

# Why is There a Need for Uncertainty Theory?

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## Abstract

Uncertainty theory is a branch of mathematics for modeling human uncertainty. This paper will answer the following questions: What is uncertainty? In what situations does uncertainty arise? What is the difference between uncertain variable and uncertain set? This paper will also discuss the relations and differences among uncertainty, fuzziness and probability.

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## 1 Introduction

Uncertainty is an extremely important feature of the real world. How do we understand uncertainty? How do we model uncertainty? In order to answer those questions, an uncertainty theory was founded by Liu [6] in 2007 and refined by Liu [10] in 2010. Nowadays uncertainty theory has become a branch of mathematics for modeling human uncertainty.

Let  $\Gamma$  be a nonempty set, and  $\mathcal{L}$  a  $\sigma$ -algebra over  $\Gamma$ . Each element  $\Lambda$  in  $\mathcal{L}$  is called an event. A set function  $\mathcal{M}$  from  $\mathcal{L}$  to  $[0, 1]$  is called an uncertain measure if it satisfies the following axioms [6]:

Axiom 1. (Normality Axiom)  $\mathcal{M}\{\Gamma\} = 1$  for the universal set  $\Gamma$ ;

Axiom 2. (Duality Axiom)  $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$  for any event  $\Lambda$ ;

Axiom 3. (Subadditivity Axiom) For every countable sequence of events  $\Lambda_1, \Lambda_2, \dots$ , we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet  $(\Gamma, \mathcal{L}, \mathcal{M})$  is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu [7], thus producing the fourth axiom of uncertainty theory:

Axiom 4. (Product Axiom) Let  $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$  be uncertainty spaces for  $k = 1, 2, \dots, n$ . Then the product uncertain measure  $\mathcal{M}$  is an uncertain measure on the product  $\sigma$ -algebra  $\mathcal{L}_1 \times \mathcal{L}_2 \times \dots \times \mathcal{L}_n$  satisfying

$$\mathcal{M}\left\{\prod_{k=1}^n \Lambda_k\right\} = \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\}.$$

An uncertain variable is a measurable function  $\xi$  from an uncertainty space  $(\Gamma, \mathcal{L}, \mathcal{M})$  to the set of real numbers, i.e., for any Borel set  $B$  of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \tag{1}$$

is an event. In order to describe an uncertain variable in practice, the concept of uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}, \quad \forall x \in \mathfrak{R}. \tag{2}$$

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Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain variables with uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If the function  $f(x_1, x_2, \dots, x_n)$  is strictly increasing with respect to  $x_1, x_2, \dots, x_m$  and strictly decreasing with respect to  $x_{m+1}, x_{m+2}, \dots, x_n$ , then

$$\xi = f(\xi_1, \xi_2, \dots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \dots, \Phi_n^{-1}(1 - \alpha)). \quad (3)$$

This is the operational law of uncertain variables.

Uncertainty theory is thus deduced from the three foundation stones: uncertain measure, uncertain variable and uncertainty distribution. For exploring the recent developments of uncertainty theory, the readers may consult my book *Uncertainty Theory* at <http://orsc.edu.cn/liu/ut.pdf>.

## 2 What is Uncertainty?

An axiomatic system called uncertainty theory has been constructed. What is uncertainty? In fact, we can answer it this way. If it happens that some phenomenon can be quantified by uncertain measure, then we call the phenomenon uncertainty. How do we verify that a phenomenon can be quantified by uncertain measure? In order to answer it, let us consider the question “how many students are there in Uncertainty Theory Laboratory”. Assume the “number of students” is not exactly known but between 7 and 9. In this case, we may derive 8 events (i.e., a  $\sigma$ -algebra) from the concept of “number of students”. The 8 events are listed as follows,

$$\emptyset, \{7\}, \{8\}, \{9\}, \{7, 8\}, \{7, 9\}, \{8, 9\}, \{7, 8, 9\}.$$

In order to indicate the belief degree that each event will occur, we must assign to each event a number between 0 and 1, for example,

$$\begin{aligned} \mathcal{M}\{7\} &= 0.6, & \mathcal{M}\{8\} &= 0.3, & \mathcal{M}\{9\} &= 0.2, \\ \mathcal{M}\{7, 8\} &= 0.8, & \mathcal{M}\{7, 9\} &= 0.7, & \mathcal{M}\{8, 9\} &= 0.4, \\ \mathcal{M}\{\emptyset\} &= 0, & \mathcal{M}\{7, 8, 9\} &= 1. \end{aligned}$$

This assignment implies that a set function  $\mathcal{M}$  is defined from those 8 events to  $[0, 1]$ . If it happens that such a set function satisfies the axioms of uncertainty theory (i.e., it is an uncertain measure), then the “number of students” is an uncertain variable.

