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Adaptive Synchronization of the Fractional-Order LÜ Hyperchaotic System with Uncertain Parameters and Its Circuit Simulation

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Abstract

In this article, the adaptive control scheme with only two controllers is applied to synchronization of the fractional-order hyperchaotic Lü system with unknown parameters. Based on fractional stability theory of fractional-order systems, adaptive controllers and the parameter updating rule are designed. Numerical simulations confirm the effectiveness of the proposed synchronization approaches. Especially, the circuit experiment simulations also demonstrate that the experimental results are in agreement with numerical simulations.

Keywords: fractional order hyperchaotic system, synchronization, adaptive controllers, circuit simulation

1 Introduction

The fractional calculus dates from 17th century, but its applications to physics, engineering and control processing are just a recent subject of interest [2, 7]. Many systems exist in interdisciplinary fields, such as dielectric polarization [15], viscoelastic systems [8], quantitative finance [10] and quantum evolution of complex systems [9]. It is known that some fractional systems behave chaotically, for example, the fractional-order Chua's circuit [6], the fractional-order unified chaotic system [17], the fractional-order Rössler system and so forth [11].

On the other hand, synchronization of chaotic fractional-order system has recently received considerable attention owing to its potential applications in secure communication and control processing, for example, Zhang et al. adopted Pecora-Carroll method, the linear feedback control and the bidirectional coupling to realize chaotic synchronization of the Rucklidge system [19]; Wang et al. analysis the synchronization conditions of the fractional order chaotic systems with activation feedback method [16]; Lu proposed to realize the synchronization of the fractional order unified chaotic systems using state observer in Ref.[12]; Wu investigate the synchronization of fractional-order Chen hyperchaotic systems based on Laplace transform theory [18].To our best knowledge, we can find that some methods such as linearization feedback control method and active control scheme, eliminate nonlinear terms of systems when designing controllers, which make the coefficient matrix of the system to be the constant matrix. Although this scheme can control the fractional-order chaotic system to synchronize, it costs too much. The adaptive control method proposed in this paper can achieve synchronization of fractional-order hyperchaotic systems only using two controllers, and we can see the feasibility of this technique. Moreover, since hyperchaotic systems have two positive Lyapunov exponents at least. The trajectories of hyperchaotic systems show more complex dynamical behaviors because they extend towards many directions. Complex hypechaotic signal can increase the reliability of encryption technique in chaotic secure communication. Therefore, study on synchronization of the fractional-order hyperchaotic systems will be a crutial question for discussion in aspects of chaos application.

In the present paper, we propose the adaptive synchronization method with only two controllers for the fractional-order hyperchaotic Lü systems with unknown parameters. This scheme, based on stability theory of

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fractional-order systems is simple, theoretically rigorous and convenient to realize synchronization. What's more, not only do numerical simulations be performed to verify the effectiveness of the proposed synchronization techniques, but circuit experiment simulation results show the effectiveness of the presented method.

2 Fractional Calculus Predictor-Corrector Algorithm

There are several definitions of a fractional differential system. The most-used one in engineering field may be the Riemann-Liouville fractional derivatives, defined by

$$D_*^{\alpha} x(t) = J^{m-\alpha} x^{(m)}(t), \alpha > 0 \tag{1}$$

where $m = [\alpha]$, i.e., m is the first integer which is not less than α . $J^{\beta}(\beta > 0)$ is the β -order Riemann-Liouville integral operator with expression:

$$J^{\beta} y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau) d\tau . \tag{2}$$

Here Γ stands for the Gamma function. The operator D_*^{α} is generally called " α -order Caputo differential operator" [3].

According to Refs.[4-5], the predictor-corrector method, namely the generalized Adams-Bashforth-Moulton method is described.

Consider the following fractal order differential equations:

$$D_*^{\alpha} y(t) = f(t, y(t)), \ 0 \le t \le T,$$
$$\text{and } y^{(k)}(0) = y_0^{(k)}, k = 0,1,\cdots,m-1. \tag{3}$$

This is equivalent to the following Volterra integral equations:

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds. \tag{4}$$

Set $h = T/N$, $t_n = nh, n = 0,1,2,\cdots N \in Z^+$. Then, one can discrete this equation as

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j)) \tag{5}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-a)(n+1)^{\alpha}, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1, \end{cases}$$

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)) ,$$

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha} ((n+1-j)^{\alpha} - (n-j)^{\alpha}) .$$

The algorithm convergence order of an error series in the above numerical scheme is p, that is $\max_{j=0,1,\cdots,N} \left| y(t_j) - y_h(t_j) \right| = O(h^p)$, where $p = \min(2, 1+\alpha)$ .Numerical solution of a fractional-order system can be determined by applying the mentioned method.

