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Intuitionistic Fuzzy Optimization Technique for Nash Equilibrium Solution of Multi-objective Bi-Matrix Games

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Abstract

In this paper, we present an application of intuitionistic fuzzy optimization model to a two person multi-objective bi-matrix games (pair of pay-offs matrices) for the Nash equilibrium solution(NES) with mixed strategies. We use linear membership and non-membership function for such computation. We introduce the intuitionistic fuzzy(IF) goal for a choice of a strategy in a pay-off matrix in order to incorporate ambiguity of human judgements and a player wants to maximize his/her degree of attainment of the IF goal. It is shown that this NES is the optimal solution of the mathematical programming problem, namely a quadratic programming problem. In addition, numerical example is also presented to illustrate the methodology.

Keywords: bi-matrix game, Nash equilibrium, intuitionistic fuzzy sets, fuzzy optimization, intuitionistic fuzzy optimization, degree of attainment

1 Introduction

Fuzziness in bi-matrix game theory was studied by various researchers [15, 7], where the goals are viewed as fuzzy sets, but they are very limited and in many cases they do not represent exactly the real problems. In practical situation, due to insufficiency in the information available, it is not easy to describe the fuzzy constraint conditions by ordinary fuzzy sets and consequently, the evaluation of membership values is not always possible up to Decision Maker (DM)'s satisfaction. Due to the same reason evaluation of non-membership values is not always possible and consequently there remains an indeterministic part of which hesitation survives. In such situation intuitionistic fuzzy set (IFS), Atanassov [2] serve better our required purpose. Also, in realistic models, the more extensive application of multiple pay-off has been found to be more rather than the simple pay-off [11]. No studies, however, have been made for NES of multi-objective bi-matrix game with IF goal which will be examined in this paper. We introduce here a new approach to solving multi-objective bi-matrix payoff matrices in IF environment. IF goal for the pay-off matrices has been formulated in order to incorporate the ambiguity of human judgement. We assume that each player has a IF goal for the choice of the strategy and players want to maximize the degree of attainment of the IF goal.

For the purpose, this paper is organized as follows: In Section 2, we shall give some basic definitions, notations and optimization model on IFS. In Section 3, a expected pay-off, IF goal for bi-matrix game is defined and a degree of attainment of the IF goal is considered. The equilibrium solution with respect to a degree of attainment of a IF goal is also defined. In Section 4, the methods of computing the NES of a single-objective bi-matrix game are proposed, when the membership, nonmembership functions of the IF goals are linear. In Section 5, the NES for multi-objective game is proposed. Lastly a numerical example is given in Section 6.

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2 Intuitionistic Fuzzy Sets

The intuitionistic fuzzy set introduced by Atanassov [2] is characterized by two functions expressing the degree of belonging and the degree of non-belongingness respectively.

Definition 1 Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universal set. An Atanassov's intuitionistic fuzzy set (IFS) A in a given universal set U is an object having the form

$$A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle : x_i \in U\} \tag{1}$$

where the functions

$$\mu_A : U \to [0,1]; \quad i.e., x_i \in U \to \mu_A(x_i) \in [0,1]$$

and

$$\nu_A : U \to [0,1] \quad i.e., x_i \in U \to \nu_A(x_i) \in [0,1]$$

define the degree of membership and the degree of nonmembership of an element $x_i \in U$ to the set $A \subseteq U$, respectively, such that they satisfy the following conditions :

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x_i \in U$$

which is known as intuitionistic condition. The degree of acceptance $\mu_A(x)$ and of nonacceptance $\nu_A(x)$ can be arbitrary.

Definition 2 For all $A \in IFS(U)$, let

$$\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i), \tag{2}$$

which is called the the Atanassov's intuitionistic index of the element x_i in the set A or the degree of uncertainty or indeterministic part of x_i. Obviously,

$$0 \leq \pi_A(x) \leq 1; \quad \text{for all } x_i \in U. \tag{3}$$

Obviously, when $\pi_A(x) = 0, \forall x \in U$, i.e., $\mu_A(x) + \nu_A(x) = 1$, the set A is a fuzzy set as follows:

$$A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in U\} = \{\langle x, \mu_A(x)\rangle : x \in U\}.$$

Therefore fuzzy set is a especial IFS.

If an Atanassov's IFS C in U has only an element, then C is written as follows

$$C = \{\langle x_k, \mu_C(x_k), \nu_C(x_k)\rangle\} \tag{4}$$

which is usually denoted by $C = \{\langle \mu_C(x_k), \nu_C(x_k)\rangle\}$ in short.

Definition 3 Let A and B be two Atanassov's IFS in the set U. $A \subset B$ if and only if

$$\mu_A(x_i) \leq \mu_B(x_i) \text{ and } \nu_A(x_i) \geq \nu_B(x_i); \text{ for any } x_i \in U. \tag{5}$$

Definition 4 Let A and B be two Atanassov's IFS in the set U. $A = B$ if and only if

$$\mu_A(x_i) = \mu_B(x_i) \text{ and } \nu_A(x_i) = \nu_B(x_i); \text{ for any } x_i \in U. \tag{6}$$

Namely, $A = B$, if and only if $A \subset B$ and $B \subset A$.

Definition 5 Let A and B be two Atanassov's IFS in the set U. The intersection of A and B is defined as

$$A \cap B = \{\langle x_i, \min(\mu_A(x_i), \mu_B(x_i)), \max(\nu_A(x_i), \nu_B(x_i))\rangle | x_i \in U\}. \tag{7}$$

Definition 6 Let A and B be two Atanassov's IFS in the set U and $\lambda > 0$ be a real number. Designate:

(i) $A + B = \{\langle x_i, \mu_A(x_i)\mu_B(x_i) - \mu_A(x_i)\mu_B(x_i), \nu_A(x_i)\nu_B(x_i)\rangle | x \in U\}$

(ii) $AB = \{\langle x_i, \mu_A(x_i)\mu_B(x_i), \nu_A(x_i) + \nu_B(x_i) - \nu_A(x_i)\nu_B(x_i)\rangle | x_i \in U\}$

(iii) $\lambda A = \{< x_i, 1 - (1 - \mu_A(x_i))^\lambda, (\nu_A(x_i))^\lambda >: x_i \in U\}$.

Note: From the above definitions we see that the numbers $\mu_A(x)$ and $\nu_A(x)$ reflect respectively the extent of acceptance and the degrees of rejection of the element x to the set A, and the numbers $\pi_A(x)$ is the extent of indeterminacy between both.

2.1 Intuitionistic Fuzzy Optimization Model

Generally, an optimization problem includes objective(s) and constraints. Let us consider an optimization problem

$$\begin{aligned} & f_i(x) \to \min; \quad i = 1, 2, \cdots, p \\ \text{subject to,} \qquad & g_j(x) \le 0; \quad j = 1, 2, \cdots, q, \end{aligned}$$

where x denotes the unknowns, $f_i(x)$ denotes the objective functions, $g_j(x)$ denotes constraints (non-equalities), p denotes the number of objectives and q denotes the number of constraints.

The solution of this crisp optimization problem satisfies all constraints exactly.