Perhaps the above approach is the unique scientific way to judge if a phenomenon is uncertainty. In summary, we may have a formal definition of uncertainty that is anything that satisfies the axioms of uncertainty theory. In other words, uncertainty is anything that can be quantified by the uncertain measure.

Please also note that the word “uncertainty” has been widely used or abused. In a wide sense, Knight [4] and Keynes [3] used uncertainty to represent any non-probabilistic phenomena. This type of uncertainty is also known as Knightian uncertainty, Keynesian uncertainty or true uncertainty. Unfortunately, it is impossible for us to develop a unified mathematical theory to deal with such an uncertainty because the concept of non-probability represents too many things. In a narrow sense, Liu [10] defined uncertainty as anything that satisfies the axioms of uncertainty theory. It is emphasized that uncertainty in the narrow sense is a scientific terminology, but uncertainty in the wide sense is not.

Some people think that *uncertainty and probability are synonymous*. This is a wrong viewpoint either in the wide sense or in the narrow sense. Uncertainty and probability are undoubtedly two different concepts. Otherwise, the terminology “uncertainty” becomes superfluous and we should use “probability” only.

Some people believe that *everything is probability or subjective probability*. When the sample size is large enough, the estimated probability may be close enough to the real one. Meanwhile, perhaps the viewpoint is somewhat true. However, we are frequently lack of observed data, and then the estimated probability may be far from the real one. Note that probability theory may lead to counterintuitive results in this case. More extensively, Hicks [2] concluded that “we should always ask ourselves, before we apply [stochastic methods], whether they are appropriate to the problem at hand. Very often they are not.”

Some people affirm that *probability theory is the only legitimate approach*. Perhaps this misconception is rooted in Cox's theorem [1] that any measure of belief is "isomorphic" to a probability measure. However, uncertain measure is considered coherent but not isomorphic to any probability measure. What goes wrong with Cox's theorem? Personally I think that Cox's theorem presumes the truth value of conjunction of two propositions is a twice differentiable function of the truth values of individual propositions, i.e.,

$$T(P \wedge Q) = f(T(P), T(Q))$$

and then excludes uncertain measure from its start. In fact, there does not exist any evidence that the truth value of conjunction is completely determined by the truth values of individual propositions, let alone a twice differentiable function.

### 3 In What Situations does Uncertainty Arise?

Frequency is the percentage of all the occurrences of an event in the experiment. An event's frequency is a factual property, and does not change with our state of knowledge. In other words, the frequency in the long run exists and is relatively invariant, no matter if it is observed by us.

A fundamental premise of applying probability theory is that the estimated probability is close enough to the long-run frequency, no matter whether the probability is interpreted as subjective or objective. Otherwise, the law of large numbers is no longer valid and probability theory is no longer applicable.

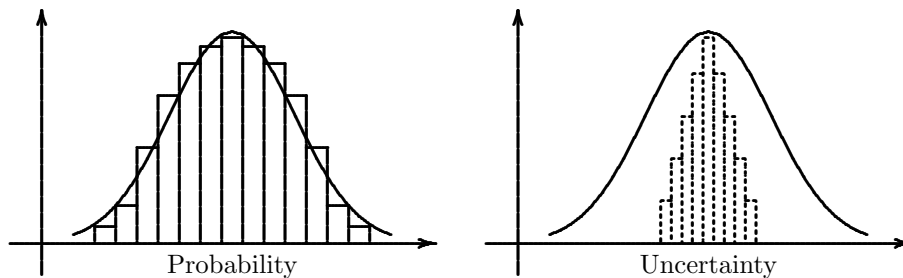


Figure 1: When the sample size is large enough, the estimated probability (curve) is close enough to the long-run frequency (solid histogram) and probability theory is applicable. When the sample size is too small (even no-sample), the belief degree (curve) usually has much larger variance than the long-run frequency (dashed histogram) and we should deal with it by uncertainty theory.

However, very often we are lack of observed data about the unknown state of nature, not only for economic reasons, but also for technical difficulties. How do we deal with this case? It seems that we have to invite some domain experts to evaluate their belief degree that each event will occur. Since human beings usually overweight unlikely events [12], the belief degree may have much larger variance than the real frequency, and we should deal with it by uncertainty theory.

Could we deal with the belief degree by probability theory when the belief degree deviates from the frequency? Some people do think so and call it subjective probability. However, it is inappropriate because probability theory may lead to counterintuitive results in this case.

