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An Extension of Risk Measures Using Non-Precise a-Priori Densities

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Abstract

In this paper we propose a fuzzy risk measure. Risk measure is an important concept in order to analyze risk under uncertainty. However, in the real world, we need more flexible risk measures including human intelligence. In order to overcome this difficulty we consider the fuzzy risk measure using non-precise a-priori densities.

Keywords: conditional value at risk, non-precise a-priori densities, Bayes' estimation

1 Introduction

We consider a stochastic model $X \sim f(\cdot|\theta), \theta \in \Theta$ with continuous parameter space Θ, a-priori density function $\pi(\theta)$, where $X \in L^1(\Omega, \mathcal{A}, P)$ on some probability space $(\Omega, \mathcal{A}, P)$. In classical statistical methods, we usually use only one density function to apply maximum likelihood estimation or Bayesian estimation. However, Viertl and Hareter [10] pointed out that this setting is insufficient because precise a-priori distributions are questionable concerning their justification. This is why they proposed non-precise a-priori densities $\tilde{\pi}(\theta)$ whose precise definition is given by Definition 1.

On the other hand, in recent years some risk measures have been generated and analyzed by an economically motivated optimization problem, for example, value at risk($V@R$), conditional value-at-risk ($CV@R$) [8], convex risk of measure [3] and so on. In particular $CV@R$ is a very useful and important criterion when dealing with real problems, see [5, 6, 9]. In this paper we propose a fuzzy conditional value-at-risk $\widetilde{CV@R}$ using a non-precise density $\tilde{\pi}$ in order to handle risk more flexibly. As far as we know, Yoshida [11] firstly investigated risk measures under fuzzy environment. However prior distributions were not considered in this paper. We believe that introducing non-precise a priori densities for risk analysis is very useful in order to analyze more practical problems.

2 Preliminaries

Let $\mathbb{R}, \mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ be the sets of real numbers, real n-dimensional column vectors and real $m \times n$ matrices, respectively. Let $\mathcal{B}(\mathbb{R})$ be all Borel sets on $\mathbb{R}$. The sets $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ are endowed with the norm $\|\cdot\|$, where for $x = (x(1), \ldots, x(n)) \in \mathbb{R}^n, \|x\| = \sum_{j=1}^n |x(j)|$ and for $y = (y_{ij}) \in \mathbb{R}^{m\times n}$, $\|y\| = \max_{1\le i\le m} \sum_{j=1}^n |y_{ij}|$. For any set X, let $\mathcal{F}(X)$ be the set of all fuzzy sets $X \to [0, 1]$. The α-cut of $\tilde{x} \in \mathcal{F}(X)$ is given by $\tilde{x}_\alpha := \{x \in X | \tilde{x}(x) \ge \alpha\} (\alpha \in (0, 1])$ and $\tilde{x}_0 := \mathrm{cl}\{x \in X | \tilde{x}(x) > 0\}$, where cl is a closure of a set. Let $\tilde{\mathbb{R}}$ be the set of all fuzzy numbers, i.e., $\tilde{r} \in \tilde{\mathbb{R}}$ means that $\tilde{r} \in \mathcal{F}(\mathbb{R})$ is normal, upper semi-continuous and fuzzy convex and has a compact support. Let $\mathcal{C}$ be the set of all bounded and closed intervals of $\mathbb{R}$. Then, for $\tilde{r} \in \mathcal{F}(\mathbb{R})$, it holds that $\tilde{r} \in \tilde{\mathbb{R}}$ if and only if $\tilde{r}$ normal and $\tilde{r}_\alpha \in \mathcal{C}$ for $\alpha \in [0, 1]$. So, for $\tilde{r} \in \tilde{\mathbb{R}}$, we write $\tilde{r}_\alpha = [\tilde{r}^-_\alpha, \tilde{r}^+_\alpha] (\alpha \in [0, 1])$. We use the extension principle [2] by Zadeh to define arithmetics with fuzzy numbers and fuzzy functions $\tilde{f}(x) \in \tilde{\mathbb{R}}$ for each $x \in \mathbb{R}$, respectively. Here, we will give a partial order $\preccurlyeq$ on $\mathcal{C}$ by the