2.1.1 Fuzzy Optimization Model

In fuzzy optimization problem such as fuzzy mathematical programming, Zimmermann [17], fuzzy optimal control, Zadeh [16], the objective(s) and/or constraints or parameters and relations are represented by fuzzy sets. These fuzzy sets explain the degree of satisfaction of the respective condition and are expressed by their membership functions [18]. Analogously as in the crisp case, fuzzy optimization problem the degree of satisfaction of the objective(s) and the constraints is maximized:

$$\begin{aligned} & f_i(x) \to \tilde{\min}; \quad i = 1, 2, \cdots, p \\ \text{subject to,} \qquad & g_j(x) \tilde{\le} 0; \quad j = 1, 2, \cdots, q, \end{aligned}$$

where $\tilde{\min}$ denotes fuzzy minimization and $\tilde{\le}$ denotes fuzzy inequality.

According to Bellman-Zadeh's approach [4] this problem can be transformed to the following optimization problem

$$\begin{aligned} & \max \mu_i(x); \quad x \in \Re^n; i = 1, 2, \cdots, p+q \\ \text{subject to,} \qquad & 0 \le \mu_i(x) \le 1, \end{aligned}$$

where $\mu_i(x)$ denotes degree of membership (acceptance) of x to the respective fuzzy sets. This application of Bellman-Zadeh's approach [4] to solve such fuzzy optimization problem realizes the *min-aggregator*.

2.1.2 Intuitionistic Fuzzy Optimization Model

Intuitionistic fuzzy optimization (IFO), a method of uncertainty optimization, is put forward on the basis of intuitionistic fuzzy sets, Atanassov [2]. It is an extension of fuzzy optimization in which the degrees of rejection of objective(s) and constraints are considered together with the degrees of satisfaction. This optimization problems similar to fuzzy optimization problems can be represented as a two stage process which includes

(i) aggregation of constraints and objective(s) and

(ii) defuzzification (maximization of aggregation function)

According to IFO theory, we are to maximize the degree of acceptance of the IF objective(s) and constraints and to minimize the degree of rejection of IF objective(s) and constraints as

$$\left.\begin{array}{c} \max\limits_{x \in \Re^n} \{\mu_k(x)\}; \\ \min\limits_{x} \{\nu_k(x)\}; \\ \mu_k(x), \nu_k(x) \ge 0; \\ \mu_k(x) \ge \nu_k(x); \\ 0 \le \mu_k(x) + \nu_k(x) \le 1; \end{array}\right\} k = 1, 2, \ldots, p+q$$

where $\mu_k(x)$ denotes the degree of acceptance of x from the k^{th} IFS and $\nu_k(x)$ denotes the degree of rejection of x from the k^{th} IFS. According to Atanassov property of IFS, the conjunction of intuitionistic fuzzy objective(s) and constraints in a space of alternatives U is defined as

$$A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle : x \in U\}, \tag{8}$$

which is defined as the *intuitionistic fuzzy decision set* (IFDS), where A denotes the integrated intuitionistic fuzzy objective/ goals and B denotes integrated intuitionistic fuzzy constraint set and they can be written as

$$\begin{aligned} A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\} = \{\langle x, \min_{i=1}^{p} \mu_i(x), \max_{i=1}^{p} \nu_i(x)\rangle : x \in U\} && (9) \\ B &= \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\} = \{\langle x, \min_{j=1}^{q} \mu_j(x), \max_{j=1}^{q} \nu_j(x)\rangle : x \in U\}. && (10) \end{aligned}$$

Let the intuitionistic fuzzy decision set (8) be denoted by C, then min-aggregator is used for conjunction and max operator for disjunction

$$C = A \cap B = \{\langle x, \mu_C(x), \nu_C(x)\}\rangle | x \in U\}, \tag{11}$$

$$\text{where,} \qquad \mu_C(x) = \min\{\mu_A(x), \mu_B(x)\} = \min_{k=1}^{p+q} \mu_k(x) \tag{12}$$

$$\text{and} \qquad \nu_C(x) = \max\{\nu_A(x), \nu_B(x)\} = \max_{k=1}^{p+q} \nu_k(x), \tag{13}$$

where $\mu_C(x)$ denotes the degree of acceptance of IFDS and $\nu_C(x)$ denotes the degree of rejection of IFDS. Therefore,

$$\mu_C(x) \leq \mu_k(x), \nu_C(x) \geq \nu_k(x); \quad 1 \leq k \leq p+q. \tag{14}$$

The formula can be transformed to the following system

$$\begin{array}{cc} \max \alpha, \quad \min \beta & \\ \alpha \leq \mu_k(x); & k = 1, 2, \ldots, p+q \\ \beta \geq \nu_k(x); & k = 1, 2, \ldots, p+q \\ \alpha \geq \beta; & \text{and } \alpha + \beta \leq 1; \alpha, \beta \geq 0 \end{array}$$

where α denotes the minimal acceptable degree of objective(s) and constraints and β denotes the maximal degree of rejection of objective(s) and constraints. The IFO model can be changed into the following certainty (non-fuzzy) optimization model as :

$$\left.\begin{array}{cc} \max(\alpha - \beta) & \\ \alpha \leq \mu_k(x); & k = 1, 2, \ldots, p+q \\ \beta \geq \nu_k(x); & k = 1, 2, \ldots, p+q \\ \alpha \geq \beta; & \text{and } \alpha + \beta \leq 1; \alpha, \beta \geq 0 \end{array}\right\} \tag{15}$$

which can be easily solved by some simplex methods.

3 Bi-Matrix Game

A bi-matrix game, Nash [8], can be considered as a natural extension of the matrix game. Let I, II denote two players and let $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ be the sets of all pure strategies available for players I, II respectively. By α_{ij} and γ_{ij} we denote the pay-offs that the player I and II receive when player I plays the pure strategy i and player II plays the pure strategy j. Then we have the following pay-off matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots \alpha_{2n} \\ \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \cdots \alpha_{mn} \end{pmatrix}; B = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots \gamma_{2n} \\ \cdots & \cdots & \\ \gamma_{m1} & \gamma_{m1} & \cdots \gamma_{mn} \end{pmatrix} \tag{16}$$

where we assume that each of the two players chooses a strategy, a pay-off for each of them is represented as a crisp number. We denote the game by $\Gamma = \langle \{I, II\}, A, B \rangle$.

For two person non zero sum multi objective game, multiple pair of $m \times n$ pay-off matrices can be written as

$$A^1 = \begin{pmatrix} \alpha^1_{11} & \alpha^1_{12} & \cdots \alpha^1_{1n} \\ \alpha^1_{21} & \alpha^1_{22} & \cdots \alpha^1_{2n} \\ \cdots & \cdots & \cdots \\ \alpha^1_{m1} & \alpha^1_{m2} & \cdots \alpha^1_{mn} \end{pmatrix}; \quad A^2 = \begin{pmatrix} \alpha^2_{11} & \alpha^2_{12} & \cdots \alpha^2_{1n} \\ \alpha^2_{21} & \alpha^2_{22} & \cdots \alpha^2_{2n} \\ \cdots & \cdots & \cdots \\ \alpha^2_{m1} & \alpha^2_{m2} & \cdots \alpha^2_{mn} \end{pmatrix}; \ldots$$

$$\ldots; A^{n1} = \begin{pmatrix} \alpha_{11}^{n1} & \alpha_{12}^{n1} & \cdots\alpha_{1n}^{n1} \\ \alpha_{21}^{n1} & \alpha_{22}^{n1} & \cdots\alpha_{2n}^{n1} \\ \cdots & \cdots & \cdots \\ \alpha_{m1}^{n1} & \alpha_{m2}^{n1} & \cdots\alpha_{mn}^{n1} \end{pmatrix}$$

