Chaos Control of a Non-linear Finance System

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Abstract

The usefulness of chaos control for a kind of nonlinear chaos finance system is studied. First, the selection of feedback gain matrix control, its revision of gain matrix control, linear feedback control and speed feedback control are used to control chaos to unstable equilibrium point. Then the Lyapunov direct method and Routh-Hurwith are used to analyze and study the gain’s scope of controlled system and the asymptotic stability at the equilibrium point. At last, the validity of the four controlling methods is proved through theoretical analysis and numerical simulations.

Keywords: chaos finance system, chaos control, Lyapunov method, linear feedback control, speed feedback control

1 Introduction

Nonlinear chaotic dynamical system research is popular problem in the nonlinear science field. Nonlinear chaotic systems have been extensively studied within scientific, engineering and mathematical communities [3, 5, 6]. Since the chaotic phenomenon in economics was first found in 1985, great impact has been imposed on the prominent economics at present, because the occurrence of the chaotic phenomenon’s in the economic system means that the macro economic operation has in itself the inherent indefiniteness. Researches on the complicated economic system by applying nonlinear methods have been fruitful [7, 8, 9]. In the field of finance, stocks and social economics, due to the interaction between nonlinear factors, with the evolution process from low dimensions to high dimensions, the diversity and complexity have manifested themselves in the internal structure of the system and there exists extremely complicated phenomenon and external characteristics in such a kind of system. So it has become more and more important to make a systematic and deep study in the internal structural characteristics in such a complicated economic system.

In this paper, we study the chaos control of a nonlinear finance system. The selection of gain matrix control, the revision of gain matrix control [4], the speed feedback control and linear feedback control [1] are used to suppress chaos to unstable equilibrium points. The Lyapunov direct method [2, 10] and Routh-Hurwith are used to analyze the asymptotic stability at the equilibrium point. Moreover, numerical simulations are applied to verify the effectiveness of chosen controllers.

This article is organized as follows. In Section 2, a new chaotic finance system is introduced. The speed feedback control and linear feedback control is considered in Section 3. In Section 4, we discuss the selection of feedback gain matrix control and it’s revision of feedback gain matrix control. Four numerical examples are given to demonstrate the effectiveness of the proposed methods in Section 5. Finally the conclusion is given in Section 6.

2 The No-linear Finance Chaotic System

The nonlinear finance chaotic system can be described by the following differential equation:

\[
\begin{align*}
\dot{x} &= z + (y - a)x \\
\dot{y} &= 1 - by - x^2 \\
\dot{z} &= -x - cz
\end{align*}
\]

(1)
where variable \( x \) represents the interest rate in the model; variable \( y \) represents the investment demand and variable \( z \) is the price exponent. The parameter \( a \) is the saving, \( b \) is the per investment cost, \( c \) is the elasticity of demands of commercials. And they are positive constants.

When \( c-b-abc \leq 0 \), system (1) has only equilibrium point \((0, 1/b, 0)\). When \( c-b-abc > 0 \), system (1) has another two equilibrium points \((\pm \theta, (1 + ac)/c, \theta \theta / c)\), where \( \theta = \sqrt{(c-b-abc)/c} \).

We have a transformation of system (1) at the equilibrium point \((\theta, (1 + ac)/c, -\theta / c)\):

\[
\begin{align*}
X &= x - \theta \\
Y &= y - (1 + ac)/c \\
Z &= z + \theta / c.
\end{align*}
\] (2)

Then system (1) is rewritten as follows:

\[
\begin{align*}
X &= X / c + \theta Y + XY + Z \\
Y &= -X^2 - 2\theta X - bY \\
Z &= -X + cZ.
\end{align*}
\] (3)

For system (3), it has three equilibrium points \( p_0 (0, 0, 0), p_1 (-2\theta, 0, 2\theta / c), p_2 (-\theta, \theta^2 / b, \theta / c) \).

When \( c-b-abc < 0, c+b-1/c<0 \), then the balance point \( P_0 (0, 0, 0) \) of system (3) under the conditions is the stable convergence.

When \( c-b-abc > 0, c+b-1/c<0 \), the balance point \( P_0 (0, 0, 0) \) of system (3) under the conditions is the saddle.