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definition: For $[a,b],[c,d] \in \mathcal{C}$,

$$[a,b] \preccurlyeq [c,d] \quad \text{if} \quad a \le c \quad \text{and} \quad b \le d,$$
$$[a,b] \prec [c,d] \quad \text{if} \quad [a,b] \preccurlyeq [c,d] \quad \text{and} \quad [a,b] \neq [c,d].$$

This partial order $\preccurlyeq$ on $\mathcal{C}$ is extended to that of $\tilde{\mathbb{R}}$, called fuzzy max order,

$$\tilde{u} \preccurlyeq \tilde{v} \quad \text{if} \quad \tilde{u}_\alpha \preccurlyeq \tilde{v}_\alpha \quad \text{for all} \quad \alpha \in [0,1],$$
$$\tilde{u} \prec \tilde{v} \quad \text{if} \quad \tilde{u} \preccurlyeq \tilde{v} \quad \text{and} \quad \tilde{u} \neq \tilde{v}.$$

Also, as a further extension, the partial order for fuzzy functions [10] can be defined similarly. The Hausdorff metric on $\mathcal{C}$ is denoted by δ, i.e.,

$$\delta([a,b],[c,d]) = |a-c| \vee |b-d| \quad \text{for} \quad [a,b],[c,d] \in \mathcal{C}.$$

This metric can be extended to $\tilde{\mathbb{R}}$ by

$$\delta(\tilde{u},\tilde{v}) = \sup_{\alpha\in[0,1]} \delta((\tilde{u})_\alpha,(\tilde{v})_\alpha)$$

for $\tilde{u},\tilde{v} \in \tilde{\mathbb{R}}$. Then, it is known that the metric space $(\tilde{\mathbb{R}}^n,\delta)$ is complete [1].

Lemma 1 ([2]) *Let* $\tilde{x}(x) = \sup_{\alpha\in[0,1]} \alpha I_{A_\alpha}(x)$, *where* $A_\beta \subset A_\gamma$ *for* $A_\beta, A_\gamma (0 \le \gamma \le \beta \le 1) \in \mathcal{C}$ *and* $\cap_{\alpha<\beta} A_\alpha = A_\beta$. *Then,* $\tilde{x} \in \tilde{\mathbb{R}}$ *and* $\tilde{x}_\alpha = A_\alpha$.

This lemma is very popular so we omit the proof.

Owing to this decomposition lemma, we can define a fuzzy number if we can construct its $\alpha-$cut. Referring to [10], we define fuzzy integral $\tilde{\mathfrak{F}}$ for the fuzzy functions $\tilde{f}$, non-precise densities $\tilde{\pi}$ and the fuzzy probability $\tilde{P}(A)$, respectively.

Definition 1 ([10]) *Assume that* $\tilde{f}_\alpha^-(x)$ *and* $\tilde{f}_\alpha^+(x)$ *are integrable functions on* $\mathbb{R}$. *Then fuzzy integral is defined by*

$$\tilde{\mathfrak{F}} = (F)\int_a^b \tilde{f}(x)dx,$$

where $\tilde{\mathfrak{F}}_\alpha^- := \int_a^b \tilde{f}_\alpha^-(x)dx$ *and* $\tilde{\mathfrak{F}}_\alpha^+ := \int_a^b \tilde{f}_\alpha^+(x)dx$ *for all* $\alpha \in (0,1]$ *and a condition for* $\tilde{\pi}$ *is*

$$(F)\int_{\mathbb{R}} \tilde{\pi}(x)dx = \tilde{1}$$

with $\tilde{1} \in \tilde{\mathbb{R}}$ *and* $1 \in \tilde{1}_1$.