3 Synchronization via Adaptive Control

Recently, Min *et al*. proposed the fractional-order Lü hyperchaotic systems in Ref. [14] which is described by

$$\begin{cases} \dfrac{d^q x}{dt^q} = a(y-x) + w \\ \dfrac{d^q y}{dt^q} = cy - xz \\ \dfrac{d^q z}{dt^q} = xy - bz \\ \dfrac{d^q w}{dt^q} = xz + dw \end{cases} \tag{6}$$

where q is the fractional-order, when q=0.95, (a, b, c, d) = (36, 3, 20, 1), the system behaves hyperchaotic. Fig.1 exhibits hyperchatioc attractor.

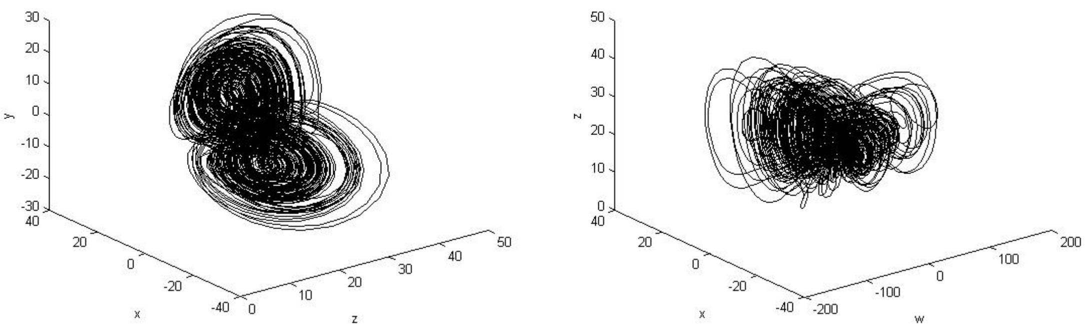


Figure 1: Hyperchaotic attractors of the fractional-order Lü system with the order q=0.95

3.1 Design of Controllers

In this section, we introduce adaptive control method synchronize two identical hyperchaotic fractional-order Lü system. Here, only two controllers are adopted. The drive system is described by

$$\begin{cases} \dfrac{d^q x_1}{dt^q} = a(y_1 - x_1) + w_1 \\ \dfrac{d^q y_1}{dt^q} = cy_1 - x_1 z_1 \\ \dfrac{d^q z_1}{dt^q} = x_1 y_1 - bz_1 \\ \dfrac{d^q w_1}{dt^q} = x_1 z_1 + dw_1 \end{cases} \tag{7}$$

whereas the response system is

$$\begin{cases} \dfrac{d^q x_2}{dt^q} = a_1(y_2 - x_2) + w_2 \\ \dfrac{d^q y_2}{dt^q} = c_1 y_2 - x_2 z_2 + u_1 \\ \dfrac{d^q z_2}{dt^q} = x_2 y_2 - b_1 z_2 \\ \dfrac{d^q w_2}{dt^q} = x_2 z_2 + d_1 w_2 + u_2 \end{cases} \tag{8}$$

where a_1, b_1, c_1, d_1 are unknown parameters, which need to be estimated in the response system, $u=[u_1, u_2]^T$ are controllers which are to be designed.

Now, we define the state errors between system (7) and (8) as $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$, $e_4 = w_2 - w_1$. Subtraction the system (7) from system (8), the error dynamical system can be expressed by

$$\begin{cases} \dfrac{d^q e_1}{dt^q} = a_1(e_2 - e_1) + e_a(y_1 - x_1) + e_4 \\ \dfrac{d^q e_2}{dt^q} = c_1 e_2 + e_c y_1 - e_1 z_2 - e_3 x_1 + u_1 \\ \dfrac{d^q e_3}{dt^q} = e_1 y_2 + e_2 x_1 - b_1 e_3 - e_b z_1 \\ \dfrac{d^q e_4}{dt^q} = d_1 e_4 + e_d w_1 + z_2 e_1 + x_1 e_3 + u_2 \end{cases} \tag{9}$$

where $e_a = a_1 - a$, $e_b = b_1 - b$, $e_c = c_1 - c$, $e_d = d_1 - d$.