$$B^1 = \begin{pmatrix} \beta_{11}^{1} & \beta_{12}^{1} & \cdots\beta_{1n}^{1} \\ \beta_{21}^{1} & \beta_{22}^{1} & \cdots\beta_{2n}^{1} \\ \cdots & \cdots & \cdots \\ \beta_{m1}^{1} & \beta_{m2}^{1} & \cdots\beta_{mn}^{1} \end{pmatrix}; \ B^2 = \begin{pmatrix} \beta_{11}^{2} & \beta_{12}^{2} & \cdots\beta_{1n}^{2} \\ \beta_{21}^{2} & \beta_{22}^{2} & \cdots\beta_{2n}^{2} \\ \cdots & \cdots & \cdots \\ \beta_{m1}^{2} & \beta_{m2}^{2} & \cdots\beta_{mn}^{2} \end{pmatrix}; \ldots$$

$$\ldots; B^{n2} = \begin{pmatrix} \beta_{11}^{n2} & \beta_{12}^{n2} & \cdots\beta_{1n}^{n2} \\ \beta_{21}^{n2} & \beta_{22}^{n2} & \cdots\beta_{2n}^{n2} \\ \cdots & \cdots & \cdots \\ \beta_{m1}^{n2} & \beta_{m2}^{n2} & \cdots\beta_{mn}^{n2} \end{pmatrix}$$

where the player I and the player II have $n1$ and $n2$ number of objectives. Let the domain for the player I be defined by $D_1 = D_1^1 \times D_1^2 \times \ldots \times D_1^{n1} \subseteq \Re^{n1}$ and that for the player II be $D_2 = D_2^1 \times D_2^2 \times \ldots \times D_2^{n2} \subseteq \Re^{n2}$.

3.1 Nash Equilibrium Solution

Nash [8] defined the concept of Nash equilibrium solutions (NES) in crisp bi-matrix games for single pair of payoff matrices and presented methodology for obtaining them.

3.1.1 Pure Strategy

Let I, II denote two players and let $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ be the sets of all pure strategies available for players I, II respectively.

Definition 7 A pair of strategies $(row\ \ r, column\ \ s)$ is said to constitute a NES to a bi-matrix game Γ if the following pair of inequalities is satisfied for all $i = 1, 2, ..., m$ and for all $j = 1, 2, ..., n$:

$$\alpha_{is} \le \alpha_{rs}; \quad \gamma_{rj} \le \gamma_{rs} \tag{17}$$

The pair $(\alpha_{rs}, \gamma_{rs})$ is known as a *Nash equilibrium outcome* of the bi-matrix game in pure strategies. A bi-matrix game can admit more than one NES, with the equilibrium outcomes being different in each case. The NES concept for bi-matrix games with single pair of IF payoffs in pure strategy has been proposed by Nayak and Pal [11].

3.1.2 Mixed Strategy

In the previous section, we have encountered only cases in which a given bi-matrix game admits a unique or a multiple of Nash equilibria. There are other cases, where the Nash equilibrium does not exist in pure strategies. We denote the sets of all mixed strategies, called strategy spaces, available for players I, II by

$$S_I = \{(x_1, x_2, ..., x_m) \in \Re_+^m : x_i \ge 0; i = 1, 2, ..., m \text{ and } \sum_{i=1}^{m} x_i = 1\}$$

$$S_{II} = \{(y_1, y_2, ..., y_n) \in \Re_+^n : y_i \ge 0; i = 1, 2, ..., n \text{ and } \sum_{i=1}^{n} y_i = 1\},$$

where $\Re_+^m$ denotes the $m-$dimensional non negative Euclidean space. Since the player is uncertain about what strategy he/she will choose, he/she will choose a probability distribution over the set of alternatives available to him/her or a mixed strategy x in terms of game theory. We shall denote this bi-matrix game by

$$\Gamma_1 = \langle \{I, II\}, S_I \times S_{II}, A, B \rangle.$$

For $x \in S_I, y \in S_{II}$, $x^T Ay$ and $x^T By$ are the expected pay-off to the player I and player II respectively.The NES concept for bi-matrix games with IF goals in mixed strategies for single pair of matrices has been proposed by Nayak and Pal [11].

Definition 8 (**Expected pay-off**): If the mixed strategies x and y are proposed by the player I and player II respectively, then the expected pay-off of the player I and player II are defined by

$$x^T Ay = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i \alpha_{ij} y_j \text{ and } x^T By = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i \gamma_{ij} y_j. \tag{18}$$

Definition 9 A pair $(x^* \in S_I, y^* \in S_{II})$ is called a NES to a bi-matrix game Γ_1 in mixed strategies, if the following inequalities are satisfied.

$$\left.\begin{array}{c} x^T Ay^* \leq x^{*T} Ay^*; \ \ \forall x \in S_I \\ x^{*T} By \leq x^{*T} By^*; \ \ \forall y \in S_{II} \end{array}\right\} \tag{19}$$

x^* and y^* are also called the optimal strategies for the player I and player II respectively. Then the pair of numbers $V = \langle x^{*T} Ay^*, x^{*T} By^* \rangle$ is said to be the Nash equilibrium outcome of Γ_1 in mixed strategies, and the triplet (x^*, y^*, V) is said to be a solution of bi-matrix game.

4 Single Objective Bi-Matrix Game with IF Goal

To derive the computational procedure for NES with respect to degree of attainment of the IF goal in single objective bi-matrix games, first, we define some terms which are useful in the solution procedure. Let the domain for the player I be defined by

$$D_1 = \{x^T Ay : (x, y) \in S_I \times S_{II} \subset \Re^m \times \Re^n\} \subseteq \Re \tag{20}$$

and that for the player II be defined by

$$D_2 = \{x^T By : (x, y) \in S_I \times S_{II} \subset \Re^m \times \Re^n\} \subseteq \Re. \tag{21}$$

Definition 10 (**IF goal**): A IF goal $\hat{G}_1$ for player I is defined as a IFS in D_1 characterized by the membership and nonmembership functions

$$\begin{array}{c} \mu_{\hat{G}_1} : D_1 \to [0,1] \text{ and } \nu_{\hat{G}_1} : D_1 \to [0,1] \\ \text{or simply, } \mu_1 : D_1 \to [0,1] \text{ and } \nu_1 : D_1 \to [0,1], \end{array} \tag{22}$$

such that $0 \leq \mu_1(x) + \nu_1(x) \leq 1$. Let $\hat{G}_1$ be the IF goal and $\hat{C}_1$ be the IF constraint in the space D_1, for the player I, then according to Atanassov's property of IFS [2], the decision $\hat{D}$, which is a IFS, is a given by

$$\begin{array}{rcl} \hat{D} & = & \hat{G}_1 \cap \hat{C}_1 = \{\langle x, \mu_{\hat{D}}(x), \nu_{\hat{D}}(x)\rangle : x \in D_1\} \\ \text{where, } \ \mu_{\hat{D}}(x) & = & \min\{\mu_{\hat{G}_1}(x), \mu_{\hat{C}_1}(x)\}; \ \ \nu_C(x) = \max\{\nu_{\hat{G}_1}(x), \nu_{\hat{C}_1}(x)\}, \end{array} \tag{23}$$

where $\mu_{\hat{D}}(x)$ denotes the degree of acceptance of IF decision set $\hat{D}$ and $\nu_{\hat{D}}(x)$ denotes the degree of rejection of IF decision set.

