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Uncertainty Linear Regression Models

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Abstract

Statistical linear regression played important roles in statistical relation modeling and inferential statistics on probability distribution theoretical foundation. However, the uncertainty phenomena in real world including human psychological behavior do have many different forms. The random uncertainty addressed by probability theory is merely one of those phenomena and is not able to handle other forms of uncertainties. In this paper, we review an axiomatic uncertain measure theory to deal with an uncertainty different from the randomness with a unique feature in uncertainty distributions. Furthermore, we propose uncertainty copula linked multivariate uncertainty distributional theory for developing an uncertainty distributional structure for uncertainty linear regression models. Under uncertainty FGM-Normal Assumptions, an uncertainty Gauss-Markov theorem is proved and thus the weighted least squares (WLS) estimator for model parameters is derived. Finally, when the FGM copula parameter $\varpi = 0$, i.e., the OLS regression modeling and distributional theory is fully developed, but when $\varpi \neq 0$, the WLS modeling and distributional theory is partially explored.

Keywords: uncertain measure, uncertainty variable, uncertainty copula, FGM-Normal multivariate uncertainty distribution, uncertainty linear regression model, WLS estimation, BLUE

1 Introduction

Uncertainty phenomena in the real world, including human psychological behavior, do have many different forms. The random uncertainty addressed by probability theory [8, 23, 24, 28] is merely one of the forms and is not able to handle other forms of uncertainties. As a new mathematical branch, Liu's uncertainty theory includes a new uncertainty calculus, uncertainty processes, uncertainty logic and inference. It has gained wide applications already, for example, uncertainty programming in system reliability, equipment allocation, and uncertainty finance, etc., see [10-17, 21]. The current research focus in uncertainty theory is the field of uncertainty statistics.

Linear regression models [3, 4, 9, 29] are the most frequently used statistical models and thus there are many commercial software packages with linear regression modeling options. It is also well-known that in probability theory the normal random variable and its properties play a fundamental role and support the linear regression modeling.

The notion of measure defines an event measuring grade system within a conceptual uncertainty environment. A measure system's establishment is not a copy of a real world phenomenon, however, it is some abstraction of a real world phenomenon to some degree and it suggests a specific partial reflection of reality. Without measure specification, there is no scientific foundation to discuss any particular form of uncertainty, say, randomness, or vagueness, or general form of uncertainty. Thus measure specification is a prerequisite for exploring any form of uncertainty with mathematical rigor.

Classical measure and probability measure belong to the class of "completely additive measures", i.e., the measure of union of disjoint events is just the sum of the measures of the individual events. These measures may be too restrictive when covering real life applications. In contrast to probability measure, capacity, belief measure, plausibility measure, possibility measure and necessity measure belong to the class of "completely non-additive

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measures". Since this class of measures does not assume the self-duality property, all of its measures are inconsistent with the law of contradiction and law of excluded middle, which dominate human thinking logic.

Zadeh [36, 37] and his fuzzy mathematics School built on the foundation of set membership assumptions. Zadeh broke with traditional deterministic as well as probabilistic thinking and gained great success in industrial applications. The critical issue in Zadeh's theory is if any phenomenon could be specified by a membership function, it can be claimed to be a fuzzy phenomenon. Then a series of contradictions arise, namely that Zadeh's school can claim rough sets, grey sets, and even random sets are all special cases of the fuzzy sets because all of them have a membership function!

The uncertain measure proposed by Liu [15] is neither a completely additive measure nor a completely non-additive measure. In fact, an uncertain measure is a "partially additive measure" because of its self-duality. Credibility measure and chance measure are special types of an uncertain measure. One advantage of this class of measures is the solid consistency with the law of contradiction and law of excluded middle. Therefore it is of extreme importance to establish a linear inference theory from an uncertain measure foundation.

Ancient Chinese educationists following Confucius claimed that the best way to learn new knowledge is by reviewing its old one forms. Liu's [15] axiomatic uncertain measure doctrine is a new theory, particularly the Axiom 3, whose σ -sub-additivitivity brings many fundamental features into the uncertainty distributional theory [23]. Therefore, in order to avoid unconsciously bringing the probabilistic thinking behavior into the uncertainty theory, and instead to sharply recognize some new features of Liu's uncertainty theory, we will note salient comparisons between the uncertainty theory and probabilistic counterpart throughout the remaining sections.

The structure of the remaining sections is stated as follows. Section 2 will be used to review Liu's axiomatic uncertain measure and uncertain variable theory. Section 3 covers the discussion on the uncertainty multivariate distributions. We propose an approach of utilizing uncertainty copula-linked given marginal uncertainty distributions, as a mechanism to define uncertainty multivariate distributions and provide the necessary platform for linear regression modeling. In Section 4, we discuss an uncertainty Gauss-Markov Theorem under uncertainty FGM-Normal multivariate distribution assumptions. Hence uncertainty WLS regression modeling and distributional theory are developed. Section 5 concludes the paper.

2 Uncertainty Variables and Distributions

As a prerequisite to investigate a general form of uncertain events, an uncertain measure must be specified. Liu's uncertain measure is an axiomatically defined set function mapping from a σ -algebra of a given space (set) to the unit interval [0,1], which provides a measuring grade system of an uncertain phenomenon and facilitates the formal definition of an uncertain variable.

Let Ξ be a nonempty set (space), and $\mathfrak{A}(\Xi)$ the σ -algebra on Ξ. Each set, say $A\subset\Xi$, $A\in\mathfrak{A}(\Xi)$, is called an uncertain event. A number, denoted by $\lambda\{A\}$, $0\le\lambda\{A\}\le 1$, is assigned to event $A\in\mathfrak{A}(\Xi)$, which indicates the uncertain measuring grade with which event $A\in\mathfrak{A}(\Xi)$ occurs. The normal set function $\lambda\{A\}$ satisfies following axioms given by Liu [17]:

Axiom 1: (Normality) $\lambda\{\Xi\}=1$.

Axiom 2: (Self-Duality) $\lambda\{\cdot\}$ is self-dual, i.e., for any $A\in\mathfrak{A}(\Xi), \lambda\{A\}+\lambda\{A^c\}=1$.

Axiom 3: (σ- Subadditivity) $\lambda\left\{\bigcup_{i=1}^{\infty}A_i\right\}\le\sum_{i=1}^{\infty}\lambda\{A_i\}$ for any countable event sequence $\{A_i\}$.

Axiom 4: (Product Measure Axiom) Let Ξ_k be nonempty sets on which the uncertainty measures $\lambda_k\{\cdot\}$ are defined, $k=1,2,\cdots,d$, respectively. Then the product measure $\lambda\{\cdot\}$ on the product σ -algebra $\mathfrak{A}(\Xi)$, where $\Xi=\Xi_1\times\Xi_2\times\cdots\times\Xi_d$, i.e., $\mathfrak{A}(\Xi)=\mathfrak{A}(\Xi_1)\times\mathfrak{A}(\Xi_2)\times\cdots\times\mathfrak{A}(\Xi_d)$, is an uncertain measure. In other words, for any measurable rectangle $\Lambda=\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_d$, where $\Lambda_k\in\mathfrak{A}(\Xi_k)$, $k=1,2,\cdots,d$,

$$\lambda\{\Lambda\}=\min_{1\le k\le d}\lambda_k\{\Lambda_k\} \tag{1}$$

i.e., for each event $\Lambda\in\mathfrak{A}(\Xi)$

$$\lambda\{\Lambda\}=\begin{cases}\sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_d\subset\Lambda}\min\limits_{1\le k\le d}\lambda_k\{\Lambda_k\} & \text{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_d\subset\Lambda}\min\limits_{1\le k\le d}\lambda_k\{\Lambda_k\}>0.5\\ 1-\sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_d\subset\Lambda^c}\min\limits_{1\le k\le d}\lambda_k\{\Lambda_k\} & \text{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_d\subset\Lambda^c}\min\limits_{1\le k\le d}\lambda_k\{\Lambda_k\}>0.5\\ 0.5 & \text{otherwise}\end{cases} \tag{2}$$

Definition 2.1 (Liu [10], [15], [17]) A set function $\lambda:\mathfrak{A}(\Xi)\to[0,1]$ which satisfies *Axioms 1-3* is called an uncertain measure. The triple $(\Xi,\mathfrak{A}(\Xi),\lambda)$ is called an uncertain measure space.

Definition 2.2 An uncertainty variable ξ is a measurable mapping, i.e., $\xi:(\Xi,\mathfrak{A}(\Xi))\to(\mathbb{R},\mathfrak{B}(\mathbb{R}))$, where $\mathfrak{B}(\mathbb{R})$ denotes the Borel σ -algebra on $\mathbb{R}=(-\infty,+\infty)$.