Definition 2 ([10]) *Let* $S_\alpha = \{f : f \text{is a probability density s.t.} \tilde{\pi}_\alpha^-(x) \le f(x) \le \tilde{\pi}_\alpha^+(x) \text{ for all } x \in \mathbb{R}\}$. *The* $\alpha-$*cut* $[\tilde{P}_\alpha^-, \tilde{P}_\alpha^+]$ *of the fuzzy probability* $\tilde{P}(A)$ *is defined by*

$$\begin{aligned} \tilde{P}_\alpha^+(A) &= \sup_{f\in S_\alpha}\int_A f(x)dx = \begin{cases} 1-\int_{A^c}\tilde{\pi}_\alpha^-(x)dx & \text{if } \int_A \tilde{\pi}_\alpha^+(x)dx + \int_{A^c}\tilde{\pi}_\alpha^-(x)dx > 1, \\ \int_A \tilde{\pi}_\alpha^+(x)dx & \text{else,} \end{cases} \\ \tilde{P}_\alpha^-(A) &= \inf_{f\in S_\alpha}\int_A f(x)dx = \begin{cases} \int_A \tilde{\pi}_\alpha^-(x)dx & \text{if } \int_A \tilde{\pi}_\alpha^-(x)dx + \int_{A^c}\tilde{\pi}_\alpha^+(x)dx > 1, \\ 1-\int_{A^c}\tilde{\pi}_\alpha^+(x)dx & \text{else} \end{cases} \end{aligned} \tag{1}$$

for $A \in \mathcal{B}(\mathbb{R})$.

We denote the calculation of fuzzy probability $\tilde{P}(A)$ by

$$\tilde{P}(A) = (FP)\int_A \tilde{\pi}(x)dx \quad \text{for } A \in \mathcal{B}(\mathbb{R}). \tag{2}$$

Recall that $X \sim f(\cdot|\theta), \theta \in \Theta$ with continuous parameter space Θ, a-priori density function $\pi(\theta)$. Then, a distribution function of X for a prior density $\pi(\theta)$ is given by

$$F_X(x|\pi) = \int_{-\infty}^x \int_\Theta \pi(\theta)f(y|\theta)d\theta dy. \tag{3}$$

In case of non-precise a-priori density $\tilde{\pi}(\theta)$,

$$\tilde{F}_X(x|\tilde{\pi}) = (FP)\int_{-\infty}^{x} \tilde{f}(y|\tilde{\pi})dy, \tag{4}$$

where $\tilde{f}(y|\tilde{\pi}) = (F)\int_{\Theta} \tilde{\pi}(\theta)f(y|\theta)d\theta$. Note that $\tilde{f}(y|\tilde{\pi})$ is clearly a non-precise density, that is, $(F)\int_{-\infty}^{\infty} \tilde{f}(y|\tilde{\pi})dy = \tilde{1}$.

Theorem 1 *For any $\tilde{\pi}$, we have the following.*
(i)$\lim_{x\to\infty} \tilde{F}_X(x|\tilde{\pi}) = 1$, $\lim_{x\to-\infty} \tilde{F}_X(x|\tilde{\pi}) = 0$.
(ii)$\tilde{F}_X(x|\tilde{\pi}) \succeq \tilde{F}_X(y|\tilde{\pi})$ *if* $x \geq y$.
(iii) $\lim_{x\to y+0} \tilde{F}_X(x|\tilde{\pi}) = \tilde{F}_X(y|\tilde{\pi})$.

Proof.(i) It is sufficient to prove that

$$\lim_{x\to\infty}\Big((FP)\int_{-\infty}^{x} \tilde{f}(y|\tilde{\pi})dy\Big)_{\alpha}^{+} = \lim_{x\to\infty}\Big((FP)\int_{-\infty}^{x} \tilde{f}(y|\tilde{\pi})dy\Big)_{\alpha}^{-} = 1$$

holds. From (1), we have

$$\begin{aligned} &\lim_{x\to\infty}\Big((FP)\int_{-\infty}^{x} \tilde{f}(y|\tilde{\pi})dy\Big)_{\alpha}^{+} \\ &= \begin{cases} \lim\limits_{x\to\infty}\big(1 - \displaystyle\int_{x}^{\infty} \tilde{f}_{\alpha}^{-}(y|\tilde{\pi})\big)dy = 1 & if\ \int_{-\infty}^{x} \tilde{f}_{\alpha}^{+}(y|\tilde{\pi})dy + \int_{x}^{\infty} \tilde{f}_{\alpha}^{-}(y|\tilde{\pi})dy > 1, \\ \lim\limits_{x\to\infty}\displaystyle\int_{-\infty}^{x} \tilde{f}_{\alpha}^{+}(y|\tilde{\pi})dy = 1 & else. \end{cases} \end{aligned}$$