Similarly, a IF goal for player II is IFS on D_2 characterized by the membership function $\mu_{\hat{G}_2} : D_2 \to [0,1]$ and nonmembership function $\nu_{\hat{G}_2} : D_2 \to [0,1]$ such that $0 \leq \mu_{\hat{G}_2}(x) + \nu_{\hat{G}_2}(x) \leq 1$.

A membership, non-membership function value for a IF goal can be interpreted as the degree of attainment of the IF goal for a strategy of a payoff.

Definition 11 (**Degree of attainment of IF goal**): For any pair of mixed strategies,$(x, y) \in S_I \times S_{II}$, let the IF goal for player I be denoted by $\hat{G}_1(x, y)$, then the degree of attainment of the IF goal is defined as

$$\max_x\{\mu_{\hat{G}_1}(x, y)\} \text{ and } \min_x\{\nu_{\hat{G}_1}(x, y)\}. \tag{24}$$

The degree of attainment of the IF goal can be considered to be a concept of a degree of satisfaction of the fuzzy decision [2, 16], when the constraint can be replaced by expected pay-off. When the player I and II choose strategies x and y respectively, the maximum degree of attainment of the IF goal is determined by the relation (24).

Let the membership and non-membership function for the aggregated IF goal be $\mu_1(x, y)$ and $\nu_1(x, y)$ respectively for the player I and that for the player II, they are $\mu_2(x, y)$ and $\nu_2(x, y)$ respectively. Thus, when a player has two different strategies, he/she prefers the strategy possessing the higher membership function value and lower nonmembership function value in comparison to the other.

Definition 12 (**Equilibrium solution of IF bi-matrix game**): Let A and B be the pay-off matrices of a single objective bi-matrix game Γ. If the player I chooses a mixed strategy x, the player II chooses a mixed strategy y, the membership and non membership functions for player I and player II are $\mu_1(x,y), \nu_1(x,y)$ and $\mu_2(x,y), \nu_2(x,y)$ respectively, and their expected pay-offs are x^TAy and x^TBy respectively, then

$$\mu_1(x,y) = \mu_1(x^TAy); \quad \nu_1(x,y) = \nu_1(x^TAy)$$

$$\mu_2(x,y) = \mu_2(x^TBy); \quad \nu_2(x,y) = \nu_2(x^TBy).$$

A pair $(x^*, y^*) \in S_I \times S_{II}$ is called a NES of the IF matrix game, with respect to the degree of attainment of the IF goal in a single objective bi-matrix game if for any other mixed strategies x and y, the pair satisfies the following inequalities

$$\mu_1(x^{*T}Ay^*) \geq \mu_1(x^TAy^*) \text{ and } \nu_1(x^{*T}Ay^*) \leq \nu_1(x^TAy^*); \forall x \in S_I$$

$$\mu_2(x^{*T}By^*) \geq \mu_2(x^TBy^*) \text{ and } \nu_2(x^{*T}By^*) \leq \nu_2(x^TBy^*); \forall y \in S_{II}.$$

$(x^{*T}Ay^*, x^{*T}By^*)$ is called the Nash equilibrium outcome of the bi-matrix game in mixed strategies. Since the membership and non-membership functions are convex and continuous [6, 2], the solution with respect to the degree of attainment of the aggregated IF goal for Γ_1 always exist.

Thus an equilibrium solution for IF games is defined with respect to the degree of attainment of the IF goals.

4.1 Optimization Problem for Player I

The linear membership and nonmembership functions [2] of the IF goal $\mu_1(x^TAy)$ and $\nu_1(x^TAy)$, for the player I can mathematically be represented as (Fig. 1):

$$\mu_1(x^TAy) = \begin{cases} 1; & x^TAy \geq \overline{a} \\ \frac{x^TAy - \underline{a}}{\overline{a} - \underline{a}}; & \underline{a} < x^TAy < \overline{a} \\ 0; & x^TAy \leq \underline{a} \end{cases}$$

$$\nu_1(x^TAy) = \begin{cases} 1; & x^TAy \leq \underline{a} \\ \frac{x^TAy - \overline{a}}{\underline{a} - \overline{a}}; & \underline{a} < x^TAy < \overline{a} \\ 0; & x^TAy \geq \overline{a} \end{cases}$$

where $\underline{a}$ and $\overline{a}$ are the tolerances of the expected pay-off x^TAy and $\mu_1(x^TAy)$ should be determined in objective allowable region $[\underline{a}, \overline{a}]$. For player I, $\underline{a}$ and $\overline{a}$ are the pay-off giving the worst and the best degree of satisfaction

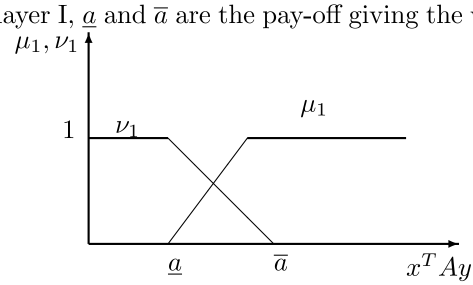

Figure 1: Membership and nonmembership functions for player I

respectively. Although $\underline{a}$ and $\overline{a}$ would be any scalars with $\overline{a} > \underline{a}$, Nishizaki [14] suggested that, parameters $\underline{a}$ and $\overline{a}$ can be taken as

$$\overline{a} = \max_x \max_y x^TAy = \max_i \max_j a_{ij}$$

$$\underline{a} = \min_x \min_y x^TAy = \min_i \min_j a_{ij}.$$

Therefore, the conditions $\underline{a} \leq \min_{i,j} a_{ij}$ and $\overline{a} \geq \max_{i,j} a_{ij}$ hold. Thus player I is not satisfied by the pay-off less than $\underline{a}$ but is fully satisfied by the pay-off greater than $\underline{a}$. Now,

$$\frac{x^T Ay - \underline{a}}{\overline{a} - \underline{a}} = (\frac{-\underline{a}}{\overline{a} - \underline{a}}) + x^T(\frac{A}{\overline{a} - \underline{a}})y = c_1^{(1)} + x^T \hat{A} y,$$

where $c_1^{(1)} = \frac{-\underline{a}}{\overline{a} - \underline{a}}$ and $\hat{A} = \frac{1}{\overline{a} - \underline{a}} A$. Similarly, if we define $c_1^{(2)} = \frac{-\overline{a}}{\underline{a} - \overline{a}}$, then the membership and nonmembership functions can be written as

$$\mu_1(x^T Ay) = \begin{cases} 1 & x^T Ay; \geq \overline{a} \\ c_1^{(1)} + x^T \hat{A} y; & \underline{a} < x^T Ay < \overline{a} \\ 0; & x^T Ay \leq \overline{a} \end{cases}$$

$$\nu_1(x^T Ay) = \begin{cases} 1; & x^T Ay; \leq \overline{a} \\ c_1^{(2)} - x^T \hat{A} y; & \underline{a} < x^T Ay < \overline{a} \\ 0; & x^T Ay \geq \overline{a} \end{cases}$$

4.2 Optimization Problem for Player II

Here we are consider player II solution with respect to the degree of attainment of his fuzzy goal. The membership and nonmembership functions of the IF goal $\mu_2(x^T By)$ and $\nu_2(x^T By)$ can be represented as (Fig. 2):

$$\mu_2(x^T By) = \begin{cases} 1; & x^T By \leq \underline{b} \\ \frac{x^T By - \underline{b}}{\overline{b} - \underline{b}}; & \underline{b} < x^T By < \overline{b} \\ 0; & x^T By \geq \overline{b} \end{cases}$$

$$\nu_2(x^T By) = \begin{cases} 1; & x^T By \geq \overline{b} \\ \frac{x^T By - \overline{b}}{\underline{b} - \overline{b}}; & \underline{b} < x^T By < \overline{b} \\ 0; & x^T By \leq \overline{b} \end{cases}$$

where $\underline{b}$ and $\overline{b}$ are the tolerances of the expected pay-off $x^T By$ and $\mu_2(x^T By)$ should be determined in objective