Our objective is to design effective controllers u and the update law of parameters to make the response system (8) and drive system (7) to achieve synchronization. We choose the controllers (10) and the update law of parameters (11) as follows:

$$\begin{cases} u_1 = -k_1 e_2 = -k_1^* e_2 - e_{k1} e_2 \\ u_2 = -k_2 e_4 = -k_2^* e_4 - e_{k2} e_4 \end{cases} \tag{10}$$

where $e_{k1} = k_1 - k_1^*$, $e_{k2} = k_2 - k_2^*$, k_i^* (i=1,2) are real constants.

And the update law of parameters:

$$\begin{cases} \dfrac{d^q k_1}{dt^q} = \alpha e_2^2, (\alpha > 0) \\ \dfrac{d^q k_2}{dt^q} = \beta e_4^2, (\beta > 0) \\ \dfrac{d^q e_a}{dt^q} = -(y_1 - x_1) e_1 \\ \dfrac{d^q e_b}{dt^q} = z_1 e_3 \\ \dfrac{d^q e_c}{dt^q} = -y_1 e_2 \\ \dfrac{d^q e_b}{dt^q} = -w_1 e_4 . \end{cases} \tag{11}$$

Theorem For any initial conditions, the drive systems (7) and response system (8) are globally asymptotically synchronized by the control law (10) and update law (11).

Proof: Combining the control law (10), update law (11) and error system, we can get such that:

$$\begin{pmatrix} \sigma \dfrac{d^q e_1}{dt^q} \\ \dfrac{d^q e_2}{dt^q} \\ \dfrac{d^q e_3}{dt^q} \\ \dfrac{d^q e_4}{dt^q} \\ \sigma \dfrac{d^q e_a}{dt^q} \\ \dfrac{d^q e_b}{dt^q} \\ \dfrac{d^q e_c}{dt^q} \\ \dfrac{d^q e_d}{dt^q} \\ \dfrac{1}{\alpha} \dfrac{d^q e_{k1}}{dt^q} \\ \dfrac{1}{\beta} \dfrac{d^q e_{k2}}{dt^q} \end{pmatrix} = \begin{pmatrix} -a_1\sigma & a_1\sigma & 0 & \sigma & (y_1 - x_1)\sigma & 0 & 0 & 0 & 0 & 0 \\ -z_2 & c_1 - k_1^* & -x_1 & 0 & 0 & 0 & y_1 & 0 & -e_2 & 0 \\ y_2 & x_1 & -b_1 & 0 & 0 & -z_1 & 0 & 0 & 0 & 0 \\ z_2 & 0 & x_1 & d_1 - k_2^* & 0 & 0 & 0 & w_1 & 0 & -e_4 \\ -(y_1 - x_1)\sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -w_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_a \\ e_b \\ e_c \\ e_d \\ e_{k1} \\ e_{k2} \end{pmatrix} .$$

Let

$$A=\begin{pmatrix} -a_1\sigma & a_1\sigma & 0 & \sigma & (y_1-x_1)\sigma & 0 & 0 & 0 & 0 & 0 \\ -z_2 & c_1-k_1^* & -x_1 & 0 & 0 & 0 & y_1 & 0 & -e_2 & 0 \\ y_2 & x_1 & -b_1 & 0 & 0 & -z_1 & 0 & 0 & 0 & 0 \\ z_2 & 0 & x_1 & d_1-k_2^* & 0 & 0 & 0 & w_1 & 0 & -e_4 \\ -(y_1-x_1)\sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -w_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and let the number λ is eigenvalue of matrix *A*, if there exists a nonzero vector $\xi=(\xi_1,\xi_2,\cdots,\xi_{10})^T$,which is an eigenvector of *A* associated with the eigenvalue λ , we can get

$$A\xi=\lambda\xi\ . \tag{12}$$

Let take the conjugate transpose on both sides of equation above, we have

$$\xi^H A^T=\overline{\lambda}\xi^H\ . \tag{13}$$

And we use $\xi^H/2$ multiply on the left of equation (12) and postmultiply $\xi/2$ in the equation (13), respectively, then make an addition after the previous process. We also have

$$\xi^H(\frac{1}{2}A+\frac{1}{2}A^T)\xi=\frac{1}{2}(\lambda+\overline{\lambda})\xi^H\xi\ . \tag{14}$$