For further discussions via comparison, let $(\Omega,\mathfrak{F},P)$ be a probability space and $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$ be a measurable space on real-line, then a real-valued function X is random variable if and only if for all $r\in\mathbb{R}$, the pre-image $\{\omega\in\Omega:X(\omega)\le r\}\in\mathfrak{F}$. For each value $r\in\mathbb{R}$ taken by a real-valued random variable X, the event $B=(-\infty,r]$ is an element of the Borel σ- algebra of real-line $\mathbb{R}$, the pre-image of event B under random variable X is

$$\{\omega\in\Omega:X(\omega)\in B\}=\{\omega\in\Omega:X(\omega)\le r\}, \tag{3}$$

event $\{\omega\in\Omega:X(\omega)\le r\}$ is an element of σ- algebra $\mathfrak{F}$ on Ω, where the probability measure P defined on this set class, i.e., σ- algebra $\mathfrak{F}$, i.e., $P:\mathfrak{F}\to[0,1]$.

Therefore every element (event) of $\mathfrak{F}$ is assigned with a probability grade, i.e., event $\{\omega\in\Omega:X(\omega)\le r\}$ is assigned a probability grade, which is $P\{\omega\in\Omega:X(\omega)\le r\}$. Overall, σ- algebra $\mathfrak{F}$ allows the formal definition of a random variable in terms of membership of the pre-image $\{\omega\in\Omega:X(\omega)\le r\}$ to the σ- algebra $\mathfrak{F}$, in which the probability measuring grade is defined and every event of σ- algebra $\mathfrak{F}$ is assigned. In probability theory each random variable on the probability space $(\Omega,\mathfrak{F},P)$ induces a probability space $(\mathbb{R},\mathfrak{B}(\mathbb{R}),\mu)$ by means of the following correspondence:

$$\forall B\in\mathfrak{B}(\mathbb{R}):\ \ \mu(B)=P\{X^{-1}(B)\}=P\{X\in B\}. \tag{4}$$

Let us denote $\mu=P\circ X^{-1}$ and specifically the distribution

$$F(r)=\mu\{(-\infty,r]\}=P\{X\le r\}. \tag{5}$$

Similarly, in the uncertainty variable definition, the measurable mapping is characterized by the membership of the pre-image of event (a Borel set) $B=(-\infty,r]$ under the uncertainty variable ξ, to the σ -algebra $\mathfrak{A}(\Xi)$. In other words,

$$\forall B\in\mathfrak{B}(\mathbb{R}),\{\tau\in\Xi:\xi\in B\}\in\mathfrak{A}(\Xi). \tag{6}$$

The measurability of uncertainty variable ξ then induces a measure on the measurable space $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$. Let us denote the induced measure as ν. Similarly to the probabilistic case in Eq. (2), for $\forall B\in\mathfrak{B}(\mathbb{R})$, the induced measure is

$$\nu\{B\}=\lambda\{\tau\in\Xi:\xi\in B\}=\lambda\{\tau\in\Xi:\xi(\tau)\le r\}. \tag{7}$$

Therefore, let $\nu=\lambda\circ\xi^{-1}$ and specifically, the distribution is defined by the induced measure

$$\Psi(r)=\nu\{(-\infty,r]\}=\lambda\{\tau\in\Xi:\xi(\tau)\le r\}. \tag{8}$$

The induced distribution by uncertainty variable ξ is just the uncertainty distribution, denoted by $\Psi(\cdot)$, which characterizes the measurement of the uncertainty associated with every event with the uncertainty variable ξ.

Definition 2.3 (Liu [10, 15, 17]) The uncertainty distribution $\Psi:\mathbb{R}\to[0,1]$ of an uncertainty variable ξ on $(\Xi,\mathfrak{A}(\Xi),\lambda)$ is

$$\Psi(x)=\lambda\{\tau\in\Xi\,|\,\xi(\tau)\le x\}. \tag{9}$$

For the uncertain measure, as an axiomatic measure development, the set class σ-algebra $\mathfrak{A}(\Xi)$ plays critical roles in defining set function - uncertain measure λ as well as roles in defining the measurability of uncertainty variable.

Remark 2.4 The "density function" of an uncertainty distribution may be "routinely" defined by the derivative of an uncertainty distribution. As usual, we may state that a nonnegative function ψ: $\mathbb{R}\to\mathbb{R}^+$ satisfying

$$\Psi(x)=\int_{-\infty}^{-\infty}\psi(s)ds,\ \lim_{x\to-\infty}\Psi(x)=0, \lim_{x\to+\infty}\Psi(x)=1.0 \tag{10}$$

is called an uncertainty density function. Nevertheless, it should be stressed here that given an uncertainty "density" $\psi(x)\ge 0$, an integral $\int_A\psi(s)ds$ qualifies as a description of an uncertainty distribution function if and only if event $A=\{\xi\le x\}$ or $A^c=\{\xi>x\}$. Fundamentally, an uncertainty distribution is characterized by an uncertainty measure $\lambda\{\xi\le x\}$, which is σ- subadditive, while an integral $\int_A\psi(s)ds$ is σ- additive for any set sequence $\cup A_n$, as long as $A_i\cap A_j=\varnothing,\ i\ne j$, i.e., $\int_{\cup A_n}\psi(s)ds=\sum\int_{A_n}\psi(s)ds$. The σ- additivity property of integration can serve to describe the cumulative distribution function perfectly in probability theory, because of the compatibility. But Liu's [10, 15, 17] uncertainty distribution is different, being defined by σ- subadditive uncertain measure. The σ- subadditivity implies that $\int_A\psi(s)ds$ is undefined. If A is an event, say $A=[a,b],\ a\le b,\ a,b$ finite, both the interpretation of the integral $\int_a^b\psi(s)ds$ and the determination of the value of the integral $\int_a^b\psi(s)ds$ are questionable and doubtful. Therefore in describing an uncertainty distribution the derivative concept may or may not make sense. It is reasonable to avoid the term "density" and from now on in our work we will no longer engage the notion of an "uncertainty density".

Definition 2.5 (Liu [10, 15, 17]) Let ξ be an uncertainty variable defined on the uncertain space $(\Xi,\mathfrak{A}(\Xi),\lambda)$, then the expectation of ξ is

$$\mathrm{E}[\xi]=\int_0^{+\infty}\lambda\{\xi\ge r\}dr-\int_{-\infty}^{0}\lambda\{\xi\le r\}dr \tag{11}$$

provided that at least one of the two integrals is finite.

Definition 2.6 (Liu [10, 15, 17]) Let ξ be an uncertainty variable with finite expectation $\mathrm{E}[\xi]$, then the variance of ξ is

$$\mathrm{V}[\xi]=\mathrm{E}\left[(\xi-\mathrm{E}[\xi])^2\right]. \tag{12}$$

Definition 2.7 (Liu [15]) Let ξ be an uncertainty variable with finite expectation $\mathrm{E}[\xi]$, then for any given positive integer $k>0$, (1) $\mathrm{E}[\xi^k]$ is called the k^{th} moment of an uncertainty ξ variable; (2) $\mathrm{E}\left[|\xi|^k\right]$ is called the k^{th} absolute moment of an uncertainty variable ξ; (3) $\mathrm{E}\left[(\xi-\mathrm{E}[\xi])^k\right]$ is called the k^{th} central moment of an uncertainty variable ξ; (4) $\mathrm{E}\left[|\xi-\mathrm{E}[\xi]|^k\right]$ is called the k^{th} central moment of an uncertainty variable ξ if these moments are finite.

Theorem 2.8 Let ξ be an uncertainty variable on uncertain measure space $(\Xi, \mathfrak{A}(\Xi), \lambda)$ and h be a monotonic non-decreasing function $h : \mathbb{R} \to \mathbb{R}^+$, then the expectation of $h(\xi)$ is

$$\mathrm{E}\left[h(\xi)\right] = \int_0^{+\infty} h(r)\lambda\{\xi \ge r\}\,dr - \int_{-\infty}^{0} h(r)\lambda\{\xi \le r\}\,dr \,. \tag{13}$$

3 The Uncertainty Multivariate Distributions

In classical statistics, a linear regression model reveals relationship between dependent or response variable y and at least one explanatory variable X. Relaxing the conditioning allows one to consider the bivariate distribution of (*X*,*Y*), and extensions to multivariate relationships. Analogously, the need arises for multivariate uncertain measure and uncertainty multivariate distributions.

Definition 3.1 (Liu [10, 15, 17]) Let multivariate uncertainty variable $(\xi_1, \xi_2, \cdots, \xi_d)$ be defined on an uncertain measure space $(\Xi, \mathfrak{A}(\Xi), \lambda)$, then the multivariate function $\Psi_{\xi_1,\xi_2,\cdots,\xi_d} : D \to [0,1]$ is called an uncertainty multivariate distribution if

$$\Psi_{\xi_1,\xi_2,\cdots,\xi_d}(x_1, x_2, \cdots, x_d) = \lambda\{\xi_1 \le x_1, \xi_2 \le x_2, \cdots, \xi_d \le x_d\} \,. \tag{14}$$

Up to now, there has been no concrete class of uncertainty multivariate distributions being proposed and investigated due to the complexity in this matter. A prerequisite of developing uncertainty statistical modeling is the specification of multivariate uncertainty distributions or "equivalent" multivariate joint measures capable of revealing the uncertainty relationship between two uncertainty variables or among a group of uncertainty variables.