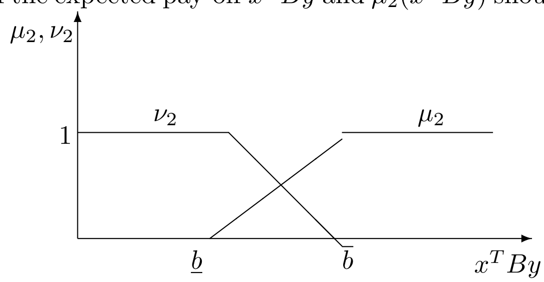

Figure 2: Membership and nonmembership functions for player II

allowable region $[\underline{b}, \overline{b}]$ and $\underline{b} \leq \min_{i,j} b_{ij}$, $\overline{b} \geq \max_{i,j} b_{ij}$. Also, the membership and nonmembership functions can be written as

$$\mu_2(x^T By) = \begin{cases} 1; & x^T By; \geq \overline{b} \\ c_2^{(1)} + x^T \hat{B} y; & \underline{b} < x^T By < \overline{b} \\ 0; & x^T By \leq \overline{b} \end{cases}$$

$$\nu_2(x^T By) = \begin{cases} 1; & x^T By; \leq \overline{b} \\ c_2^{(2)} - x^T \hat{B} y; & \underline{b} < x^T By < \overline{b} \\ 0; & x^T By \geq \overline{b} \end{cases}$$

where $\hat{B} = \frac{1}{\overline{b} - \underline{b}} B, c_2^{(1)} = \frac{-\underline{b}}{\overline{b} - \underline{b}}$ and $c_2^{(2)} = \frac{-\overline{b}}{\underline{b} - \overline{b}}$.

Since all the membership and non-membership functions of the IF goals are linear, the optimal solution is equal to degree of attainment of the fuzzy goal for the matrix game. Since S_I and S_{II} are convex polytopes, for the choice of such unimodal, linear membership and non-membership functions, the existence of Nash equilibrium of the game is guaranteed.

Thus the NES is equal to optimal solution of the following mathematical problem

$$\begin{aligned}
P1: \quad & \max_x \mu_1(x^T\hat{A}y^*) \text{ and } \min_x \nu_1(x^T\hat{A}y^*)\\
s.t. \quad & \sum_{i=1}^{m} x_i = 1; \quad x_i \geq 0; i = 1,2,\ldots,m.\\
\text{and} & \\
P2: \quad & \max_y \mu_2(x^{*T}\hat{B}y) \text{ and } \min_y \nu_2(x^{*T}\hat{B}y)\\
s.t. \quad & \sum_{j=1}^{n} y_j = 1; \quad y_i \geq 0; i = 1,2,\ldots,n.
\end{aligned}$$

Let (x^*,y^*) are the optimal solution to $P1$ and $P2$ respectively. From the above two problems, we see that the constraints are separable in the decision variables, x and y. Hence the two problems yield the following single objective mathematical programming problem

$$\begin{aligned}
P3: \quad & \max_{x,y}\{\mu_1(x^T\hat{A}y^*) + \mu_2(x^{*T}\hat{B}y)\} + \min_{x,y}\{\nu_1(x^T\hat{A}y^*) + \nu_2(x^{*T}\hat{B}y)\}\\
s.t. \quad & \sum_{i=1}^{m} x_i = 1 \text{ and } \sum_{j=1}^{n} y_j = 1; \quad x_i \geq 0; \quad y_i \geq 0.
\end{aligned}$$

Theorem 1 *If all the membership, non membership functions of the IF goals are linear, the NES with respect to the degree of attainment of the IF goal aggregated by a minimum component is equal to the optimal solution of the non-linear programming problem*

$$\begin{aligned}
P4: \quad & \max_{x,y,p_1,p_2,q_1,q_2,\lambda_1,\lambda_2,\beta_1,\beta_2} \{\lambda_1 + \lambda_2 + p_2 + q_2 - \beta_1 - \beta_2 - p_1 - q_1\} && (25)\\
\textit{subject to} \quad & \hat{A}y + c_1^{(1)}e^m \leq p_1 e^m; \quad -\hat{A}y + c_1^{(2)}e^m \geq p_2 e^m && (26)\\
& \hat{B}x + c_2^{(1)}e^n \leq q_1 e^n; \quad -\hat{B}x + c_2^{(2)}e^n \geq q_2 e^n && (27)\\
& x^T\hat{A}y + c_1^{(1)} \geq \lambda_1; \quad -x^T\hat{A}y + c_1^{(2)} \leq \beta_1 && (28)\\
& x^T\hat{B}y + c_2^{(1)} \geq \lambda_2; \quad -x^T\hat{B}y + c_2^{(2)} \leq \beta_2 && (29)\\
& \lambda_1 + \beta_1 \leq 1; \quad \lambda_2 + \beta_2 \leq 1; \quad \lambda_1 \geq \beta_1; \quad \lambda_2 \geq \beta_2 && (30)\\
& x_1 + x_2 + x_3 = 1; \quad y_1 + y_2 + y_3 = 1 && (31)\\
& x_i, y_i, \lambda_i, \beta_i, p_i, q_i \geq 0, &&
\end{aligned}$$

where $e^m = (1,1,\ldots,1)^T; e^n = (1,1,\ldots,1)^T$.

Theorem 2 *If all the membership and non membership functions are linear, as introduced in Sections* 4.4 *and* 4.5 *and* (x^*,y^*) *is a NES for* $\Gamma = \langle\{I,II\},A,B\rangle$*, then* (x^*,y^*) *is a NES for* Γ *with respect to the degree of attainment of the IF goal.*

Corollary: Let the conditions $\underline{a} \leq \min_{i,j} \alpha_{ij}$, $\overline{a} \geq \max_{i,j} \alpha_{ij}$ and $\underline{b} \leq \min_{i,j} \gamma_{ij}$, $\overline{b} \geq \max_{i,j} \gamma_{ij}$ hold, then, (x^*,y^*) is a NES for $\Gamma = \langle\{I,II\},A,B\rangle$ if and only if (x^*,y^*) is a NES with respect to the degree of attainment of the IF goal.

Thus, once an optimal solution $(x^*,y^*,p_1^*,p_2^*,q_1^*,q_2^*,\lambda_1^*,\lambda_2^*,\beta_1^*,\beta_2^*)$ of the non-linear programming problem (as in Theorem 1) has been obtained, (x^*,y^*) gives an equilibrium solution of the bi-matrix game. The degree of attainment of IF goals G_1 and G_2 can then be determined by evaluating $x^{*T}Ay^*$ and $x^{*T}By^*$, then employing the membership and non-membership function μ_1,ν_1 and μ_2,ν_2.