Owing to the attractiveness of the attractor, there exists $m>0$, such that $\max(|x_1|,|y_1|,|z_1|,|w_1|)\le m<\infty$, $\max(|x_2|,|y_2|,|z_2|,|w_2|)\le m<\infty$, to our best knowledge, $\xi_i^*\xi_j+\xi_j^*\xi_i\le|\xi_i^*||\xi_j|+|\xi_j^*||\xi_i|$. Thus,

$$\frac{1}{2}(\lambda+\overline{\lambda})\xi^H\xi=\xi^H(\frac{1}{2}A+\frac{1}{2}A^T)\xi\ =(\xi_1^*\ \ \xi_2^*\ \ \xi_3^*\ \ \xi_4^*)\begin{pmatrix} -a_1\sigma & \frac{a_1\sigma-z_2}{2} & \frac{y_2}{2} & \frac{\sigma+z_2}{2} \\ \frac{a_1\sigma-z_2}{2} & c_1-k_1^* & 0 & 0 \\ \frac{y_2}{2} & 0 & -b_1 & \frac{x_1}{2} \\ \frac{\sigma+z_2}{2} & 0 & \frac{x_1}{2} & d_1-k_2^* \end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{pmatrix}$$

$$\le-(|\xi_1^*|\,|\xi_2^*|\,|\xi_3^*|\,|\xi_4^*|)\begin{pmatrix} a_1\sigma & -\frac{a_1\sigma+m}{2} & -\frac{m}{2} & -\frac{\sigma+m}{2} \\ -\frac{a_1\sigma+m}{2} & k_1^*-c_1 & 0 & 0 \\ -\frac{m}{2} & 0 & b_1 & -\frac{m}{2} \\ -\frac{\sigma+m}{2} & 0 & -\frac{m}{2} & k_2^*-d_1 \end{pmatrix}\begin{pmatrix}|\xi_1|\\ |\xi_2|\\ |\xi_3|\\ |\xi_4|\end{pmatrix}$$

$$=-(|\xi_1^*|\,|\xi_2^*|\,|\xi_3^*|\,|\xi_4^*|)P\begin{pmatrix}|\xi_1|\\ |\xi_2|\\ |\xi_3|\\ |\xi_4|\end{pmatrix}$$

Hence, if we choose $\sigma>\dfrac{m^2}{4a_1b_1}$, $k_1^*>c_1+\dfrac{(a_1\sigma+m)^2b_1}{4a_1b_1\sigma-m^2}$,

$$k_2^* > d_1 + \frac{(k_1^* - c_1)[m^2 a_1 \sigma + m^2(\sigma + m) + b_1(\sigma + m)^2] - m^2(a_1\sigma + m)^2/4}{(k_1^* - c_1)(4a_1 b_1 \sigma - m^2) - b_1(a_1\sigma + m)^2},$$