3.1 Uncertainty Copulas

The fundamental starting point in addressing uncertainty multivariate distributions is the multivariate joint uncertain measure in Definition 3.1. The uncertainty multivariate joint distribution may be approached from several directions. One of these directions is from the uncertainty copula.

In probability theory, the copula is an important function for a constructing multivariate distribution, see [5, 20]. Now, we propose to develop an uncertainty copula theory. For the simplification of later discussions, it is assumed that for multivariate uncertainty variable $(\xi_1, \xi_2, \cdots, \xi_d)$ with joint uncertainty distribution, all the marginal distributions $\Psi_{\xi_1}(\cdot), \Psi_{\xi_2}(\cdot), \cdots, \Psi_{\xi_d}(\cdot)$ exist, are available, and are regular (i.e., $\Psi_{\xi_i}^{-1}(\cdot)$ exists, $i = 1, 2, \cdots, d$). To prepare for uncertainty copula theory, we state and prove some results.

Theorem 3.2 Let ξ be a regular uncertainty variable with an uncertainty distribution $\Psi_{\xi}(\cdot)$, then $\Psi_{\xi}(\xi)$ is an uncertainty linear variable on $[0,1]$.

Proof: Since $\Psi_{\xi}(\cdot)$ is an uncertainty distribution function , $0 \le \Psi_{\xi}(x) \le 1$. Further, $\Psi_{\xi}(\cdot)$ is regular, it is strictly monotone increasing such that $\Psi_{\xi}^{-1}(\alpha)$ exists. In terms of Liu's [15] Theorem 1.17 operational law, $\Psi_{\xi}(\xi)$ is an uncertainty variable with an inverse uncertainty distribution

$$\Psi^{-1}_{\Psi_{\xi}(\xi)}(\alpha) = \Psi_{\xi}\left(\Psi_{\xi}^{-1}(\alpha)\right) = \alpha, \ \alpha \in [0,1] \tag{15}$$

which implies that the uncertainty distribution of $\Psi_{\xi}(\xi)$

$$\Psi_{\Psi_{\xi}}(x) = \Psi_{\xi}\left(\Psi_{\xi}^{-1}(x)\right) = x, \ x \in [0,1]. \tag{16}$$

The proof concludes here.

Definition 3.3 Let $(\xi_1, \xi_2, \cdots, \xi_d)$ be a multivariate uncertainty variable with joint uncertainty distribution $\Psi_{\xi_1,\xi_2,\cdots,\xi_d}(x_1, x_2, \cdots, x_d)$, in which all the marginal distributions $\Psi_{\xi_1}(\cdot), \Psi_{\xi_2}(\cdot), \cdots, \Psi_{\xi_d}(\cdot)$ exist and are regular (i.e., $\Psi_{\xi_i}^{-1}(\cdot)$ exists, $i = 1, 2, \cdots, d$). Then the multivariate uncertainty copula is defined by

$$C\left(u_1,u_2,\cdots,u_d\right)=\Psi_{\xi_1,\xi_2,\cdots,\xi_d}\left(\Psi_{\xi_1}^{-1}\left(u_1\right),\Psi_{\xi_2}^{-1}\left(u_2\right),\cdots,\Psi_{\xi_d}^{-1}\left(u_d\right)\right). \tag{17}$$

Theorem 3.4 Given an uncertainty copula $C\left(u_1,u_2,\cdots,u_d\right)$, $\left(u_1,u_2,\cdots,u_d\right)'\in\left[0,1\right]^d$, then

(1) As an uncertainty distribution, $C:\left[0,1\right]^d\to\left[0,1\right]$ is non-decreasing with respect to every component u_k, $k=1,2,\cdots,d$;

(2) For $\forall\left(u_1,u_2,\cdots,0_k,\cdots,u_d\right)'\in\left[0,1\right]^d$, $C\left(u_1,u_2,\cdots,0_k,\cdots,u_d\right)=0$, $k=1,2,\cdots,d$;

(3) For $\forall\left(1,1,\cdots,u_k,\cdots,1\right)'\in\left[0,1\right]^d$, $C\left(1,1,\cdots,u_k,\cdots,1\right)=u_k$, $k=1,2,\cdots,d$.

Remark 3.5 Definition 3.3 and Theorem 3.4 together permit a necessary condition [22] for a multivariate function $C:\left[0,1\right]^d\to\left[0,1\right]$ to be a copula which links the elements of $\left(\xi_1,\xi_2,\cdots,\xi_d\right)$ and defines a joint uncertainty distribution $\Psi_{\xi_1,\xi_2,\cdots,\xi_d}\left(x_1,x_2,\cdots,x_d\right)$. To gain further insight, let us start with a few bivariate uncertainty copulas.

Definition 3.6 Let a bivariate uncertainty variable $\left(\xi_1,\xi_2\right)$ have uncertainty marginal distributions $\Psi_{\xi_1}\left(\cdot\right)$ and $\Psi_{\xi_2}\left(\cdot\right)$ respectively. The Farlie-Gumbel-Morgenstern (FGM) copula is defined by

$$C_{\varpi}^{FGM}\left(u_1,u_2\right)=u_1u_2\left(1+\varpi\left(1-u_1\right)\left(1-u_2\right)\right),\ \varpi\in\left[-1,1\right]. \tag{18}$$

Theorem 3.7 Let a bivariate uncertainty variable $\left(\xi_1,\xi_2\right)$ have uncertainty marginal distributions $\Psi_{\xi_1}\left(\cdot\right)$ and $\Psi_{\xi_2}\left(\cdot\right)$ respectively, where

$$\Psi_{\xi_i}\left(x_i\right)=\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}\left(x_i-\theta_i\right)\right)},\ i=1,2. \tag{19}$$

Then the FGM bivariate uncertainty normal joint distribution (uncertainty bivariate FGM-Normal distribution) is

$$\Psi_{\xi_1,\xi_2}\left(x_1,x_2\right)=\prod_{i=1}^{2}\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}\left(x_i-\theta_i\right)\right)}\left(1+\varpi\prod_{i=1}^{2}\frac{\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}\left(x_i-\theta_i\right)\right)}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}\left(x_i-\theta_i\right)\right)}\right). \tag{20}$$

Proof: Notice that

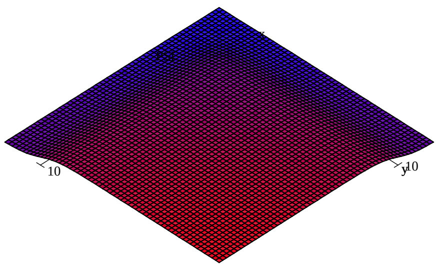
$$u_i=\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}\left(x_i-\theta_i\right)\right)},\ i=1,2, \tag{21}$$

then

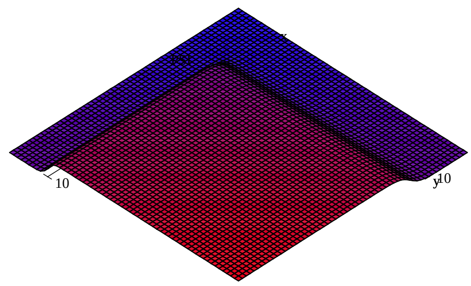
$$\begin{aligned}
&\Psi_{\xi_1,\xi_2}\left(x_1,x_2\right)\\
&=C_{\varpi}^{FGM}\left(\Psi_1\left(x_1\right),\Psi_2\left(x_2\right)\right)\\
&=\Psi_1\left(x_1\right)\Psi_2\left(x_2\right)\left(1+\varpi\left(1-\Psi_1\left(x_1\right)\right)\left(1-\Psi_2\left(x_2\right)\right)\right)\\
&=\prod_{i=1}^{2}\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}\left(x_i-\theta_i\right)\right)}\left(1+\varpi\prod_{i=1}^{2}\frac{\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}\left(x_i-\theta_i\right)\right)}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}\left(x_i-\theta_i\right)\right)}\right)
\end{aligned} \tag{22}$$

which concludes the proof.

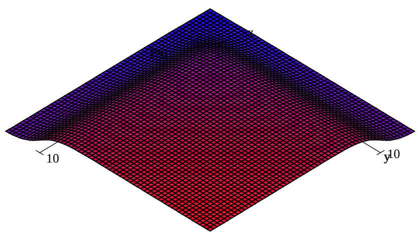
Remark 3.8 The probabilistic FGM copula is an extended version of Gumbel copula, see [1, 2, 19]. We take the functional forms into an uncertainty distribution**.** It is necessary to emphasize that given the functional triple $\left(\Psi_{\xi_1}, \Psi_{\xi_2}, C\right)$, as long as it satisfies the necessary conditions, the triple $\left(\Psi_{\xi_1}, \Psi_{\xi_2}, C\right)$ specifies a bivariate joint uncertainty measure ƛ$\left\{A_1, A_2\right\}$, which can be represented by ƛ$\left\{\xi_1 \le x_1, \xi_2 \le x_2\right\}$ in the sense of an equivalence class. Furthermore, every functional triple $\left(\Psi_{\xi_1}, \Psi_{\xi_2}, C\right)$ defines a family of uncertainty bivariate joint distributions $\left\{\Psi_{\xi_1,\xi_2}\left(x_1, x_2; \underline{\gamma}\right)\right\}, \underline{\gamma} \in \Gamma \subseteq \mathbb{R}^h$, where h is the dimensionality of parameter vector including any parameters specifying marginals and copula. For example, the parameter vector of the FGM bivariate uncertainty normal joint distribution family is $\left(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi\right)$,i.e., $h = 5$, $\Gamma = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times [-1,1]$. Figure 1 shows the plots of uncertainty bivariate FGM-Normal distributions specified by various parameter combinations.