5 Multi Objective IF Games

For a single objective bi-matrix games, the NES is already defined in Section 3 in this paper. Here we have developed the computational procedure for NES with respect to degree of attainment of the IF goal in multi objective bi-matrix games. Let us consider the multi objective bi-matrix games $(A^k, B^l); k = 1, 2, \dots, n1$ and $l = 1, 2, \dots, n2$. Let player I's membership, non membership functions of the IF goal of the k^{th} pay-off matrix be $\mu_1^k(x^T\hat{A}^k y); \nu_1^k(x^T\hat{A}^k y); k = 1, 2, \dots, n1$ and that of player II's membership, non membership functions of the IF goal of the l^{th} pay-off matrix be $\mu_2^l(x^T\hat{B}^l y); \nu_2^l(x^T\hat{B}^l y); l = 1, 2, \dots, n2$, where

$$\begin{aligned}\hat{A}^k &= \frac{1}{\overline{a}^k - \underline{a}^k}A^k; \quad \hat{c}_1^k = \frac{-\underline{a}^k}{\overline{a}^k - \underline{a}^k}; k = 1, 2, \dots, n1\\ \hat{B}^k &= \frac{1}{\overline{b}^l - \underline{b}^l}B^l; \quad \hat{c}_1^l = \frac{-\underline{b}^l}{\overline{b}^k - \underline{b}^k}; l = 1, 2, \dots, n2.\end{aligned}$$

Definition 13 (**IF goal**) A IF goal $\hat{G}_1^k$ with respect to the k^{th} pay-off matrix for player I is defined as a IFS on D_1 characterized by the membership and nonmembership functions

$$\begin{gathered}\mu_{\hat{G}_1^k} : D_1^k \to [0, 1] \text{ and } \nu_{\hat{G}_1^k} : D_1^k \to [0, 1]\\ \text{or simply, } \mu_1^k : D_1^k \to [0, 1] \text{ and } \nu_1^k : D_1^k \to [0, 1]\end{gathered}$$

such that $0 \leq \mu_{\hat{G}_1^k}(x) + \nu_{\hat{G}_1^k}(x) \leq 1$. Similarly, a IF goal $\hat{G}_2^l$ with respect to the l^{th} pay-off matrix for player II is IFS on the set D_2 characterized by the membership function $\mu_{\hat{G}_2^l} : D_2^l \to [0, 1]$ and nonmembership function $\nu_{\hat{G}_2^l} : D_2^l \to [0, 1]$ such that $0 \leq \mu_{\hat{G}_2^l}(x) + \nu_{\hat{G}_2^l}(x) \leq 1$.

For a bi matrix game, if the player I chooses the mixed strategy x and the player II chooses the mixed strategy y, the k^{th} pay-off for the player I is represented by $P_1^k = x^T A^k y$ and that the l^{th} pay-off for the player II is represented by $P_2^l = x^T B^l y$.

5.1 Aggregation Rule

Let us consider the NES with respect to the degree of attainment of the IF goal aggregated by a minimum component, which is used in multiple criteria decision making, Bellman and Zadeh [4]. A membership, non-membership function value for a IF goal can be interpreted as the degree of attainment [14] of the IF goal for a strategy of a payoff. According to Atanassov's property [2] of IFS, the intersection of IF objective(s) and constraints is defined as (23). Basically, in IF multiple criteria decision making the aggregation corresponds to the union of all the IFS intersection, and the solution is determined by maximizing the membership function and minimizing the non-membership function of aggregated function of the intersection. Thus the player I and the player II IF goals aggregated by minimum component are respectively as

$$\begin{gathered}\mu_1(x, y) = \min_k \mu_1^k(x^T\hat{A}^k y); \quad \nu_1(x, y) = \max_k \nu_1^k(x^T\hat{A}^k y)\\ \mu_2(x, y) = \min_l \mu_2^l(x^T\hat{B}^l y); \quad \nu_2(x, y) = \max_l \nu_1^l(x^T\hat{B}^l y).\end{gathered}$$

A pair of strategies (x^*, y^*) is said to NES with respect to the degree of attainment o f the fuzzy goal aggregated by the minimum component for multi objective games $(\hat{A}^k, \hat{B}^l); k = 1, 2, \dots, n1$ and $l = 1, 2, \dots, n2$, if for any other mixed strategies x and y,

$$\begin{gathered}\min_k \mu_1^k(x*^T\hat{A}^k y^*) \geq \min_k \mu_1^k(x^T\hat{A}^k y^*); \quad \max_k \nu_1^k(x*^T\hat{A}^k y^*) \leq \max_k \nu_1^k(x^T\hat{A}^k y^*)\\ \min_l \mu_2^l(x*^T\hat{B}^k y^*) \geq \min_l \mu_2^l(x^T\hat{B} y^*); \quad \max_l \nu_2^l(x*^T\hat{B}^k y^*) \leq \max_l \nu_2^l(x^T\hat{B}^k y^*).\end{gathered}$$

Thus the NES is equal to optimal solution of the following mathematical problem

$$\begin{aligned}P5: &\quad \max_x \min_k \mu_1^k(x^T\hat{A}^k y^*) \text{ and } \min_x \max_k \nu_1^k(x^T\hat{A}^k y^*)\\ s.t. &\quad \sum_{i=1}^{m} x_i = 1; \quad x_i \geq 0; i = 1, 2, \dots, m.\\ P6: &\quad \max_y \min_l \mu_2^l(x^{*T}\hat{B}^l y) \text{ and } \min_y \max_l \nu_2^l(x^{*T}\hat{B}^l y)\\ s.t. &\quad \sum_{j=1}^{n} y_j = 1; \quad y_i \geq 0; i = 1, 2, \dots, n.\end{aligned}$$

Therefore, the NES is equal to the optimal solution of the following mathematical programming problem

$$\begin{aligned} \text{P7}: \quad & \max_{x,y}\{\min_k \mu_1^k(x^T\hat{A}y^*) + \min_l \mu_2^l(x*^T\hat{B}y) - \min_k \nu_1^k(x^T\hat{A}y^*) - \min_l \nu_2^l(x*^T\hat{B}y)\} \\ \text{s. t.} \quad & \sum_i x_i = 1; \ \sum_j y_j = 1 \\ & x_i, y_j \geq 0, \end{aligned}$$

where x^* and y^* are the optimal solutions. The process of solving IFO model can be divided into two steps, which include aggregation of IF objective(s) and constraints, aggregated by Atanassov's property [1] of IFS AND defuzzification of the aggregated function so that the IFO model is changed into crisp one. Let us consider the solution of the game with respect to the degree of attainment of the IF goal aggregated by minimum component [2]. Such rule is often used in decision making [16].

Theorem 3 *If all the membership functions of the IF goals are linear, the NES with respect to the degree of attainment of the IF goal aggregated by a minimum component is equal to the optimal solution of the non-linear programming problem :*

$$P8: \max_{x,y,p_1,p_2,q_1,q_2,\lambda_1,\lambda_2,\beta_1,\beta_2}\{\lambda_1 + \lambda_2 + p_2 + q_2 - \beta_1 - \beta_2 - p_1 - q_1\} \tag{32}$$

subject to

$$\begin{aligned} \hat{A}^k y + c_1^{(1)k} e^m &\leq p_1 e^m; \ -\hat{A}^k y + c_1^{(2)k} e^m \geq p_2 e^m & (33) \\ \hat{B}^l x + c_2^{(1)l} e^n &\leq q_1 e^n; \ -\hat{B}^l x + c_2^{(2)l} e^n \geq q_2 e^n & (34) \\ x^T\hat{A}^k y + c_1^{(1)k} &\geq \lambda_1; \ -x^T\hat{A}^k y + c_1^{(2)k} \leq \beta_1 & (35) \\ x^T\hat{B}^l y + c_2^{(1)l} &\geq \lambda_2; \ -x^T\hat{B}^l y + c_2^{(2)l} \leq \beta_2 & (36) \\ \lambda_1 + \beta_1 \leq 1; \ \lambda_2 &+ \beta_2 \leq 1; \ \lambda_1 \geq \beta_1; \ \lambda_2 \geq \beta_2 & (37) \\ \sum_i x_i &= 1; \ \sum_j y_j = 1 & (38) \\ x_i, y_i, \lambda_i, \beta_i, p_i, q_i &\geq 0, \forall i \end{aligned}$$

where $e^m = (1,1,\ldots,1)^T; e^n = (1,1,\ldots,1)^T, k = 1,2,\ldots,n1$ *and* $l = 1,2,\ldots,n2$.