When \( c-b-abc = 0, 0<\theta<1 \), then according to Ref [4], the equilibrium point \( P_0 (0, 0, 0) \) is the unstable point of the non-hyperbola.

When \( c-b-abc = 0, \theta>1 \), According to the central manifold theorem, the balance point \( P_0 (0, 0, 0) \) is also gradually inclined to be stable.

When \( bc^3 + b^2 c^3 - 2ab^2 c^2 + (2ab - 2 - 3b^2)c + 3b = 0 \), Bifurcation occurs at \( P_1 (-2\theta, 0, 2\theta / c) \), and it is unstable equilibrium point.

When \( a=0.9, b=0.2, c=1.2 \), the system (3) has a new chaotic attractor, as shown in Fig.1.

![Figure 1: The new attractor of system (3)](image)

### 3 Controlling Chaotic Attractor to Equilibrium \( P_0 \)

In this section, we will control chaotic system (3) to equilibrium point \( P_0 (0, 0, 0) \) by using two different feedback methods, suppose that the controlled system is the following form:
In which \( u_i \) \((i=1, 2, 3)\) are external control inputs. By using the suitable input function \( u_i \) \((i=1, 2, 3)\), the chaotic trajectory \((x, y, z)\) of system (4) will be dragged to \( P_0 (0, 0, 0)\).

3.1 Linear Feedback Control

We assume that \( u_3 = 0, u_1, u_2 \) are of the nonlinear forms, \( u_1 = -k_1 x, u_2 = -k_2 y \), where \( k_i \) \((i=1,2)\) are feedback coefficients. We can get the controlled system (3) as follows:

\[
\begin{align*}
\dot{X} &= X / c + \theta Y + Z + XY - k_1 X \\
\dot{Y} &= -2 \theta X - b Y - X^2 + u_2 \\
\dot{Z} &= -X - c Z + u_1.
\end{align*}
\]

(4)

Theorem 1 When \( k_1 + k_2 + 1 / (c + \theta / 2 - b) > 0 \), the controlled system (5) is asymptotically stable at equilibrium point \( P_0 (0, 0, 0) \).

Proof: we divide system (5) into subsystems:

\[
\begin{align*}
\dot{X} &= X / c + \theta Y + Z + XY - k_1 X \\
\dot{Y} &= -2 \theta X - b Y - X^2 - k_2 Y \\
\dot{Z} &= -X - c Z.
\end{align*}
\]

(5)

The time derivative of \( V \) along the trajectories of system (6) is

\[
\dot{V}_1 = (1 / c - k_1) X^2 - (b + k_1) Y^2 - \theta XY + XZ.
\]

Let \( \left\| X, Y \right\| = \sqrt{X^2 + Y^2} \). When \( k_1 = k_2 + b + 1 / c > 0 \),

\[
\dot{V}_1 \leq (1 / c - k_1) X^2 - (b + k_1) Y^2 + \theta |X| |Y| + |X||Z| \\
\leq (1 / c - k_1) X^2 - (b + k_2) Y^2 + \theta (X^2 + Y^2) / 2 + \sqrt{X^2 + Y^2} |Z| \\
= -(b + k_2 - \theta / 2)(X^2 + Y^2) + \sqrt{X^2 + Y^2} |Z|.
\]

When \( \sqrt{X^2 + Y^2} \geq |Z| / (k_2 + b - \theta / 2 - \epsilon) \), then \( \left\| X, Y \right\| \geq |Z| / (k_2 + b - \theta / 2 - \epsilon) \),

\[
\dot{V}_1 = -\epsilon (X^2 + Y^2) = -\epsilon \left\| X, Y \right\|^2.
\]

Let \( a_1 (r) = \epsilon r^2 \). Then \( \chi_1 (r) = r / (k_2 + b - \theta / 2 - \epsilon) \).

Let \( a_2 (r) = a_1 (r) = r^2 / 2 \). Then \( \gamma_1 (r) = a_1^{-1} \times a_1 \times \chi_1 (r) = r / (k_2 + b - \theta / 2 - \epsilon) \).