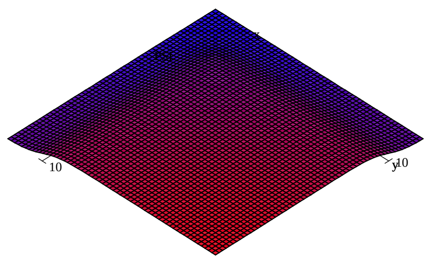
$\left(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi\right) = \left(0,1,0,1,0.25\right)$



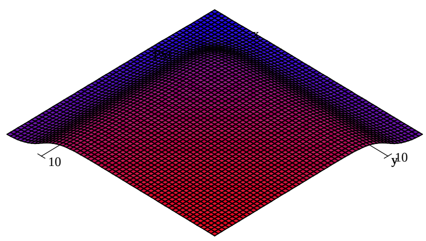
$\left(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi\right) = \left(1, 0.5, 0, 0.25, 0.5\right)$



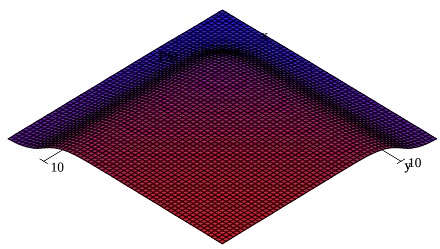
$\left(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi\right) = \left(0,1,0,1,1\right)$



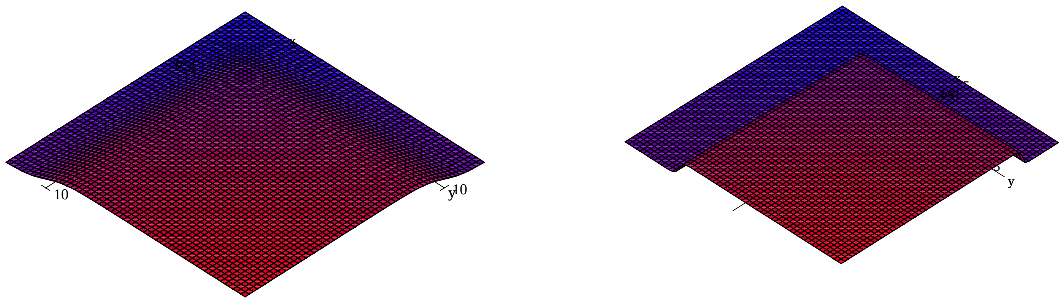
$\left(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi\right) = \left(0,1,0,1,-0.25\right)$



$\left(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi\right) = \left(0,1,0,1,-0.5\right)$



$\left(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi\right) = \left(0,1,0,1,-1\right)$



$(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi) = (0,1,0,1,0)$ $\quad$ $(\theta_1, \sigma_1, \theta_2, \sigma_2, \varpi) = (-1, 0.1, -3, 0.2, 0.75)$

Figure 1: Plots of uncertainty bivariate FGM-Normal distributions

Lemma 3.9 The uncertainty bivariate Farlie-Gumbel-Morgenstern (FGM) copula is symmetric.

Proof: Because the components of the uncertainty FGM copula, u_1 and u_2 are interchangeable, i.e., $C(u_1,u_2) = C(u_2,u_1)$, $C(\cdot,\cdot)$ is thus symmetric.

Definition 3.10 A Type I uncertainty extended FGM (EFGM) copula is an extension to anuncertainty FGM copula, which is

$$C(u_1,u_2) = u_1u_2\left(1+\varpi\Lambda(u_1)\Lambda(u_2)\right),\ \varpi \in [-1,1] \tag{23}$$

where $\Lambda:[0,1]\to[0,1]$ satisfyies Condition (1) $\Lambda(0)=\Lambda(1)=0$; Condition (2) for $\forall(u_1,u_2)\in\mathbb{I}^2=[0,1]\times[0,1]$, $\left|\Lambda(u_1)-\Lambda(u_2)\right| \le |u_1-u_2|$ (Lipschitz condition).

Example 3.11 Assuming that (a) $\Lambda(x) = x\wedge(1-x)$ (b) $\Lambda(x) = x(1-x)$, (c) $\Lambda(x) = \sin(\pi x)/\pi$, three extended FGM families are defined.

Definition 3.12 Let the bivariate uncertainty variable (ξ_1,ξ_2) have uncertainty marginal distributions $\Psi_{\xi_1}(\cdot)$ and $\Psi_{\xi_2}(\cdot)$ respectively. Then an EFGM bivariate uncertainty joint distribution is

$$C_{\varpi,\Lambda}^{EFGM}(u_1,u_2) = \prod_{i=1}^{2} u_i \left(1+\varpi\prod_{i=1}^{2}\Lambda(u_i)\right). \tag{24}$$

Definition 3.13 Let the bivariate uncertainty variable (ξ_1,ξ_2) have uncertainty marginal distributions $\Psi_{\xi_1}(\cdot)$ and $\Psi_{\xi_2}(\cdot)$ respectively, where

$$\Psi_{\xi_i}(x_i) = \frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}(x_i-\theta_i)\right)},\ i=1,2\,. \tag{25}$$

Then a Type I EFGM bivariate uncertainty normal joint distribution (EFGM-Normal) is given by

$$\Psi_{\xi_1,\xi_2}(x_1,x_2) = \prod_{i=1}^{2}\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}(x_i-\theta_i)\right)}\left(1+\varpi\prod_{i=1}^{2}\Lambda\left(\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_i}(x_i-\theta_i)\right)}\right)\right). \tag{26}$$

Remark 3.14 Similarly, a non-symmetric copula can be defined. In uncertainty financial engineering and risk analysis, non-symmetric copulas can play a vital role. For example, a Frank copula,

$$C_{\varpi}\left(u_1,u_2\right)=\begin{cases}-\frac{1}{\varpi}\ln\left(1+\frac{1}{e^{-\varpi}-1}\prod_{i=1}^{2}\left(e^{-\varpi u_i}-1\right)\right) & \varpi\in\mathbb{R}\setminus\{0\}\\ u_1u_2 & \varpi=0.\end{cases} \tag{27}$$

Also, a Gumbel copula

$$C_{\varpi}\left(u_1,u_2\right)=\exp\left(-\left(\sum_{i=1}^{2}\left(-\ln\left(u_i\right)\right)^{\varpi}\right)^{\frac{1}{\varpi}}\right),\varpi\in[1,+\infty). \tag{28}$$

Definition 3.15 An d -dimensional FGM uncertainty copula is defined by

$$C_{\underline{\varpi}}^{FGM}\left(u_1,u_2,\cdots,u_d\right)=\prod_{i=1}^{d}u_i\left(1+\sum_{k=2}^{d}\sum_{1\le j_1<j_2<\cdots<j_k}\varpi_{j_1j_2,\cdots,j_k}\prod_{i=1}^{k}\left(1-u_{j_i}\right)\right),\quad \underline{\varpi}\in[-1,1]^d. \tag{29}$$

Definition 3.16 Assuming that d marginal distributions $\Psi_{\xi_1}(\cdot),\Psi_{\xi_2}(\cdot),\cdots,\Psi_{\xi_d}(\cdot)$ are given and regular, the d -dimensional multivariate FGM uncertainty joint distribution $\Psi_{\xi_1,\xi_2,\cdots,\xi_d}\left(x_1,x_2,\cdots,x_d\right)$ can be defined by

$$\begin{aligned}\Psi_{\xi_1,\xi_2,\cdots,\xi_d}\left(x_1,x_2,\cdots,x_d\right)&=C_{\underline{\varpi}}^{FGM}\left(\Psi_{\xi_1}\left(x_1\right),\Psi_{\xi_2}\left(x_2\right),\cdots,\Psi_{\xi_d}\left(x_d\right)\right)\\&=\prod_{i=1}^{d}\Psi_{\xi_i}\left(x_i\right)\left(1+\sum_{k=2}^{d}\sum_{1\le j_1<j_2<\cdots<j_k}\varpi_{j_1j_2,\cdots,j_k}\prod_{i=1}^{k}\left(1-\Psi_{\xi_{j_i}}\left(x_{j_i}\right)\right)\right),\quad \underline{\varpi}\in[-1,1]^d.\end{aligned} \tag{30}$$