Assume the weight of a truck is 90 tons and the strengths of 50 bridges are iid normal random variables  $\mathcal{N}(100, 1)$  in tons (I am afraid this fact cannot be verified without the help of God). For simplicity, it is admitted that there is only one truck on the bridge at every time, and a bridge collapses whenever its real strength is less than the weight of truck. Now let us have the truck cross over the 50 bridges one by one. It is easy to verify that

$$\Pr\{\text{"the truck can cross over the 50 bridges"}\} \approx 1. \quad (4)$$

That is to say, the truck may cross over the 50 bridges successfully.

However, when there do not exist any observed data for the strength of bridge at the moment, we have to invite some domain experts to evaluate the belief degree about it. As we stated before, usually the belief degree has much larger variance than the real strength of bridge. In other words, the real strength of bridge

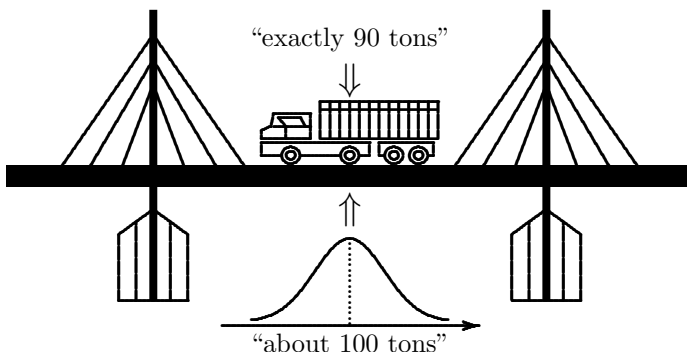


Figure 2: A truck is crossing over a bridge

is relatively invariant compared with the human belief degree. Let us imagine what will happen if the belief degree is treated as probability.

Assume the belief degree looks like a normal probability distribution  $\mathcal{N}(100, 100)$ . If we want to deal with this problem by probability theory, then we have no choice but to regard the strengths of the 50 bridges as iid normal random variables with expected value 100 and variance 100 in tons. Now we have the truck cross over the 50 bridges one by one. Then we immediately have

$$\Pr\{\text{"the truck can cross over the 50 bridges"}\} \approx 0. \quad (5)$$

Thus it is almost impossible that the truck crosses over the 50 bridges successfully. The results (4) and (5) are at opposite poles. In other words, it leads to a contradictory result if we deal with the belief degree by probability theory. Hence probability theory is not applicable to human uncertainty.

Within the research area of information science, we should follow a basic principle that *a possible proposition cannot be judged impossible*. In other words, if a proposition is possibly true, then its truth value should not be zero. Equivalently, if a proposition is possibly false, then its truth value should not be unity. Thus probability theory is not appropriate to human uncertainty on the basis of such a principle.

## 4 What is the Difference Between Uncertain Variable and Uncertain Set?

Uncertain variable and uncertain set are two basic tools in uncertainty theory. What is the difference between them? As their names suggest, both of them belong to the same broad category of uncertain concepts. However, they are differentiated by their mathematical definitions: an uncertain set is a set-valued function while an uncertain variable is a real-valued function. In other words, the former refers to a collection of values, while the latter to one value.

Essentially, the difference between uncertain variable and uncertain set focuses on the property of *exclusivity*. In fact, the same word can be either uncertain variable or uncertain set. They will be distinguished from the context. If the concept has exclusivity, then it is an uncertain variable. Otherwise, it is an uncertain set. A few examples will illustrate the difference between the two concepts.

**Example 1:** Consider the statement “John is a young man”. Is “young” an uncertain variable or an uncertain set? If we are interested in John’s real age, then “young” is an uncertain variable rather than an uncertain set because it is an exclusive concept (John’s age cannot be more than one value). For example, if John is 20 years old, then it is impossible that John is 25 years old. In other words, “John is 20 years old” does exclude the possibility that “John is 25 years old”. By contrast, if we are interested in what ages can be regarded “young”, then “young” is an uncertain set rather than an uncertain variable because the concept now has no exclusivity. For example, both 20-year-old and 25-year-old men can be considered “young”. In other words, “a 20-year-old man is young” does not exclude the possibility that “a 25-year-old man is young”.