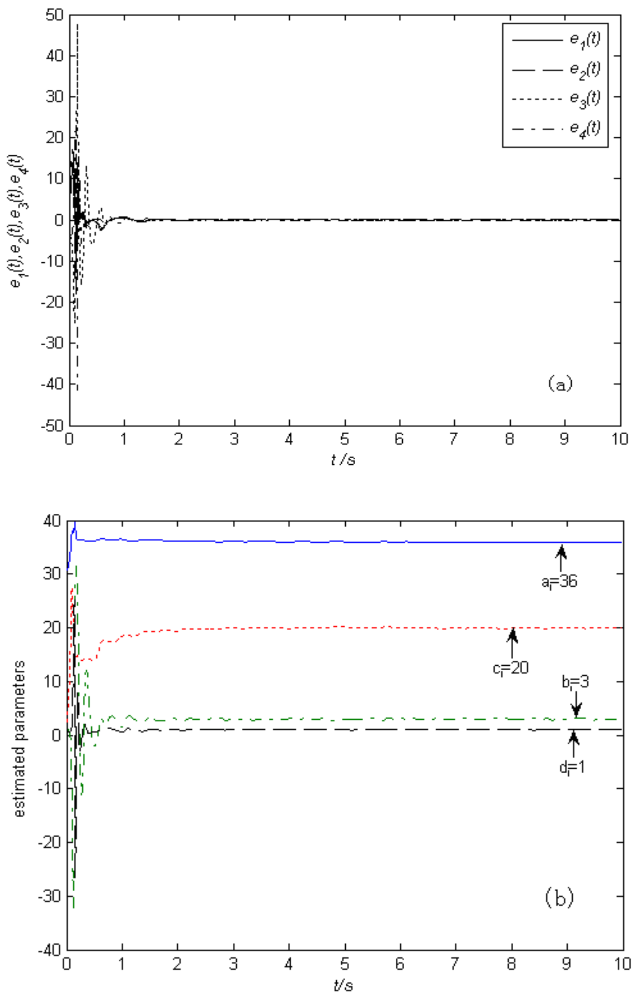
and we can get that P is positive definite matrix. Therefore

$$\frac{1}{2}(\lambda + \overline{\lambda})\xi^H \xi \le -(|\xi_1^*||\xi_2^*||\xi_3^*||\xi_4^*|)P\begin{pmatrix} |\xi_1| \\ |\xi_2| \\ |\xi_3| \\ |\xi_4| \end{pmatrix} \le 0 .$$

Obviously, $\lambda + \overline{\lambda} \le 0$, and any eigenvalue of matrix A is satisfying $|\arg(\lambda_i)| > 0.5q\pi$, based on the stability theory of the fractional-order systems [13], the error $e_i(i=1,2,3,4)$ will converge to zero as $t \to \infty$ which implies that the system (7) and (8) are globally synchronized.

3.2 Numerical Simulation

In the numerical simulation with the predictor-corrector method, select the true values of "unknown" parameters of the drive system as a=36, b=3, c=20, and d=1, and take the initial estimated parameters as $a_1(0)$=-5, $b_1(0)$=-5/3, $c_1(0)$=-18, and $d_1(0)$=-0.9; we choose $\alpha = \beta = 1$, $k_1(0)$= $k_2(0)$=0. Also, the initial condition $x_1(0)$=3, $y_1(0)$=-4, $z_1(0)$=2, $w_1(0)$=2, $x_2(0)$=-3, $y_2(0)$=4, $z_2(0)$=-2, $w_2(0)$=-2. From Fig.2.(a), it can be seen that the synchronization error converges to zero. System (7) and (8) have achieved absolute synchronization. In Fig.2 (b) and (c), the estimates a_1, b_1, c_1, d_1 of the unknown parameters converge to a_1=a=36, b_1=b=3, c_1=c=20, d_1=d=1, gradually with time increase, and k_1, k_2 will converge to constant, respectively, as $t \to \infty$.



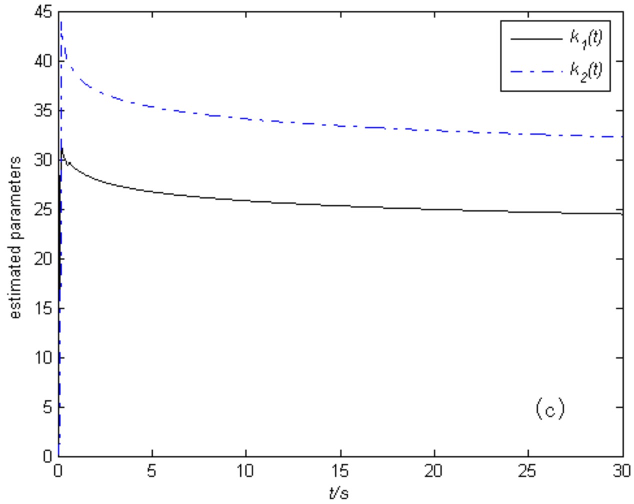


Figure 2: The synchronization errors between system (7) and (8) and the estimate value of parameters a_1, b_1, c_1, d_1, k_1 and k_2 with time t