Remark 3.17 A d -dimensional uncertainty FGM-*T* multivariate distribution can be defined by

$$\Psi_{\xi_1,\xi_2,\cdots,\xi_d}\left(x_1,x_2,\cdots,x_d\right)=\prod_{i=1}^{d}\Psi_{\xi_i}\left(x_i\right)\left(1+\sum_{k=2}^{d}\sum_{1\le j_1<j_2<\cdots<j_k}\varpi_{j_1j_2,\cdots,j_k}\prod_{i=1}^{k}\left(1-\Psi_{\xi_{j_i}}\left(x_{j_i}\right)\right)\right),\underline{\varpi}\in[-1,1]^d \tag{31}$$

where the marginals are *T* uncertainty distributions:

$$\Psi_{\xi_i}\left(x_i\right)=\sup_{x_1/x_2=t}\left(\min\left(\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}}x_1\right)},\frac{1}{1+\exp\left(-\frac{\pi}{n\sqrt{3}}\frac{1}{x_2}\right)}\right)\right),\ i=1,2,\cdots,d. \tag{32}$$

3.2 Uncertainty Linear Correlation

In classical regression modeling, linear correlation plays an important role. The definition of a measure of linear relationship between the two random variables, X_1 and X_2, is

$$\operatorname{corr}\left(X_1,X_2\right)=\frac{\mathrm{E}\left[\left(X_1-\mathrm{E}\left[X_1\right]\right)\left(X_2-\mathrm{E}\left[X_2\right]\right)\right]}{\sqrt{V\left[X_1\right]V\left[X_2\right]}}, \tag{33}$$

where $-1\le\operatorname{corr}\left(X_1,X_1\right)\le 1$. When $\operatorname{corr}\left(X_1,X_1\right)=0$, we say X_1 and X_2 are uncorrelated.

In uncertainty statistics, inevitably we need an uncertainty linear correlation construct. Let $\left(\xi_1,\xi_2\right)$ be a bivariate uncertainty variable with FGM-Normal bivariate distribution. We define an uncertainty variable

$$\varsigma=\frac{\left(\xi_1-\theta_1\right)\left(\xi_2-\theta_2\right)}{\sigma_1\sigma_2} \tag{34}$$

and investigate its uncertainty distribution.

Theorem 3.18 Let $\left(\xi_1,\xi_1\right)$ be a bivariate uncertainty variable with FGM-Normal bivariate distribution. Then the uncertainty variable $\varsigma:\mathbb{R}^2\to[-1,1]$ has an uncertainty distribution

$$\Psi_\varsigma(x) = \sup_{\frac{(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}=x} \left(\Psi_{\xi_1}(x_1)\Psi_{\xi_2}(x_2)\left(1+\varpi\left(1-\Psi_{\xi_1}(x_1)\right)\left(1-\Psi_{\xi_2}(x_2)\right)\right)\right). \quad (35)$$

Proof: Note that

$$\begin{aligned}\Psi_\varsigma(x) &= \sup_{\frac{(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}=x} \left(\Psi^{FGM}_{\xi_1,\xi_2}(x_1,x_2)\right)\\ &= \sup_{\frac{(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}=x} \left(\Psi_{\xi_1}(x_1)\Psi_{\xi_2}(x_2)\left(1+\varpi\left(1-\Psi_{\xi_1}(x_1)\right)\left(1-\Psi_{\xi_2}(x_2)\right)\right)\right).\end{aligned} \quad (36)$$

Theorem 3.19 The expected value of uncertainty variable ς is

$$\mathrm{E}[\varsigma] = \int_0^{+\infty} r\left(1-\Psi_\varsigma(r)\right)dr + \int_{-\infty}^0 r\Psi_\varsigma(r)dr. \quad (37)$$

Proof:

$$\mathrm{E}[\varsigma] = \int_0^{+\infty} r\lambda\{\varsigma \ge r\}dr + \int_{-\infty}^0 r\lambda\{\varsigma \le r\}dr = \int_0^{+\infty} r\left(1-\Psi_\varsigma(r)\right)dr + \int_{-\infty}^0 r\Psi_\varsigma(r)dr. \quad (38)$$

Remark 3.20 $\mathrm{E}[\varsigma]$ is the uncertainty linear correlation coefficient, denoted as $\rho(\varpi)$. The linear correlation coefficient's dependence on FGM copula parameter ϖ, is indicated clearly by Eq. (35).

Theorem 3.21 Under FGM-Normal assumptions, if $\varpi = 0$, i.e., ξ_1 and ξ_2 are independent each other, then $\mathrm{E}[\varsigma] = 0$, i.e., ξ_1 and ξ_2 are uncorrelated. If $\varpi \ne 0$ then $\mathrm{E}[\varsigma] = \rho(\varpi) \ne 0$.

Proof: If $\varpi = 0$, then the FGM copula becomes product copula, i.e., $C(u_2,u_2) = u_1u_2$, thus

$$\mathrm{E}[\varsigma] = \mathrm{E}\left[\frac{\Psi^{-1}_{\xi_1}(U_1)-\theta_1}{\sigma_1}\right]\mathrm{E}\left[\frac{\Psi^{-1}_{\xi_2}(U_2)-\theta_2}{\sigma_2}\right] = 0. \quad (39)$$

Notice, the converse is not necessarily true.

4 The Uncertainty Gauss-Markov Theorem

Once the uncertainty multivariate distributional mechanism is established, then uncertainty linear regression modeling can be readily discussed. Let us review classical modeling first.

4.1 Classical Linear Regression Model

In classical linear regression theory, the basic assumptions are:
Assumption 1: The model

$$y_i = \beta_0 + \beta_1 x_{1,i} + + \beta_{p,i}x_{p,i} + \varepsilon_i,\ i = 1,2,\cdots,n. \quad (40)$$

Assumption 2: The error terms satisfy the following properties:

(i) zero mean

$$\mathrm{E}[\varepsilon_i] = 0,\ i = 1,2,\cdots,n; \quad (41)$$

(ii) constant variance (homoscedasticity)

$$V[\varepsilon_i] = \sigma^2,\ i = 1,2,\cdots,n; \quad (42)$$

(iii) mutual uncorrelation

$$\mathrm{E}[\varepsilon_i\varepsilon_j] = 0,\ i \ne j,\ i,j = 1,2,\cdots,n. \quad (43)$$

Assumption 3: $x_1, x_2, \cdots, x_p$ are not random variables.

Assumption 4: $\varepsilon_i \overset{d}{\sim} N\left(0,\sigma^2\right)$, $i=1,2,\cdots,n$.

In matrix representation, Eq. (36) can be

$$\underline{y} = X\underline{\beta} + \underline{\varepsilon} \tag{44}$$

where

$$X = \left(x_{ij}\right)_{n\times(p+1)} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix} \quad \underline{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} \quad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \ \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \tag{45}$$

With the Assumption 1-3, the famous Gauss-Markov Theorem applies:

Theorem 4.1 For a linear regression model in Eq. (40) or Eq. (44), if $X'X$ is invertible and satisfies Assumption 1-3, then the ordinary least-squares (OLS) estimator of parameter vector $\underline{\beta}$ is

$$\underline{b} = \left(X'X\right)^{-1}X'\underline{y} \tag{46}$$

where

$$\underline{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_p \end{bmatrix}. \tag{47}$$

Moreover, the OLS estimator $\underline{b}$ is BLUE (an abbreviation of best (minimum variance) linear unbiased estimator).

For uncertainty statistical modeling, we face more complicated situations because the complexity of the uncertainty observational data requires a multivariate uncertainty distribution to describe the observations. In classical statistics an error term distribution is an optional assumption, while in uncertainty modeling, an error term distribution is a prerequisite.

We also need to emphasize that many concepts, or terms in classical statistics are well-defined through their evolution over decades, while uncertainty statistics is a new subject, and it is necessary to define the concepts.

4.2 Concepts in Uncertainty Linear Regression Modeling

Definition 4.2 Given a set of uncertainty responses, denoted as $\underline{y}' = \left(y_1, y_2, \cdots, y_n\right)$, and a constant vector, denoted as $\underline{l}' = \left(l_1, l_2, \cdots, l_n\right)$, the inner product

$$i\left(\underline{l}, \underline{y}\right) \triangleq \underline{l} \cdot \underline{y} = \underline{l}'\underline{y} = \sum_{i=1}^{n} l_i y_i \tag{48}$$

is called an uncertainty linear statistic if the value of vector $\underline{l}$ is known. If an inner product

$$\underline{l}'\underline{1} = \sum_{i=1}^{n} l_i = 0, \tag{49}$$

then $\underline{l}$ is called a contrast vector.

Definition 4.3 Let $\underline{y}' = \left(y_1, y_2, \cdots, y_n\right)$ be a set of uncertainty responses and $\underline{l}' = \left(l_1, l_2, \cdots, l_n\right)$ be a known constant vector. Let $\underline{y}'_{(n)} = \left(y_{(1)}, y_{(2)}, \cdots, y_{(n)}\right)$ be an uncertainty order statistic vector in ascending order, i.e.,

$$y_{(1)} = \min_{1\le i\le n}\left(y_i\right) \le y_{(2)} \le \cdots \le y_{(n)} = \max_{1\le i\le n}\left(y_i\right). \tag{50}$$

An uncertainty linear function

$$R = \sum_{i=1}^{n} l_i y_{(i)} \tag{51}$$

is called an uncertainty *R* statistic.