Similarly we can prove that $\lim_{x\to\infty}\Big((FP)\int_{-\infty}^{x} \tilde{f}(y|\tilde{\pi})dy\Big)_{\alpha}^{-} = 1$ holds. The remaining part can be proved similarly. (ii), (iii) Also, we can prove straightforward from the definition. This completes the proof. □

According to Zadeh's extension principle [2], we define the fuzzy value at risk $\widetilde{V@R}_{\gamma}(X|\tilde{\pi})$ and conditional value at risk $\widetilde{CV@R}_{\gamma}(X|\tilde{\pi})(\gamma \in (0,1))$. Let $V@R_{\gamma}(F) = \inf\{y|F(y) \geq \gamma\}, CV@R_{\gamma}(F) = \frac{1}{1-\gamma}\int_{\gamma}^{1} V@R_p(F)dp(\gamma \in (0,1))$, respectively, where F are distribution functions.

Definition 3 *For a given $\tilde{\pi}$ and a density function $f(x|\theta)$ we define the fuzzy value at risk $\widetilde{V@R}_{\gamma}(X|\tilde{\pi})$ and conditional value at risk $\widetilde{CV@R}_{\gamma}(X|\tilde{\pi})(\gamma \in (0,1))$ as follows:*

$$\begin{aligned} \widetilde{V@R}_{\gamma}(X|\tilde{\pi})(x) &= \sup_{V@R_{\gamma}(F)=x} \inf_{y} \tilde{F}_X(y|\tilde{\pi})(F(y)), \\ \widetilde{CV@R}_{\gamma}(X|\tilde{\pi})(x) &= \sup_{CV@R_{\gamma}(F)=x} \inf_{y} \tilde{F}_X(y|\tilde{\pi})(F(y)). \end{aligned} \tag{5}$$

Lemma 2 *The $\alpha-$cut of the fuzzy value at risk $\widetilde{V@R}_{\gamma}(X|\tilde{\pi})$ and conditional value at risk $\widetilde{CV@R}_{\gamma}(X|\tilde{\pi})(\gamma \in (0,1))$ are given by*

$$\begin{aligned} \widetilde{V@R}_{\gamma,\alpha}^{+}(X|\tilde{\pi}) &= \inf\{x|\tilde{F}_{X,\alpha}^{-}(x|\tilde{\pi}) \geq \gamma\}, \\ \widetilde{V@R}_{\gamma,\alpha}^{-}(X|\tilde{\pi}) &= \inf\{x|\tilde{F}_{X,\alpha}^{+}(x|\tilde{\pi}) \geq \gamma\}, \\ \widetilde{CV@R}_{\gamma,\alpha}^{+}(X|\tilde{\pi}) &= \frac{1}{1-\gamma}\int_{\gamma}^{1} \widetilde{V@R}_{p,\alpha}^{+}(X|\tilde{\pi})dp, \\ \widetilde{CV@R}_{\gamma,\alpha}^{-}(X|\tilde{\pi}) &= \frac{1}{1-\gamma}\int_{\gamma}^{1} \widetilde{V@R}_{p,\alpha}^{-}(X|\tilde{\pi})dp. \end{aligned} \tag{6}$$

Proof.

$$
\begin{aligned}
\widetilde{V@R}_{\gamma,\alpha}(X|\tilde{\pi}) &= \{x|\widetilde{V@R}_{\gamma}(X|\tilde{\pi})(x) \geq \alpha\} \\
&= \{V@R_{\gamma}(F)|\tilde{F}_X(y|\tilde{\pi})(F(y)) \geq \alpha \text{ for all } y\} \\
&= \{V@R_{\gamma}(F)|F(y) \in \tilde{F}_{X,\alpha}(y|\tilde{\pi}) \text{ for all } y\} \\
&= \{V@R_{\gamma}(F)|F(y) \in [\tilde{F}^{-}_{X,\alpha}(y|\tilde{\pi}), \tilde{F}^{+}_{X,\alpha}(y|\tilde{\pi})] \text{ for all } y\} \\
&= [V@R_{\gamma}(\tilde{F}^{+}_{X,\alpha}(y|\tilde{\pi})), V@R_{\gamma}(\tilde{F}^{-}_{X,\alpha}(y|\tilde{\pi}))].
\end{aligned}
$$