3.3 Circuit Experimental Verification

As is well known, using the standard integer order operators to approximate the fractional operators is an effective scheme to solve this kind of problem. In Ref.[1], approximations for $1/s^q$ with q=0.1-0.9 in step size 0.1 were given with errors of approximately 2dB. Here, we take q=0.95 as fractional order of hyperchaotic system. The following transfer function approximation method presented in Ref. [11]:

$$\frac{1}{s^{0.95}} = \frac{1.281s^2 + 18.6004s + 2.0833}{s^3 + 18.4738s^2 + 2.7574s + 0.003}$$

and the corresponding circuit unit is displayed in Fig.3. Thus, the transfer function $H(s)$ between A and B can be expressed as follows:

$$H(s) = \frac{1}{C_0}\frac{\left(\frac{C_0}{C_1}+\frac{C_0}{C_2}+\frac{C_0}{C_3}\right)\left[s^2 + \frac{\left(\frac{C_2+C_3}{R_1}+\frac{C_1+C_3}{R_2}+\frac{C_1+C_2}{R_3}\right)s + \frac{R_1+R_2+R_3}{R_1R_2R_3}}{C_1C_2+C_2C_3+C_1C_3}\right]}{(s+1/R_1C_1)(s+1/R_2C_2)(s+1/R_3C_3)}. \tag{15}$$

where C_0 is the unit parameter. Letting C_0=1 μF and $F(s)=H(s)C_0= 1/s^{0.95}$ and comparing Eq.(15) with Expression of $1/s^{0.95}$, we can obtain the values of resistances and capacitances as follows: R_1=717.04 MΩ, R_2=1514 kΩ, R_3=15 kΩ, C_1=1.2678 μF, C_2=4.5933 μF, C_3=3.6423 μF. AD633 is used as multiplier with an output coefficient of 0.1, LM741 chip is an operational amplifier. Other values of resistances in circuit are indicated in Fig.4. For simplicity, we choose the parameters of the drive system and response system with R_a= R_{a1}=2.78 kΩ, R_b= R_{b1}=33.3 kΩ, R_c= R_{c1}=5 kΩ, R_d= R_{d1}=100 kΩ. The parts in the dashed line are the circuitry of controller u, R_{k1}, R_{k2} can be adjustable. The circuit simulation results are depicted in Fig.5. From it, we can find that the slope of state variables equals 1. It is clear that the drive system (7) and response system (8) achieve synchronization.