Definition 4.4 Let $\underline{y}^{'} = (y_1, y_2, \cdots, y_n)$ be a set of uncertainty responses and $\underline{l}^{'} = (l_1, l_2, \cdots, l_n)$ be an arbitrary constant vector. Assuming that $\underline{l}^{'}\underline{y}$ is an uncertainty estimator for parameter α , if

$$\mathrm{E}\left[\underline{l}^{'}\underline{y}\right] = \alpha , \tag{52}$$

then $\underline{l}^{'}\underline{y}$ is called an uncertainty unbiased estimator for α , and α is called an estimable parameter.

Definition 4.5 Let $\underline{y}^{'} = (y_1, y_2, \cdots, y_n)$ be a set of uncertainty responses and $\underline{l}^{'} = (l_1, l_2, \cdots, l_n)$ be a known constant vector. Assuming that $\underline{l}^{'}\underline{y}$ is any uncertainty estimator for parameter α , if $\underline{l}_{*} \in \{\underline{l}\}$,

$$V\left[\underline{l}_{*}^{'}\underline{y}\right] \le V\left[\underline{l}^{'}\underline{y}\right], \tag{53}$$

then $\underline{l}_{*}^{'}\underline{y}$ is called an uncertainty linear unbiased estimator for α with minimum variance (i.e., the best variance).

Definition 4.6 Let $\underline{y}^{'} = (y_1, y_2, \cdots, y_n)$ be a set of uncertainty observations. The function

$$Q(\underline{\beta}) = (\underline{y} - X\underline{\beta})^{'} (\underline{y} - X\underline{\beta}) \tag{54}$$

is called a quadratic form in $\underline{\beta}$, where

$$X = (x_{ij})_{n\times(p+1)} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix} \quad \underline{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \tag{55}$$

Lemma 4.7 (Singular Value Decomposition Theorem) Let $A = (a_{ij})_{m\times n}$ with $\mathrm{rank}(A) = r$, $0 < r \le \min(m, n)$. Then there exist matrices $U = (u_{ij})_{m\times m}$, $V = (v_{ij})_{n\times n}$ and $\Delta_{m\times n} = \mathrm{diag}(d_i)$, where $d_1 \ge d_2 \ge \cdots \ge d_r > 0$ such that

$$A = U\Delta V^{'} . \tag{56}$$

Corollary 4.8 Assuming that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ are *i.i.d.*. from an uncertainty distribution Ψ_ε with zero mean and constant standard deviation $\sigma > 0$, then there exists a matrix $W = (w_{ij})_{n\times n}$ such that

$$\Sigma = \sigma^2 W W^{'} . \tag{57}$$

4.3 Uncertainty Response

In classical statistics, (see [18, 23, 30, 31, 32]), data refers typically to randomly sampled observations. The notion of data includes natural observational data, survey data, and other observational data (e.g. experimental data). Data in classical statistics are observable and repeatedly and objectively collectable. Randomness is intrinsic to all data in statistics, see [28]. Data, according to the degree of abstraction may be classified into three levels (raw) data, information, and knowledge.

However, in uncertainty modeling, impreciseness is intrinsic to uncertainty data, (i.e., expert’s knowledge) due to variations and imperfections in human observation and abstraction abilities, as well as to the complexity of the real world. In other words, data in uncertainty statistics is narrowed to the knowledge level, i.e., expert’s knowledge, no matter what its original character, objective or subjective.

Uncertainty linear regression reveals the linear uncertainty relation between the expert’s response knowledge and factor(s) without uncertainty. The expert’s response knowledge elements are inter-related in some degree, hence, it is unrealistic to assume the expert's response variables are independent. In other words, the multivariate uncertainty must be assumed in order to permit the modeling of the expert's knowledge responses.

The expert's knowledge responses are recorded in real number format. Nevertheless, it is necessary to emphasize that the expert's knowledge responses should never be regarded as an ordinary number. Instead, the response number should be treated as a representative of expert's knowledge response population or equivalently of an uncertainty distribution of an expert's knowledge response. The uncertainty multivariate joint distribution is just the mechanism underlying the expert's knowledge response variable.

In classical statistics, the Gaussian multivariate plays a central role in regression modeling, see [18]. As an illustrative example, the bivariate normal probability distribution is

$$\Phi_{X_1X_2,\rho}\left(x_1,x_2\right)=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{\left(r-\mu_1\right)^2}{\sigma_1^2}-\frac{2\rho\left(\left(r-\mu_1\right)\left(s-\mu_2\right)\right)}{\sigma_1\sigma_2}+\frac{\left(s-\mu_2\right)^2}{\sigma_2^2}\right)\right)drds. \quad (58)$$

It is obvious the bivariate normal copula is

$$C_\rho^{Gauss}\left(u_1,u_2\right)=\int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{\left(r-\mu_1\right)^2}{\sigma_1^2}-\frac{2\rho\left(\left(r-\mu_1\right)\left(s-\mu_2\right)\right)}{\sigma_1\sigma_2}+\frac{\left(s-\mu_2\right)^2}{\sigma_2^2}\right)\right)drds. \quad (59)$$

Notice that $C_\rho^{Gauss}\left(u_1,u_2\right)$ involves a function expressed by a double integration, which is σ-additive in nature. Therefore it is impossible to "borrow" a Gaussian copula to construct an uncertainty multivariate distribution with the symmetry property. In contrast the FGM copula or Type I EFGM copula possesses a symmetry property, and hence the FGM-Normal uncertainty multivariate distribution can play an analogous role. However a new challenge arises: the "variance-covariance" is no longer an identity matrix with a constant coefficient σ^2, but a matrix $\Sigma=\sigma^2\left(\left(1-\rho\right)I_n+\rho J_n\right)$, where I_n is an $n\times n$ identity matrix and J_n is an $n\times n$ matrix having all elements equal to one.

4.4 Uncertainty Distributions under FGM-Normal Assumptions

The next problem to be addressed is the uncertainty distributions for quadratic forms under correlated distributions, i.e., FGM-Normal multivariate uncertainty distribution assumptions.

Definition 4.9 Assume that a FGM-Normal multivariate uncertainty distribution is given by

$$\Psi_{Y_1Y_2\cdots Y_n}\left(y_1,y_2,\cdots,y_n\right)=\prod_{i=1}^{d}\Psi_{Y_i}\left(y_i\right)\left(1+\theta\prod_{i=1}^{n}\left(1-\Psi_{Y_i}\left(y_i\right)\right)\right),\ \theta\in\left[-1,1\right]. \quad (60)$$

Let $\mathfrak{O}_n^d$ be the marginalization operator which reduces an *n*-dimensional multivariate uncertainty distribution to a *d*-dimensional multivariate uncertainty distribution within the same family. Then the d - dimensional multivariate uncertainty distribution is given by

$$\begin{aligned}\Psi_{Y_1Y_2\cdots Y_D}\left(y_1,y_2,\cdots,y_d\right)&=\mathfrak{O}_n^d\left(\prod_{i=1}^{n}\Psi_{Y_i}\left(y_i\right)\left(1+\theta\prod_{i=1}^{n}\left(1-\Psi_{Y_i}\left(y_i\right)\right)\right)\right)\\&=\prod_{i=1}^{d}\Psi_{Y_i}\left(y_i\right)\left(1+\theta\prod_{i=1}^{d}\left(1-\Psi_{Y_i}\left(y_i\right)\right)\right),\ \theta\in\left[-1,1\right]\end{aligned} \quad (61)$$

where

$$\Psi_{Y_l}\left(y_l\right)=\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_l}y_l\right)},\ l=1,2,\cdots,d. \quad (62)$$

Remark 4.10 The introduction of operator $\mathfrak{O}_n^d$ serves the purpose of maintenance of distributional consistency.