Similarly, in case of $\widetilde{CV@R}_{\gamma}(X|\tilde{\pi})$, we can prove. □

3 Main Results

Proposition 1 *For any random variables X, Y and $\tilde{\pi}$, $\widetilde{CV@R}_{\gamma}$ has the following* (i)-(iv)*:*
(i) *(Monotonicity) If $X \leq Y$, $\widetilde{CV@R}_{\gamma}(X|\tilde{\pi}) \preceq \widetilde{CV@R}_{\gamma}(Y|\tilde{\pi})$.*
(ii) *(Translation invariance) For X and $c \in \mathbb{R}$, $\widetilde{CV@R}_{\gamma}(X+c|\tilde{\pi}) = \widetilde{CV@R}_{\gamma}(X|\tilde{\pi}) + c$.*
(iii) *(Homogeneity) For X and $\lambda > 0$, $\widetilde{CV@R}_{\gamma}(\lambda X|\tilde{\pi}) = \lambda\widetilde{CV@R}_{\gamma}(X|\tilde{\pi})$.*
(iv) *(Convexity) For X, Y and $0 \leq \lambda \leq 1$, $\widetilde{CV@R}_{\gamma}(\lambda X+(1-\lambda)Y|\tilde{\pi}) \preceq \lambda\widetilde{CV@R}_{\gamma}(X|\tilde{\pi})+(1-\lambda)\widetilde{CV@R}_{\gamma}(Y|\tilde{\pi})$.*

Proof.(i) It is sufficient to prove the following. For all $\alpha \in [0,1]$, if $X \leq Y$, then

$$
\begin{aligned}
\widetilde{CV@R}^{-}_{\gamma,\alpha}(X|\tilde{\pi}) \leq \widetilde{CV@R}^{-}_{\gamma,\alpha}(Y|\tilde{\pi}), \\
\widetilde{CV@R}^{+}_{\gamma,\alpha}(X|\tilde{\pi}) \leq \widetilde{CV@R}^{+}_{\gamma,\alpha}(Y|\tilde{\pi}).
\end{aligned}
$$

Above relationships are derived from a property of non-fuzzy conditional value at risk, see [7] and Lemma 2.
(ii) Since $\tilde{F}_{X+c,\alpha}(x|\tilde{\pi}) = \tilde{F}_{X,\alpha}(x-c|\tilde{\pi})$ holds, the assertion follows. Similarly the other parts can be proved. □

Here we introduce a fuzzy version of the acceptance set [4] which play an important role when considering risk measures.

Definition 4 *For any $\tilde{\pi}$, the acceptance set of $\widetilde{CV@R}_{\gamma}$ is defined by*

$$
\mathfrak{A}_{\widetilde{CV@R}_{\gamma}} := \{X|\widetilde{CV@R}_{\gamma}(X|\tilde{\pi}) \preceq \tilde{0}\}. \tag{7}
$$

Proposition 2 *Let $\mathfrak{A} := \mathfrak{A}_{\widetilde{CV@R}_{\gamma}}$. Then,*
(i) *For $X \in \mathfrak{A}$, if it holds that $Y \leq X$, $Y \in \mathfrak{A}$.*
(ii) *$\mathfrak{A}$ is convex cone.*

Proof.(i) From (i) of Proposition 1 and the definition of $\mathfrak{A}$, it holds that

$$
\widetilde{CV@R}_{\gamma}(Y|\tilde{\pi}) \preceq \widetilde{CV@R}_{\gamma}(X|\tilde{\pi}) \preceq \tilde{0}.
$$