Proof : The Kuhn-Tucker conditions to the problem is satisfied. Let S be the set of all feasible solutions of the above non-linear programming problem $P8$, then $S \neq \phi$, the null set. From the first constraints (33), we have,

$$\begin{aligned} \hat{A}^k y &+ c_1^{(1)k} e^m \leq p_1 e^m; \ -\hat{A}^k y + c_1^{(2)k} e^m \geq p_2 e^m \\ \Rightarrow x^T\hat{A}^k y &+ c_1^{(1)k} x^T e^m \leq p_1 x^T e^m; \ -x^T\hat{A}^k y + c_1^{(2)k} x^T e^m \geq p_2 x^T e^m \\ \Rightarrow \min_k [x^T\hat{A}^k y &+ c_1^{(1)k}] \leq p_1; \ \max_k [-x^T\hat{A}^k y + c_1^{(2)k}] \geq p_2, \text{ as } x^T e^m = 1. \end{aligned}$$

Similarly, from constraints (34) we can write,

$$\min_l [x^T\hat{B}^l y + c_2^{(1)l}] \leq q_1; \ \max_l [-x^T\hat{B}^l y + c_2^{(2)l}] \geq q_2.$$

From constraints (35) and (36), we can write,

$$\begin{aligned} \min_k [x^T\hat{A}^k y + c_1^{(1)k}] &\geq \lambda_1; \ \max_k [-x^T\hat{A}^k y + c_1^{(2)k}] \leq \beta_1 \\ \min_l [x^T\hat{B}^l y + c_2^{(1)l}] &\geq \lambda_2; \ \max_l [-x^T\hat{B}^l y + c_2^{(2)l}] \leq \beta_2. \end{aligned}$$

Now, for arbitrary $\alpha = (x, y, p_1, p_2, q_1, q_2, \lambda_1, \lambda_2, \beta_1, \beta_2) \in S$, we have,

$$\begin{aligned}
\lambda_1 + \lambda_2 \quad + & \quad p_2 + q_2 - \beta_1 - \beta_2 - p_1 - q_1 \\
\leq & \quad \min_k [x^T \hat{A}^k y + c_1^{(1)k}] + \min_l [x^T \hat{B}^l y + c_2^{(1)l}] + \max_k [-x^T \hat{A}^k y + c_1^{(2)k}] \\
& \qquad + \max_l [-x^T \hat{B}^l y + c_2^{(2)l}] - \beta_1 - \beta_2 - p_1 - q_1 \\
\leq & \quad \min_k [x^T \hat{A}^k y + c_1^{(1)k}] + \min_l [x^T \hat{B}^l y + c_2^{(1)l}] + \max_k [-x^T \hat{A}^k y + c_1^{(2)k}] \\
& \qquad + \max_l [-x^T \hat{B}^l y + c_2^{(2)l}] - \max_k [-x^T \hat{A}^k y + c_1^{(2)k}] \\
& \quad - \max_l [-x^T \hat{B}^l y + c_2^{(2)l}] - \min_k [x^T \hat{A}^k y + c_1^{(1)k}] - \min_l [x^T \hat{B}^l y - c_2^{(1)l}] \\
\leq & \quad 0,
\end{aligned}$$

which shows that the optimal value of the objective function is non positive. Thus, the solution (x^*, y^*),

$$\begin{aligned}
p_1^* &= \min_k [x^{*T} \hat{A}^k y^* + c_1^{(1)}]; \quad p_2^* = \max_k [-x^{*T} \hat{A}^k y^* + c_1^{(2)}]; \\
q_1^* &= \min_l [x^{*T} \hat{B}^l y^* + c_2^{(1)}]; \quad q_2^* = \max_l [-x^{*T} \hat{B}^l y^* + c_2^{(2)}]; \\
\lambda_1^* &= \min_k [x^{*T} \hat{A}^k y^* + c_1^{(2)}]; \quad \beta_1^* = \max_k [-x^{*T} \hat{A}^k y^* + c_1^{(2)}]; \\
\lambda_2^* &= \min_l [x^{*T} \hat{B}^l y^* + c_2^{(2)}]; \quad \beta_2^* = \max_l [-x^{*T} \hat{B}^l y^* + c_2^{(2)}];
\end{aligned}$$

of the quadratic programming problem are feasible and optimal. Therefore

$$\alpha^* = (x^*, y^*, p_1^*, p_2^*, q_1^*, q_2^*, \lambda_1^*, \lambda_2^*, \beta_1^*, \beta_2^*) \in S.$$

The optimal value of the objective function of the non-linear programming problem $P8$ is 0. Conversely, let $\alpha^* \in S$ be the optimal solution to the non-linear programming problem $P8$. Since the optimal value of the problem $P8$ is zero, we get,

$$\lambda_1^* + \lambda_2^* + p_2^* + q_2^* - \beta_1^* - \beta_2^* - p_1^* - q_1^* = 0. \tag{39}$$

From (38) and (39), we have,

$$\min_k [x^T \hat{A}^k y^* + c_1^{(1)k}] \quad \leq \quad p_1^* x^T e^m = p_1^*; \quad \max_k [-x^T \hat{A}^k y^* + c_1^{(2)k}] \geq p_2^* \tag{40}$$

$$\min_l [x^T \hat{B}^l y^* + c_2^{(1)l}] \quad \leq \quad q_1^*; \quad \max_l [-x^T \hat{B}^l y^* + c_2^{(2)l}] \geq q_2^* \tag{41}$$

and similarly, from (40) and (41), we have,

$$\min_k [x^{*T} \hat{A}^k y^* + c_1^{(2)k}] \quad \geq \quad \lambda_1^*; \quad \max_k [-x^{*T} \hat{A}^k y^* + c_1^{(2)k}] \leq \beta_1^* \tag{42}$$

$$\min_l [x^{*T} \hat{B}^l y^* + c_2^{(2)l}] \quad \geq \quad \lambda_2^*; \quad \max_l [-x^{*T} \hat{B}^l y^* + c_2^{(2)l}] \leq \beta_2^*. \tag{43}$$

Using (39),(42) and (43), we write,

$$\begin{aligned}
& p_1^* + q_1^* - p_2^* - q_2^* = \lambda_1^* + \lambda_2^* - \beta_1^* - \beta_2^* \\
& \qquad \leq \min_k [x^{*T} \hat{A}^k y^* + c_1^{(1)k}] + \min_l [x^{*T} \hat{B}^l y^* + c_2^{(2)l}] \\
& \qquad\qquad - \max_k [x^{*T} \hat{A}^k y^* + c_1^{(2)k}] - \min_l [-x^{*T} \hat{B}^l y^* + c_2^{(2)l}] \\
\Rightarrow \quad & \min_l [x^{*T} \hat{B}^l y^* + c_2^{(1)l}] - \min_l [-x^{*T} \hat{B}^l y^* + c_2^{(2)l}] \\
& \qquad \geq -\min_k [x^{*T} \hat{A}^k y^* + c_1^{(1)k}] + \max_k [-x^{*T} \hat{A}^k y^* + c_1^{(2)k}] + p_1^* + q_1^* - p_2^* - q_2^*.
\end{aligned}$$