Theorem 4.11 Assume that an *n*-dimensional uncertainty vector, denoted by $\left(\xi_{0,1},\xi_{0,2},\cdots,\xi_{0,n}\right)'$, follows a FGM-Standard Normal multivariate uncertainty distribution

$$\Psi_{\xi_{0,1}\xi_{0,2}\cdots\xi_{0,n}}\left(z_1,z_2,\cdots,z_n\right)=\prod_{i=1}^{n}\Psi_{\xi_{0,i}}\left(z_i\right)\left(1+\theta\prod_{i=1}^{n}\left(1-\Psi_{\xi_{0,i}}\left(z_i\right)\right)\right),\ \theta\in\left[-1,1\right] \tag{63}$$

where

$$\Psi_{\xi_{0,i}}\left(z_l\right)=\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}}y_i\right)},\ i=1,2,\cdots,n\ , \tag{64}$$

i.e., Liu's standard normal uncertainty variable. Define a chi-square statistic

$$\chi^2_{[d]}=\sum_{j=1}^{d}\xi^2_{0,i_j} \tag{65}$$

where $\chi^2_{[\cdot]}$ is used to symbolize the chi-square statistic obtained as the sum of squares of *d* components from a FGM-Standard Normal distributed uncertainty vector (i.e., correlated components). In contrast, in expression $\chi^2_{(d)}=\sum_{i=1}^{d}\xi^2_{0,i}$ (, i.e., independent components, see [6]), $(\cdot)$ refers to the sum of squares of *d i.i.d.* Liu's standard normal uncertainty variables. Then

$$\Psi_{\chi^2_{[d]}}\left(v\right)=\sup_{\sum_{j=1}^{d}z^2_{i_j}=v}\left(\prod_{j=1}^{d}\Psi_{\xi_{0,i_j}}\left(z_j\right)\left(1+\theta\prod_{i=1}^{d}\left(1-\Psi_{\xi_{0,i_j}}\left(z_j\right)\right)\right)\right). \tag{66}$$

Proof: Based on Definition 4.9, it is easy to establish that the joint distribution of *d* components from an *n*-dimensional FGM-Standard Normal distributed uncertainty vector can be obtained by

$$\begin{aligned}\Psi_{\xi_{0,1}\xi_{0,2}\cdots\xi_{0,d}}\left(z_1,z_2,\cdots,z_d\right)&=\mathfrak{D}_n^d\left(\prod_{i=1}^{n}\Psi_{\xi_{0,i}}\left(z_i\right)\left(1+\theta\prod_{i=1}^{n}\left(1-\Psi_{\xi_{0,i}}\left(z_i\right)\right)\right)\right)\\&=\prod_{i=1}^{d}\Psi_{\xi_{0,i}}\left(z_i\right)\left(1+\theta\prod_{i=1}^{d}\left(1-\Psi_{\xi_{0,i}}\left(z_i\right)\right)\right).\end{aligned} \tag{67}$$

Then the chi-square statistic under FGM-Standard Normal distribution is

$$\Psi_{\chi^2_{[d]}}\left(v\right)=\sup_{\sum_{i=1}^{d}z_i=v}\left(\prod_{i=1}^{d}\Psi_{\xi_{0,i}}\left(z_i\right)\left(1+\theta\prod_{i=1}^{d}\left(1-\Psi_{\xi_{0,i}}\left(z_i\right)\right)\right)\right), \tag{68}$$

which concludes the proof.

4.5 Uncertainty Gauss-Markov Theorem

A critical step toward uncertainty regression modeling is the estimator of the coefficient vector $\underline{b}$, which characterizes the uncertainty statistical relation. In classical regression modeling, the Gauss-Markov Theorem addresses the estimator of the coefficient vector $\underline{b}$. Now, let us examine whether an uncertainty Gauss-Markov Theorem holds and plays similar role.

Theorem 4.12 Let $\underline{\varepsilon}'=\left(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n\right)$ be an uncertainty *n*-dimensional multivariate variable with FGM-Normal distribution $\Psi_{\xi_1,\xi_2,\cdots,\xi_n}\left(x_1,x_2,\cdots,x_n\right)$, with a FGM copula

$$C^{FGM}_{\varpi}\left(u_1,u_2,\cdots,u_d\right)=\prod_{i=1}^{d}u_i\left(1+\varpi\prod_{i=1}^{n}\left(1-u_i\right)\right),\varpi\in\left[-1,1\right]^d \tag{69}$$

and the marginal distributions $\Psi_{\xi_1}\left(\cdot\right),\Psi_{\xi_2}\left(\cdot\right),\cdots,\Psi_{\xi_d}\left(\cdot\right)$, where

$$\Psi_{\varepsilon_i}(z_i)=\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_0}z_i\right)},\ i=1,2,\cdots,n. \tag{70}$$

Then (i) every pair of $(\varepsilon_i,\varepsilon_j)$, $i\neq j$, $i,j=1,2,\cdots,n$, follows a bivariate uncertainty FGM-Normal distribution of the form:

$$\Psi_{\varepsilon_i,\varepsilon_j}(x_1,x_2)=\prod_{i=1}^{2}\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma}x_i\right)}\left(1+\varpi\prod_{i=1}^{2}\frac{\exp\left(-\frac{\pi}{\sqrt{3}\sigma}x_i\right)}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma}x_i\right)}\right) \tag{71}$$

and the linear correlation coefficient is $\rho(\varpi)=\mathrm{E}\left[\varepsilon_i\varepsilon_j\right]/\sigma^2$, $i\neq j$, $i,j=1,2,\cdots,n$; (ii) the "variance-covariance" matrix

$$\Sigma=\mathrm{E}\left[\underline{\varepsilon}\underline{\varepsilon}'\right]=\sigma^2\left(\rho_{ij}\right)_{n\times n}=\sigma^2\left((1-\rho)I_n+\rho J_n\right). \tag{72}$$

Proof: Proof of (i) is demonstrated in Section 4.2. As to the proof of (ii), notice that since $\mathrm{E}\left[\underline{\varepsilon}\right]=\underline{0}$,

$$\Sigma=\mathrm{E}\left[\underline{\varepsilon}\underline{\varepsilon}'\right]=\sigma^2\left(\rho_{ij}\right)_{n\times n}=\sigma^2\left((1-\rho)I_n+\rho J_n\right) \tag{73}$$

i.e., the off-diagonal elements in matrix $\sigma^2\left(\rho_{ij}\right)_{n\times n}$ arise from the linear relationship between uncertainty variable pair $(\varepsilon_i,\varepsilon_j)$, while the diagonal elements are the variances of uncertainty variables.

Theorem 4.13 Let $\underline{y}'=(y_1,y_2,\cdots,y_n)$ be a set of experts' knowledge responses. Assuming that

$$y_i=\beta_0+\beta_1x_{i1}+\cdots+\beta_px_{ip}+\varepsilon_i,\ i=1,2,\cdots,n \tag{74}$$

where $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ are specified by an uncertainty n-dimensional FGM-Normal joint distribution $\Psi_{\xi_1,\xi_2,\cdots,\xi_n}(x_1,x_2,\cdots,x_n)$, with a FGM copula

$$C_\varpi(u_1,u_2,\cdots,u_d)=\prod_{i=1}^{d}u_i\left(1+\varpi\prod_{i=1}^{n}(1-u_i)\right),\varpi\in[-1,1]^d \tag{75}$$

and the all the regular marginal distributions $\Psi_{\xi_1}(\cdot),\Psi_{\xi_2}(\cdot),\cdots,\Psi_{\xi_d}(\cdot)$,

$$\Psi_{\varepsilon_i}(z_i)=\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_0}z_i\right)},\ i=1,2,\cdots,n \tag{76}$$

exist, then the weighted least-squares (WLS) estimator of parameter $\underline{\beta}$ is

$$\underline{b}=\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}\underline{y} \tag{77}$$

where

$$\underline{b}'=(b_0,b_1,\cdots,b_p). \tag{78}$$

And the WLS estimator $\underline{b}$ is BLUE (an abbreviation of best (minimum variance) linear unbiased estimator).

Proof: (i) $\underline{b}$ is a linear estimator because

$$\begin{aligned}\underline{b}&=\underline{l}'\underline{y}\\ \underline{l}'_{(p+1)\times n}&=\underbrace{\left(X'_{(p+1)\times n}\Sigma^{-1}_{n\times n}X_{n\times(p+1)}\right)^{-1}}_{(p+1)\times(p+1)}X'_{(p+1)\times n}\Sigma^{-1}{}_{n\times n}.\end{aligned} \tag{79}$$

(ii) Unbiasedness.

$$\begin{aligned} &\mathrm{E}[\underline{b}] \\ &=\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}\mathrm{E}[\underline{y}] \\ &=\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}\mathrm{E}[X\underline{\beta}+\underline{\varepsilon}] \\ &=\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}X\underline{\beta}+\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}\mathrm{E}[\underline{\varepsilon}] \\ &=\underline{\beta} \end{aligned} \tag{80}$$

since $\mathrm{E}[\underline{\varepsilon}]=\underline{0}_{(p+1)\times 1}$.

(iii) Let $\underline{b}^*$ be another weighted linear unbiased estimator,

$$\underline{b}^*=\underline{c}'\underline{y} \tag{81}$$

where

$$\underline{c}'=\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}+\underline{d}'. \tag{82}$$

Then

$$\begin{aligned} &\mathrm{E}[\underline{b}^*]=\mathrm{E}[\underline{c}'\underline{y}] \\ &=\left(\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}+\underline{d}'\right)\mathrm{E}[\underline{y}] \\ &=\left(\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}+\underline{d}'\right)\mathrm{E}[X\underline{\beta}+\underline{\varepsilon}] \\ &=\left(\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}X+\underline{d}'X\right)\underline{\beta} \\ &=\left(I+\underline{d}'X\right)\underline{\beta} \end{aligned} \tag{83}$$

which implies $\underline{b}^*$ is unbiased linear estimator if and only if $X'\underline{d}=\underline{0}$. Furthermore,

$$\begin{aligned} &V[\underline{b}^*]=V\left[\left(\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}+\underline{d}'\right)\underline{y}\right] \\ &=V\left[\left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}\underline{y}+\underline{d}'\underline{y}\right]=V[\underline{b}+\underline{d}'\underline{y}] \\ &=V[\underline{b}]+\underline{d}'\mathrm{E}[\underline{\varepsilon}\underline{\varepsilon}']\underline{d}=V[\underline{b}]+\underline{d}'\Sigma^{-1}\underline{d}\geq V[\underline{b}] \end{aligned} \tag{84}$$

because $\underline{d}'\Sigma^{-1}\underline{d}$ is a quadratic form, $\underline{d}'\Sigma^{-1}\underline{d}\geq 0$. Then the proof is concluded.