(ii) Clearly, homogeneity of $\widetilde{CV@R}_{\gamma}$ implies that $\mathfrak{A}$ is a cone. For $X, Y \in \mathfrak{A}$, since $\widetilde{CV@R}_{\gamma}(X|\tilde{\pi}) \preceq \tilde{0}$ and $\widetilde{CV@R}_{\gamma}(Y|\tilde{\pi}) \preceq \tilde{0}$ we have

$$
\begin{aligned}
\widetilde{CV@R}_{\gamma}(\lambda X + (1-\lambda)Y|\tilde{\pi}) &\preceq \lambda\widetilde{CV@R}_{\gamma}(X|\tilde{\pi}) + (1-\lambda)\widetilde{CV@R}_{\gamma}(Y|\tilde{\pi}) \\
&\preceq \lambda\tilde{0} + (1-\lambda)\tilde{0} = \tilde{0}.
\end{aligned}
$$

□

4 A Numerical Example

Recall the example by Viertl [10]. Let $\Theta = \mathbb{R}$ and let $\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta)$ be the density of a classical gamma distribution $\gamma(2, \frac{1}{4})$, i.e.

$$\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta) = 4^2 \times \theta \times e^{-4\theta} \times I_{[0,\infty)}(\theta)$$

and we consider triangle fuzzy numbers for each $\theta \in \Theta$, that is,

$$\tilde{\pi}_\alpha(\theta) = \Big[\frac{(\alpha+1)\tilde{\pi}_1^-(\theta)}{2}, \frac{(3-\alpha)\tilde{\pi}_1^+(\theta)}{2}\Big] \text{ for each } \theta \in \Theta.$$

In Fig.1, real line shows $\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta)$ and dashed lines show $\tilde{\pi}_0^-(\theta)$ and $\tilde{\pi}_0^+(\theta)$, respectively.

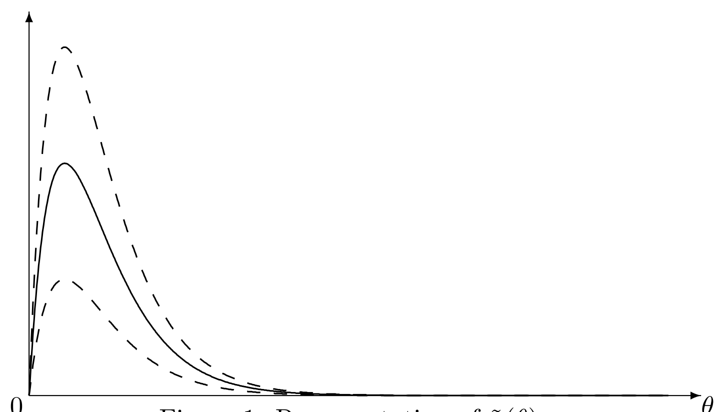

Figure 1: Representation of $\tilde{\pi}(\theta)$

Also, $f(x|\theta)$ is the density of the exponential distribution, i.e.

$$f(x|\theta) = \theta e^{-\theta x} \qquad x \geq 0, \quad x \in \mathbb{R}.$$

We can obtain $F_X(x|\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta)) = 1 - \frac{16}{(x+4)^2}$ from (3) and

$$\widetilde{V@R}^{+}_{\gamma,1}(X|\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta)) = \widetilde{V@R}^{-}_{\gamma,1}(X|\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta)) = \frac{4}{\sqrt{1-\gamma}} - 4,$$
$$\widetilde{CV@R}^{+}_{\gamma,1}(X|\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta)) = \widetilde{CV@R}^{-}_{\gamma,1}(X|\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta)) = \frac{8\sqrt{1-\gamma}}{1-\gamma} - 4$$

from the definition of non-fuzzy $V@R_\gamma$ and $CV@R_\gamma$. From (1), we can get the following for each α.

$$\tilde{F}^+_{X,\alpha}(y|\tilde{\pi}) = \begin{cases} 1 - \frac{8(1+\alpha)}{(4+y)^2} & \text{if } y > 4\sqrt{2} - 4, \\ (3-\alpha)(\frac{1}{2} - \frac{8}{(4+y)^2}) & \text{else,} \end{cases}$$
$$\tilde{F}^-_{X,\alpha}(y|\tilde{\pi}) = \begin{cases} (1+\alpha)(\frac{1}{2} - \frac{8}{(4+y)^2}) & \text{if } 0 \leq y < 4\sqrt{2} - 4, \\ 1 - \frac{8(3-\alpha)}{(4+y)^2} & \text{else.} \end{cases} \tag{8}$$