**Example 2:** Consider the statement “James is a tall man”. Is “tall” an uncertain variable or an uncertain set? If we are interested in James’ real height, then “tall” is an uncertain variable rather than an uncertain set

because the concept now is exclusive (James' height cannot be more than one value). For example, if James is 180cm in height, then it is impossible that James is 185cm in height. In other words, "James is 180cm in height" does exclude the possibility that "James is 185cm in height". By contrast, if we are interested in what heights can be considered "tall", then "tall" is an uncertain set rather than an uncertain variable because the concept in this case has no exclusivity. For example, both 180cm and 185cm can be considered "tall". In other words, "a 180cm-tall man is tall" does not exclude the possibility that "a 185cm-tall man is tall".

**Example 3:** Consider the statement "most students are boys". Is "most" an uncertain variable or an uncertain set? If we are interested in what percentage of students are boys, then "most" is an uncertain variable rather than an uncertain set because the concept in this case is exclusive. For example, if 80% of students are boys, then it is impossible that 85% of students are boys. By contrast, if we are interested in what percentages can be considered "most", then "most" is an uncertain set rather than an uncertain variable because the concept now has no exclusivity. For example, both 80% and 85% can be considered "most".

## 5 Why is Fuzzy Variable not Suitable for Uncertain Quantities?

A fuzzy variable is a function from a possibility space to the set of real numbers [11]. Some people think that fuzzy variable is a suitable tool for modeling uncertain quantities. Is it really true? Unfortunately, the answer is negative.

Let us reconsider the strength of bridge. If "about 100 tons" is regarded as a fuzzy concept, then we may assign it a membership function, say

$$\mu(x) = \begin{cases} (x - 80)/20, & \text{if } 80 \leq x \leq 100 \\ (120 - x)/20, & \text{if } 100 \leq x \leq 120. \end{cases} \quad (6)$$

This membership function represents a triangular fuzzy variable (80, 100, 120).

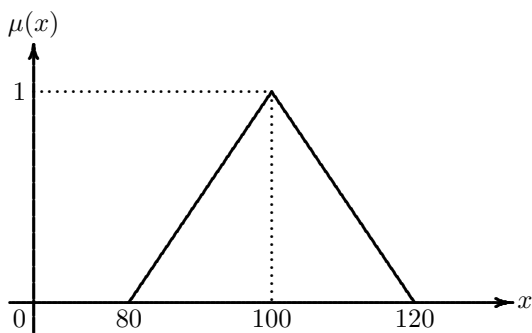


Figure 3: Membership function of triangular fuzzy variable (80, 100, 120)

Please do not argue why I choose such a membership function because it is not important for the focus of debate. Based on the membership function  $\mu$  and the definition of possibility measure

$$\text{Pos}\{B\} = \sup_{x \in B} \mu(x), \quad (7)$$

the possibility theory will immediately conclude the following three propositions:

- (a) the strength is "exactly 100 tons" with possibility measure 1,
- (b) the strength is "not 100 tons" with possibility measure 1,
- (c) "exactly 100 tons" and "not 100 tons" are equally likely.

However, it is doubtless that the belief degree of "exactly 100 tons" is almost zero. Nobody is so naive to expect that "exactly 100 tons" is the true strength of the bridge. On the other hand, "exactly 100 tons" and "not 100 tons" have the same belief degree in possibility measure. Thus we have to regard them "equally likely". It seems that no human being can accept this conclusion. This paradox shows that those imprecise quantities like "about 100 tons" cannot be quantified by possibility measure and then they are not fuzzy concepts.

## 6 Why is Fuzzy Set Not Suitable for Unsharp Concepts?

A fuzzy set is defined by its membership function  $\mu$  which assigns to each element  $x$  a real number  $\mu(x)$  in the interval  $[0, 1]$ , where the value of  $\mu(x)$  represents the grade of membership of  $x$  in the fuzzy set. This definition was given by Zadeh [13] in 1965. However, there are too many fuzzy sets that share the same membership functions. Perhaps this is the root of debate about fuzzy set theory. A more precise definition states that a fuzzy set is a function from a possibility space to a collection of sets.

Some people believe that fuzzy set is a suitable tool to model unsharp concepts. Unfortunately, it is not true. In order to convince the reader, let us examine the concept of “young”. Without loss of generality, assume “young” has a trapezoidal membership function  $(15, 20, 30, 40)$ , i.e.,

$$\mu(x) = \begin{cases} 0, & \text{if } x \leq 15 \\ (x - 15)/5, & \text{if } 15 \leq x \leq 20 \\ 1, & \text{if } 20 \leq x \leq 30 \\ (40 - x)/10, & \text{if } 30 \leq x \leq 40 \\ 0, & \text{if } x \geq 40. \end{cases}$$

Note that “young” may take any values of  $\alpha$ -cut of  $\mu$ . It follows from fuzzy set theory that we may immediately conclude two propositions:

- (a) “young” includes  $[20\text{yr}, 30\text{yr}]$  with possibility measure 1,
- (b) “young” is just  $[20\text{yr}, 30\text{yr}]$  with possibility measure 1.