Remark 4.14 The WLS estimator of $\underline{\beta}$ is BLUE under FGM-Normal multivariate uncertainty distribution assumptions for the uncertainty copula parameter $\varpi\neq 0$; however, if $\varpi=0$, which implies the errors $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ become *i.i.d.* from an uncertainty population, then the WLS estimator of $\underline{\beta}$ is reduced into an OLS estimator, hence it is BLUE.

4.6 Uncertainty Regression Inferences

Let us discuss the inferences of OLS estimator, i.e., $\varpi=0$. It is necessary to address the residuals from an uncertainty linear regression model.

Definition 4.15 Let $\underline{y}=X\underline{\beta}+\underline{\varepsilon}$ be an uncertainty linear regression model with FGM-Normal multivariate uncertainty distribution. Furthermore, it is assumed that the uncertainty copula parameter $\varpi=0$ so the errors $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ are *i.i.d.* from an uncertainty population. Then vector

$$\underline{r} = \left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\underline{y} \tag{85}$$

is defined as the modeling residual.

Theorem 4.16 Let $\underline{r}' = (r_1, r_2, \cdots, r_n)$ be modeling residual. Then,

(i) $\mathrm{E}[\underline{r}] = \underline{0}$;

(ii) $V[\underline{r}] = \sigma^2\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)$;

(ii) $\underline{1}'\underline{r} = \sum_{i=1}^{n} r_i = 0$;

(iii) $SS_r = \underline{r}'\underline{r} = \sum_{i=1}^{n} r_i^2 = \underline{y}'\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\underline{y}$;

(iv) Define $SS_y = \underline{y}'\underline{y}$ and $SS_X = \underline{b}'\underline{b} = \underline{y}'X\left(X'X\right)^{-1}X'\underline{y}$, then $SS_y = SS_X + SS_r$.

Proof: (i)

$$\mathrm{E}[\underline{r}] = \left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\mathrm{E}\left[\underline{y}\right] = \underline{0}. \tag{86}$$

(ii)

$$\begin{aligned} V[\underline{r}] &= \left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\mathrm{E}\left[\underline{\varepsilon\varepsilon}'\right]\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)' \\ &= \sigma^2\left(\mathrm{I} - 2X\left(X'X\right)^{-1}X' + X\left(X'X\right)^{-1}X'X\left(X'X\right)^{-1}X'\right) \\ &= \sigma^2\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right). \end{aligned} \tag{87}$$

(iii) Note the matrix equality $\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\underline{1} = \underline{0}$, thus

$$\underline{1}'\underline{r} = \underline{1}'\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\underline{y} = \left(\underline{1}'\mathrm{I} - \underline{1}'X\left(X'X\right)^{-1}X'\right)\underline{y} = \underline{0}'\underline{y} = 0. \tag{88}$$

(iii) Note that

$$SS_r = \underline{r}'\underline{r} = \left(\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\underline{y}\right)'\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\underline{y} = \underline{y}'\left(\mathrm{I} - X\left(X'X\right)^{-1}X'\right)\underline{y}. \tag{89}$$

(iv) It is obvious the equality holds.

Theorem 4.17 Let $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$ be an uncertainty linear regression model with FGM-Normal multivariate uncertainty distribution. Furthermore, it is assumed that the uncertainty copula parameter $\varpi = 0$ and the errors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ are *i.i.d.* from an uncertainty normal population. Then $SS_y/\sigma^2 \overset{d}{\sim} \chi^2_{(n)}$, $SS_X/\sigma^2 \overset{d}{\sim} \chi^2_{(p+1)}$, and $SS_r/\sigma^2 \overset{d}{\sim} \chi^2_{(n-p-1)}$ respectively.

Proof: With the results of Theorem 4.15 and recalling Section 3, it is a straightforward matter to establish the uncertainty distributional results for the three sums of squares statistics.

Theorem 4.18 Let the OLS estimator of $\underline{\beta}' = (\beta_0, \beta_1, \cdots, \beta_p)$ be $\underline{b}' = (b_0, b_1, \cdots, b_p)$. Further, let

$$L' = \begin{pmatrix} l_0' \\ l_1' \\ \vdots \\ l_p' \end{pmatrix} = X\left(X'X\right)^{-1}X' \tag{90}$$

where

$$\underline{l}_j' = \left(l_{j1}, l_{j2}, \cdots, l_{jn}\right), \quad j = 0, 1, \cdots, p. \tag{91}$$

Then (i) $b_j = \underline{l}_j'\underline{y}$, $j = 0, 1, \cdots, p$, follow uncertainty normal distributions; (ii) $\left(b_j - \beta_j\right)\Big/\sqrt{SS_r/\sigma^2} \overset{d}{\sim} T_{(n-p-1)}$, $j = 0, 1, \cdots, p$; (iii) $\left(b_j - \beta_j\right)^2\Big/\left(SS_r/\sigma^2\right) \overset{d}{\sim} F_{1,(n-p-1)}$, $j = 0, 1, \cdots, p$; and (iv) $\sum_{j=0}^{p}\left(b_j - \beta_j\right)^2\Big/\left(SS_r/\sigma^2\right) \overset{d}{\sim} F_{(p+1),(n-p-1)}$.

Proof: A straightforward extension to the statistic's distributional theory developed by Guo et al [6].

Remark 4.19 For uncertainty FGM-Normal with parameter $\varpi = 0$, which leads to OLS estimation, the hypothesis testing and confidence interval are easily carried out, particularly, the hypothesis testing the significance of each individual parameter β_{j_0} . However, if $\varpi \neq 0$,i.e., the WLS estimation will lead to complications in the relevant uncertainty statistics. The next theorem states the associated WLS results, i.e., $\varpi \neq 0$.

Definition 4.20 Let $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$ be an uncertainty linear regression model with FGM-Normal multivariate uncertainty distribution. Furthermore, it is assumed that the uncertainty copula parameter $\varpi \neq 0$ and the errors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ are from a FGM-Normal multivariate population. Then vector

$$\underline{r} = \left(\mathrm{I} - X\left(X'\Xi^{-1}X\right)^{-1} X'\Xi^{-1}\right)\underline{y} \tag{92}$$

is defined as the WLS modeling residual.

Theorem 4.20 Let $\underline{r}' = (r_1, r_2, \cdots, r_n)$ be WLS modeling residual. Then

(i) $\mathrm{E}[\underline{r}] = \underline{0}$;

(ii) $V[\underline{r}] = \sigma^2\left(\mathrm{I} - X\left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}\right)$;

(ii) $\underline{1}'\underline{r} = \sum_{i=1}^{n} r_i = 0$;

(iii) $SS_r = \underline{r}'\underline{r} = \sum_{i=1}^{n} r_i^2 = \underline{y}'\left(\mathrm{I} - X\left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}\right)\underline{y}$;

(iv) Define $SS_y = \underline{y}'\underline{y}$ and $SS_X = \underline{b}'\underline{b} = \underline{y}'X\left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}\underline{y}$, then $SS_y = SS_X + SS_r$;

(v) $SS_y \big/ \sigma_0^2 \overset{d}{\sim} \chi^2_{[n]}$, $SS_X \big/ \sigma_0^2 \overset{d}{\sim} \chi^2_{[p+1]}$, and $SS_r \big/ \sigma_0^2 \overset{d}{\sim} \chi^2_{[n-p-1]}$.

Proof: Statement (i) – Statement (iv) are straightforward algebraic manipulations. As to Statement (v) is a result of Theorem 4.11.

5 Conclusion

In this paper, we emphasize the intrinsic features of uncertainty data – the impreciseness in expert's knowledge. We take a comparative approach to discuss uncertainty regression modeling. By contrasting with classical regression modeling, from the aspects of response, factor, and statistical relation building to the error distributional assumptions, we examine uncertainty regression modeling structure. Whereas classical regression uses *i.i.d.* modeling and OLS estimation, uncertainty regression modeling assumes a correlated distribution mechanism, i.e., under the uncertainty multivariate distribution. Therefore, to establish a suitable uncertainty multivariate distribution for linear regression we introduce an uncertainty copula concept. Thus we specify the first time the concrete forms of uncertainty multivariate distributions. We propose a symmetric copula-linked multivariate distribution – the FGM-Normal multivariate distribution, under which we investigate linear regression modeling and statistical inferences. We note that the uncertainty regression assumptions are far complicated than those in probabilistic regression modeling, for instance, heteroscedasticity assumptions. Nevertheless, the uncertainty copula approach needs more theoretical justifications beyond our initial consistency development.

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