Here, let $\gamma = 0.99$, then we have $\widetilde{V@R}^{+}_{0.99,\alpha}(X|\tilde{\pi}) = \frac{\sqrt{8(3-\alpha)}}{\sqrt{1-0.99}} - 4$, $\widetilde{V@R}^{-}_{0.99,\alpha}(X|\tilde{\pi}) = \frac{\sqrt{8(\alpha+1)}}{\sqrt{1-0.99}} - 4$, $\widetilde{CV@R}^{+}_{0.99,\alpha}(X|\tilde{\pi}) = \frac{4\sqrt{2(3-\alpha)(1-0.99)}}{1-0.99} - 4$ and $\widetilde{CV@R}^{-}_{0.99,\alpha}(X|\tilde{\pi}) = \frac{4\sqrt{2(1+\alpha)(1-0.99)}}{1-0.99} - 4$ from Lemma 2. Therefore we can estimate $\widetilde{V@R}_\gamma$ and $\widetilde{CV@R}_\gamma$ from Lemma 1. Fig.2 and 3 show $\widetilde{V@R}_{0.99}$ and $\widetilde{CV@R}_{0.99}$, respectively.

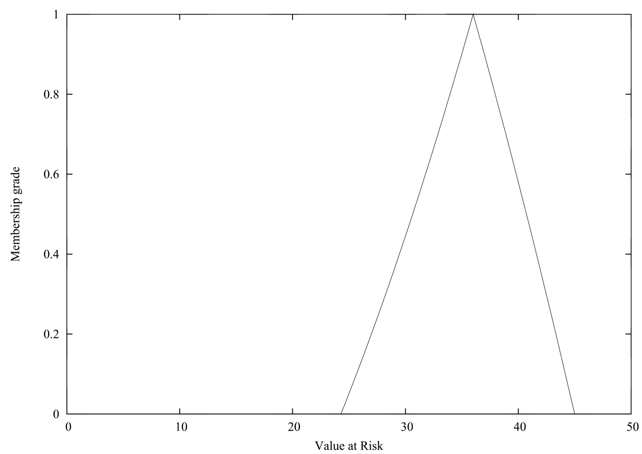

Figure 2: Representation of $\widetilde{V@R}_{0.99}$

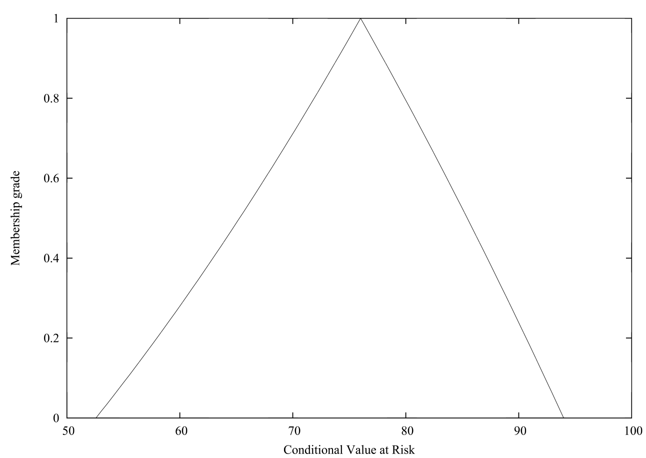

Figure 3: Representation of $\widetilde{CV@R}_{0.99}$

5 Conclusion

In managing complex environmental systems, there are two types of uncertainties. One is a probabilistic uncertainty which has been considered by a lot of authors [3, 4, 5, 6, 9] in order to represent risk under uncertainty. However the another uncertainty, fuzziness is also very important concept. In this paper we proposed fuzzy risk measures using non-precise a-priori densities. Owing to this approach, we can design more flexible mathematical models. In the future work, referring to [6], we will construct fuzzy risk models in Markov decision processes.

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