The first conclusion sounds good. However, the second conclusion seems unacceptable because it is almost impossible that “young” does mean the ages just from 20 to 30. This fact says that “young” cannot be regarded as a fuzzy set.

We should always ask ourselves, before we apply fuzzy set theory, whether it is appropriate to the problem at hand. Almost it is not.

## 7 What is the Difference Between Uncertainty and Fuzziness?

Possibility theory [14] is a branch of mathematics for studying the behavior of fuzzy phenomena. What is difference between uncertainty theory and possibility theory? The essential difference is that possibility theory assumes

$$\text{Pos}\{A \cup B\} = \text{Pos}\{A\} \vee \text{Pos}\{B\} \quad (8)$$

for any events  $A$  and  $B$  no matter if they are independent or not, and uncertainty theory assumes

$$\mathcal{M}\{A \cup B\} = \mathcal{M}\{A\} \vee \mathcal{M}\{B\} \quad (9)$$

only for independent events  $A$  and  $B$ . However, a lot of surveys showed that the measure of union of events is not necessarily maxitive, or more exactly,

$$\mathcal{M}\{A \cup B\} > \mathcal{M}\{A\} \vee \mathcal{M}\{B\} \quad (10)$$

when the events  $A$  and  $B$  are not independent. This fact states that human brains do not behave fuzziness.

Both uncertainty theory and possibility theory attempt to model human belief degree, where the former uses the tool of uncertain measure and the latter uses the tool of possibility measure. Thus they are complete competitors.

## 8 What is the Difference Between Uncertainty and Randomness?

Probability theory [5] is a branch of mathematics for studying the behavior of random phenomena. What is the difference between uncertainty theory and probability theory? The main difference is that the product uncertain measure is the minimum of uncertain measures of uncertain events, i.e.,

$$\mathcal{M}\{A \times B\} = \mathcal{M}\{A\} \wedge \mathcal{M}\{B\}, \quad (11)$$

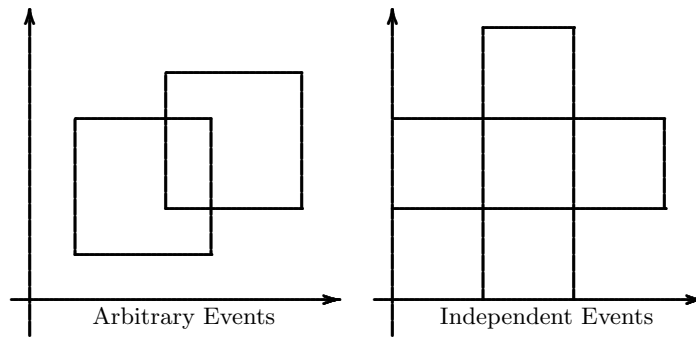


Figure 4: Union of two events

and the product probability measure is the product of probability measures of random events, i.e.,

$$\Pr\{A \times B\} = \Pr\{A\} \times \Pr\{B\}. \tag{12}$$

This difference implies that uncertain variables and random variables obey different operational laws.

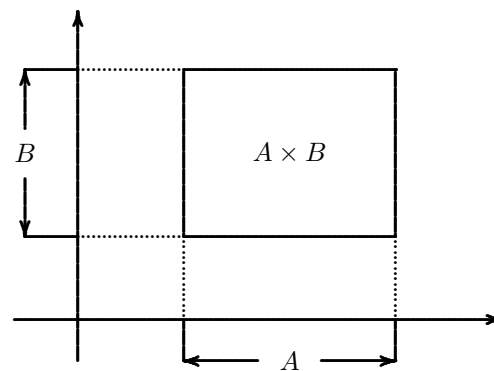


Figure 5: Product of two events (intersection of independent events)

Probability theory and uncertainty theory are complementary mathematical systems that provide two acceptable mathematical models to deal with the phenomena whose outcomes cannot be exactly predicted in advance. Personally I think probability measure should be interpreted as frequency while uncertain measure should be interpreted as belief degree. Probability theory is not an appropriate tool to model belief degree, just like that uncertainty theory is not an appropriate tool to model frequency.

## 9 Conclusion

When the sample size is too small (even no-sample) to estimate a probability distribution, we have to invite some domain experts to evaluate their belief degree that each event will occur. Since human beings usually overweight unlikely events, the belief degree may have much larger variance than the real frequency and then probability theory is no longer valid. In this situation, we should deal with it by uncertainty theory.

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