If we consider $x = x^*, y = y^*$ as a particular case in equations (40) and (41), we get,

$$\begin{aligned}
\min_k [x^{*T} \hat{A}^k y^* \quad + \quad c_1^{(1)k}] \leq p_1^*; \qquad \max_k [-x^{*T} \hat{A}^k y^* + c_1^{(2)k}] \geq p_2^* \\
\min_l [x^{*T} \hat{B}^l y^* \quad + \quad c_2^{(1)l}] \leq q_1^*; \qquad \max_l [-x^{*T} \hat{B}^l y^* + c_2^{(2)l}] \geq q_2^*.
\end{aligned}$$

Therefore,

$$\begin{aligned} q_1^* - q_2^* &\geq -\min_k[x^{*T}\hat{A}^k y^* + c_1^{(1)k}] + \max_k[-x^{*T}\hat{A}^k y^* + c_1^{(2)k}] + p_1^* + q_1^* - p_2^* - q_2^* \\ \Rightarrow p_1^* - p_2^* &\leq \min_k[x^{*T}\hat{A}^k y^* + c_1^{(1)k}] - \max_k[-x^{*T}\hat{A}^k y^* + c_1^{(2)k}] \\ \Rightarrow p_1^* &\leq \min_k[x^{*T}\hat{A}^k y^* + c_1^{(1)k}];\;\; p_2^* \geq \max_k[-x^{*T}\hat{A}^k y^* + c_1^{(2)k}]. \qquad (44) \\ \text{Similarly, } q_1^* &\leq \min_l[x^{*T}\hat{B}^l y^* + c_2^{(1)l}];\;\; q_2^* \geq \max_l[-x^{*T}\hat{B}^l y^* + c_2^{(2)l}]. \qquad (45) \end{aligned}$$

Hence from (40), (41), (44) and (45), we write,

$$\begin{aligned} p_1^* = \min_k[x^{*T}\hat{A}^k y^* + c_1^{(1)}];\quad p_2^* = \max_k[-x^{*T}\hat{A}^k y^* + c_1^{(2)}]; \\ q_1^* = \min_l[x^{*T}\hat{B}^l y^* + c_2^{(1)}];\quad q_2^* = \max_l[-x^{*T}\hat{B}^l y^* + c_2^{(2)}]. \end{aligned}$$

Finally, using (40) and (41) and above equations, we can write

$$\begin{aligned} \min_k[x^T\hat{A}^k y^* + c_1^{(1)k}] \leq \min_k[x^{*T}\hat{A}^k y^* + c_1^{(1)k}]; \\ \min_l[x^T\hat{B}^l y^* + c_2^{(1)l}] \leq \min_l[x^{*T}\hat{B}^l y^* + c_2^{(1)l}]. \end{aligned}$$

Therefore, we conclude that the pair (x^*, y^*) is the NES with respect to the degree of attainment of the IF goal aggregated by the minimum component for a bi-matrix game. □

Solution Procedure: The process of solving IFO model can be divided into two steps, which include aggregation of IF objective(s) and constraints, aggregated by Atanassov's property [1] of IFS and defuzzification of the aggregated function so that the IFO model is changed into crisp one. Let us consider the solution of the game with respect to the degree of attainment of the IF goal aggregated by minimum component. Basically, in IF decision making the aggregation corresponds to the union of all the IFS intersection, and the solution is determined by maximizing the membership function and minimizing the non-membership function of aggregated function of the intersection.

6 Illustrative Example

Let us consider the player I and player II to have three strategies and three objectives which are defined in the following pair of matrices as

$$\begin{aligned} A^1 &= \begin{pmatrix} 2 & 4 & 5 \\ 5 & 8 & 9 \\ 6 & 2 & 7 \end{pmatrix}; A^2 = \begin{pmatrix} 8 & 0 & 1 \\ 3 & 2 & 4 \\ 6 & 8 & 1 \end{pmatrix}; A^3 = \begin{pmatrix} 8 & 7 & 5 \\ 3 & 6 & 6 \\ 0 & 4 & 1 \end{pmatrix} \\ B^1 &= \begin{pmatrix} 7 & 9 & 0 \\ 5 & 8 & 9 \\ 3 & 2 & 1 \end{pmatrix}; B^2 = \begin{pmatrix} 5 & 0 & 1 \\ 9 & 2 & 4 \\ 2 & 8 & 4 \end{pmatrix}; B^3 = \begin{pmatrix} 1 & 8 & 5 \\ 0 & 6 & 2 \\ 7 & 4 & 1 \end{pmatrix}. \end{aligned}$$

The tolerances are arbitrary fixed for each matrices for each membership and non-membership functions of the players. Let the IF goals for the first objective of the player I be represented by the following linear membership and non-membership functions,

$$\mu_1^1(x^TA^1y) = \begin{cases} 1; & x^TA^1y \geq 9 \\ \frac{x^TA^1y-2}{7}; & 2 < x^TA^1y < 9 \\ 0; & x^TA^1y \leq 2 \end{cases}$$

$$\nu_1^1(x^TA^1y) = \begin{cases} 1; & x^TA^1y \leq 2 \\ \frac{x^TA^1y-9}{-7}; & 2 < x^TA^1y < 9 \\ 0; & x^TA^1y \geq 9 \end{cases}$$

Similarly, let the IF goals for the first objective of the player II be represented by the following linear membership and non-membership functions,

$$\mu_2^1(x^TB^1y) = \begin{cases} 1; & x^TB^1y \geq 9 \\ \frac{x^TB^1y-0}{9}; & 0 < x^TB^1y < 9 \\ 0; & x^TB^1y \leq 0 \end{cases}$$

$$\nu_2^1(x^TB^1y) = \begin{cases} 1; & x^TB^1y \leq 0 \\ \frac{x^TB^1y-9}{-9}; & 0 < x^TB^1y < 9 \\ 0; & x^TB^1y \geq 9 \end{cases}$$

Similarly for others. The solutions are obtained using LINGO software with formulating the problem by the help of $P8$ giving NES. The results for some combinations is shown in the following table:

NES	x_1	x_2	x_3	y_1	y_2	y_3	V
(A^1, B^1)	0.8234	0.1176	0.0590	0.8181	0.0000	0.1819	$\langle 3.11, 5.52\rangle$
(A^2, B^2)	0.0000	0.1000	0.9000	0.0000	0.3925	0.6075	$\langle 3.69, 5.33\rangle$
(A^3, B^3)	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	$\langle 6, 6\rangle$

7 Conclusion

In this paper, we have presented a model for studying multi-objective bi-matrix games with IF goals. In this approach, the degree of acceptance and the degree of rejection of objective and constraints are introduced together, which both cannot be simply considered as a complement each other and the sum of their value is less than or equal to 1. Since the strategy spaces of player I and player II could be polyhedral sets, we may also conceptualize constrained IF bi-matrix game on the lines of crisp constrained bi-matrix games. On the basis, we have defined the solution in terms of degree of attainment of IF goal in IFS environment, and find it by solving an mathematical programming problem. A numerical example has illustrated the proposed methods. This theory can be applied in decision making theory such as economics, operation research, management, war science, etc.

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