For system (7), let \( \dot{V}_2 (X, Y) = Z^2 / 2 \). The time derivative of \( V \) along the trajectories of system (7) is

\[
\dot{V}_2 = -c Z^2 - XZ \leq -c Z^2 + |Z| \sqrt{X^2 + Y^2}.
\]

When \( |Z| \geq \sqrt{X^2 + Y^2} / c - \epsilon = \left\| X, Y \right\| / c - \epsilon \), \( \dot{V}_2 \leq -\epsilon Z^2 \), let \( a_2 (r) = \epsilon r^2 \). Then \( \chi_2 (r) = r / c - \epsilon \). Let \( a_3 (r) = a_2 (r) = r^2 / 2 \). Then \( \gamma_2 (r) = a_2^{-1} \times a_2 \times \chi_2 (r) = r / (c - \epsilon) \).

When \( k_2 > 1 / c + \theta / 2 - b > 0 \), we can get the gain function

\[
\gamma_1 (\gamma_2 (r)) = r / (c - \epsilon) (k + b - \theta / 2 - \epsilon) < r.
\]

According to Ref [11], system (5) is asymptotically stable at equilibrium point \( p_0 (0, 0, 0) \).
3.2 Speed Feedback Control

We assume that \( u_1 = u_2 = 0 \), and \( u_3 \) is the speed form \( u_3 = -k_1 X \), where \( k_1 \) is a speed feedback coefficient, the controlled chaotic system (4) is rewritten as:

\[
\begin{align*}
\dot{X} &= X / c + \theta Y + XY + Z \\
\dot{Y} &= -2\theta X - bY - X^2 \\
\dot{Z} &= -X - cZ - k_1 (X / c + \theta Y + Z + XY).
\end{align*}
\]  

(8)

The Jacobi matrix of system (8) is

\[
J = \begin{bmatrix}
\frac{1}{c} & \theta & 1 \\
-2\theta & -b & 0 \\
-k_1 / c - 1 & -k_1 \theta & -c - k_1
\end{bmatrix}.
\]  

(9)

The characteristic equation of matrix (9) is

\[
\lambda^3 + p_1 \lambda^2 + p_2 \lambda + p_3 = 0
\]

where

\[
\begin{align*}
p_1 &= b + c - 1 / c + k_1, \\
p_2 &= bc - b / c + bk_1 + 2\theta^2, \\
p_3 &= 2c - 2b - 2abc.
\end{align*}
\]

When \( a = 0.9, b = 0.2, c = 1.2 \), it is easy to see that \( p_1 = 1.568 > 0 \), according to Routh-Hurwitz criterion,

\[
p_1 > 0, p_1 p_2 > p_3,
\]  

(10)

i.e., \( k_1 > 0.4937 \).

The real parts \( \lambda \) of Jacobi matrix (9) with the equilibrium point \( p_0(0,0,0) \) are all negative, when \( a = 0.9, b = 0.2, c = 1.2 \) and \( k_1 \) satisfy (10), the controlled chaotic system (8) is asymptotically stable at equilibrium point \( p_0(0,0,0) \).

4 Controlling Chaotic Attractor to Equilibrium \( P_1 \)

In this section, we will control chaotic system to equilibrium \( p_1(-2\theta,0,2\theta / c) \) by using different feedback methods. Suppose that the control system is the following form:

\[
\begin{align*}
\dot{X} &= X / c + \theta Y + XY + Z + u_1 \\
\dot{Y} &= -2\theta X - bY - X^2 + u_2 \\
\dot{Z} &= -X - cZ + u_3
\end{align*}
\]  

(11)

where

\[
\begin{align*}
u_1 &= -k_{13} (X - \bar{X}), u_2 = -k_{22} (Y - \bar{Y}), u_3 = -k_{33} (Z - \bar{Z}).
\end{align*}
\]

There \( k_i (i = 1,2,3) \) are feedback coefficients.