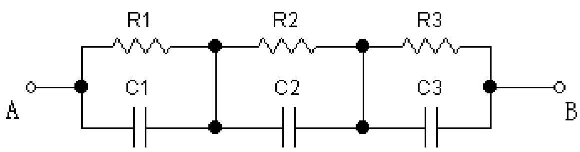


Figure 3: Circuit diagram of fractional-order $1/s^{0.95}$

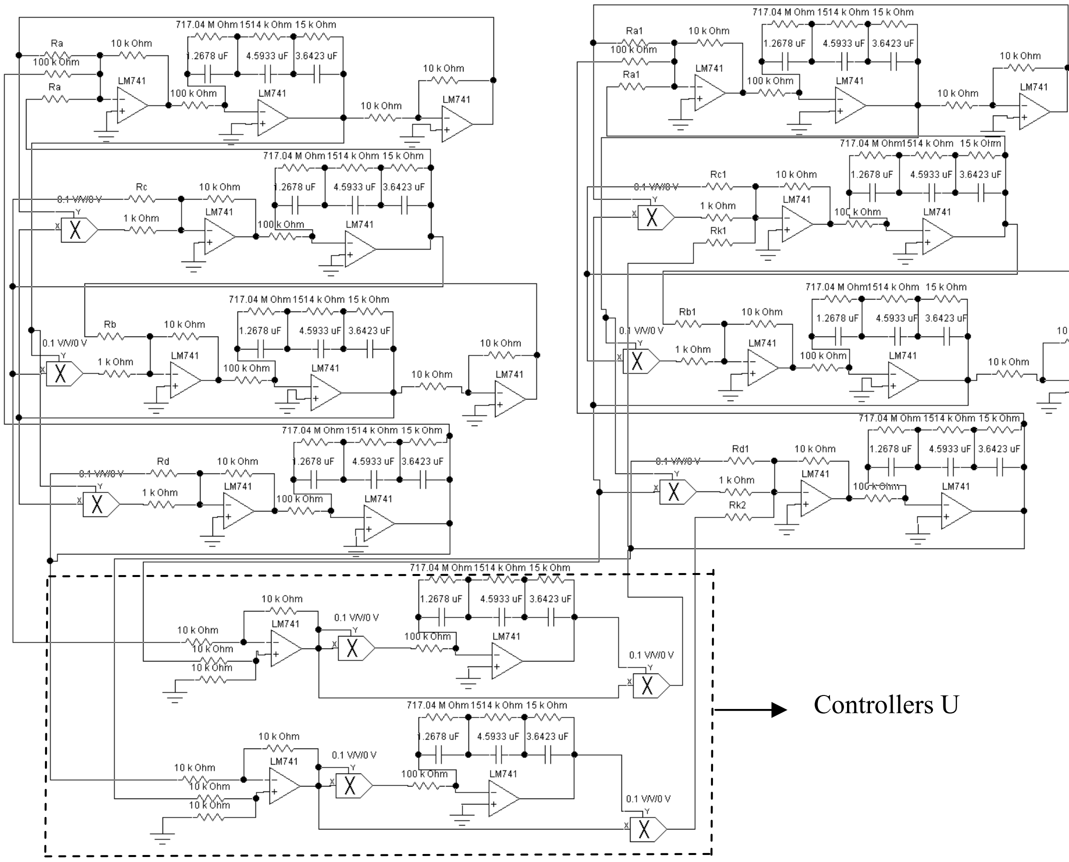
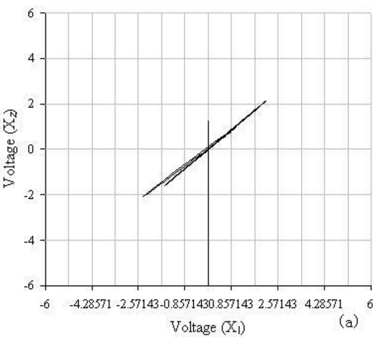
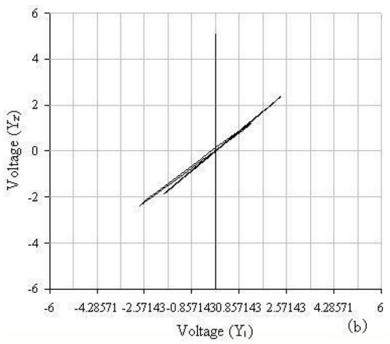


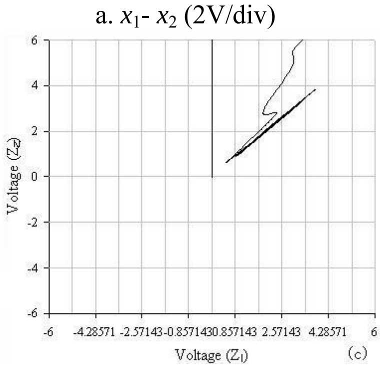
Figure 4: Circuit of synchronization between system (7) and (8)



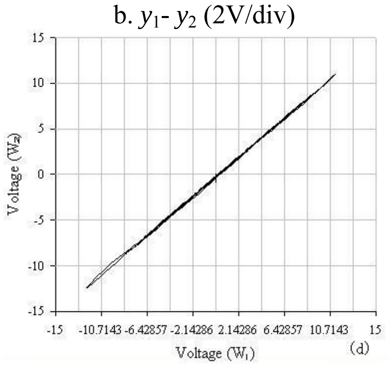
a. x_1- x_2 (2V/div)



b. y_1- y_2 (2V/div)



c. z_1- z_2 (2V/div)



d. w_1- w_2 (5V/div)

Figure 5: Circuit experiment of states synchronization between system (7) and (8)

4 Conclusion

In this paper, we have studied the synchronization of the fractional-order Lü hyperchaotic systems applying adaptive approaches based on the stability theory of the fractional-order system. The synchronization techniques are simple, theoretically rigorous and convenient to realize synchronization. Numerical simulations are given to verify the effectiveness of the proposed synchronization schemes, and also, circuit experiment simulation results are in agreement with numerical simulations. Also, this method can also be extended to other hyperchaotic fractional-order differential systems.

Acknowledgements

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