1$$

where the particles ξ_k are independent identically distributed random variables with "the same" law as x_t; these particles are indeed samples drawn from the Euler discretization of the SDE (3). Here we use the well known Euler scheme since there isn't a significant gain with more sophisticated discretization schemes. Then, for any function f on $\mathbb{R}_+$,

$$\mathbb{E}\left[f(x_t) | y_1, ..., y_{n-1}\right] \approx \sum_{k=1}^{K} w_k f(\xi_k) \quad t_{n-1} \le t < t_n \quad n \ge 1.$$

The weights $\{w_k\}_{k=1,...,K}$ are updated only as and when an observation y_n proceeds, each one according to the likelihood of its corresponding particle, i.e., at each observation time t_n,

$$w_k = \frac{\Pi_{y_n|y_1,\ldots,y_{n-1},x(t_n)=\xi_k}(y_n)}{\sum_{\ell=1}^{K} \Pi_{y_n|y_1,\ldots,y_{n-1},x(t_n)=\xi_\ell}(y_n)}$$

where $\{\xi_k\}_{k=1,...,K}$ are samples with the same law as $x(t_n)$.

Besides sampling, there may be (importance) resampling at each observation time: the set of particles is updated for removing particles with small weights and duplicating those with important weights. We simulate K new iid random variables according to the distribution

$$\sum_{k=1}^{K} w_k \, \epsilon_{\xi_k}.$$

Obviously, the new particles have new weights and thus give a new approximation for the posterior distribution. On the other hand, these new particles are used to initialize the Euler discretization scheme for the next sampling.

The following is the remainder of implementation details of the Monte-Carlo particle filter.

- Number of particles: $K = 1000$
- Time step of the Euler discretization: 0.01 time unit
- In practice the distribution for the initial volatility x_0 is not available, here we take a uniform distribution on $[\varepsilon, 1]$ ($\varepsilon > 0$ must be small); its density satisfies the imposed condition (9).

The sample path of the square root volatility (in percent) is displayed in Figure 3. This sample path exhibits relatively high volatilities that are clustered together round the 697th trading day; this corresponds to the Asian financial crisis of October 1997.

6 Conclusion

Probabilistic management of uncertainty in dynamical systems can be illustrated with a financial engineering application: volatility estimation. We treat volatility as a stochastic process and construct a filter that is recursive and pathwise in observations. These two aspects are designated with the term online, or real-time, filtering. The filter output is one particular sample path of the volatility process. The main feature that makes online particle filtering possible is analytic resolution of a Fokker-Planck equation. Our method does not require data transformation, such as removing seasonality. The conformity between the implementation result—within a low simulation cost—and some practical issues prove the performance of the method to my satisfaction.

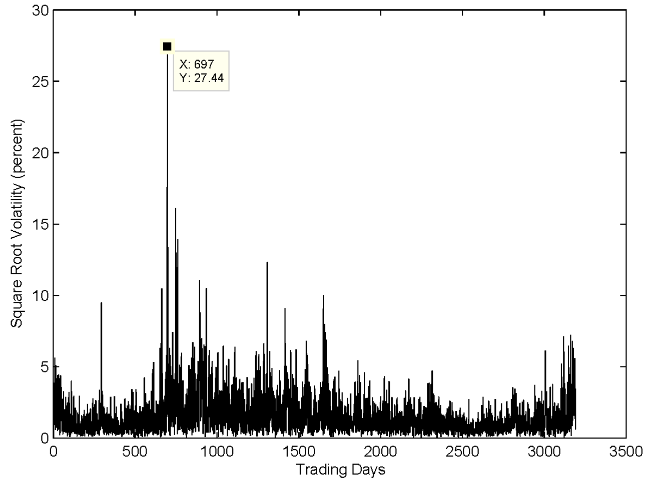

Figure 3: The estimated sample path for the volatility of the Hang Seng index

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