4.1 The Selection of Gain Matrix Control

Diverted the controlled system to the equilibrium point \( p_1(-2\theta,0,2\theta / c) \), then the controlled system is rewritten as

\[
\begin{align*}
\dot{X} &= X / c + XY + \theta Y + Z - k_{13} (X + 2\theta) \\
\dot{Y} &= -X^2 - 2\theta X - bY - k_{22} Y \\
\dot{Z} &= -X - cZ - k_{33} (Z - 2\theta / c)
\end{align*}
\]  

(12)

Let \( x = X + 2\theta, y = Y, z = Z - 2\theta / c \),

Then system (12) is rewritten as
\[
\begin{align*}
\dot{x} &= x/c + xy - \theta y + z - k_1 x \\
\dot{y} &= -x^2 + 2\theta x - by - k_{22} y \\
\dot{z} &= -x - cz - k_{33} z .
\end{align*}
\] (13)

Then the equilibrium point \( p_1(-2\theta, 0, 2\theta/c) \) of system (12) turns into \((0, 0, 0)\) of system (13).

**Theorem 2** When \( k_i (i = 1, 2, 3) \) are positive, the equilibrium \( p_1(-2\theta, 0, 2\theta/c) \) of system (12) is asymptotically stability when \( a = 0.9, b = 0.2, c = 1.2 \).

**Proof:** The Jacobi matrix of system (13) is

\[
\begin{bmatrix}
1/c - k_{11} & -\theta & 1 \\
2\theta & -b - k_{22} & 0 \\
-1 & 0 & -c - k_{33} \\
\end{bmatrix}.
\] (14)

The characteristic equation of matrix (6) is

\[
\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0 ,
\]

where

\[
\begin{align*}
B_1 &= k_{11} + k_{22} + k_{33} + b + c + 1/c , \\
B_2 &= bc - b/c + 2\theta^2 + c(k_{11} + k_{22}) + b(k_{11} + k_{33}) + k_{11} k_{22} + k_{11} k_{33} + k_{22} k_{33} - k_{22}/c - k_{33}/c , \\
B_3 &= 2\theta(c + k_{33}) + bk_{11}(c + k_{33}) + ck_{11} k_{33} + k_{11} k_{22} - k_{22} k_{33}/c - k_{33}/c.
\end{align*}
\]

According to Routh-Hurwitz criterion, the real parts \( \lambda \) are negative if and only if \( B_1 > 0, B_2 > 0, B_2 B_3 > B_1 \), \( k_i (i = 1, 2, 3) > 0 \).

When \( a = 0.9, b = 0.2, c = 1.2 \) and \( k_{11} = k_{33} = 0, k_{22} > 0 \), the controlled chaotic system (12) is asymptotically stable at equilibrium point \( P_1(-2\theta, 0, 2\theta/c) \).

With similar method, when \( k_i \) get different feedback coefficients, we can get the same effect and then \( P_2 \) is asymptotically stable.

### 4.2 Revision of Gain Matrix Control

We assume that \( u_2 = u_3 = 0, u_x = k_1(-x - cz) + k_2(x - x) \), where \( k_1, k_2 \) are feedback coefficients, then system (11) are is rewritten as:

\[
\begin{align*}
\dot{X} &= X/c + \theta Y + XY + Z + k_1(-X - cz) - k_2(X - X) \\
\dot{Y} &= -2\theta X - b Y - X^2 \\
\dot{Z} &= -X - cz .
\end{align*}
\] (15)

Diverted the controlled system (15) to equilibrium point \( P_1(-2\theta, 0, 2\theta/c) \), then system (15) is rewritten as:

\[
\begin{align*}
\dot{X} &= X/c + \theta Y + XY + Z + k_1(-X - 2\theta) - k_2(X + 2\theta) \\
\dot{Y} &= -2\theta X - b Y - X^2 \\
\dot{Z} &= -X - cz .
\end{align*}
\] (16)

Let \( x = X + 2\theta, y = Y, z = Z - 2\theta/c \). Then system (16) is rewritten as

\[
\begin{align*}
\dot{x} &= x/c + xy - \theta y + z - k_1 x - k_2 x \\
\dot{y} &= -x^2 + 2\theta x - by \\
\dot{z} &= -x - cz .
\end{align*}
\] (17)

**Theorem 3** When \( k_i (i = 1, 2) \) are positive, the equilibrium \( p_2(-2\theta, 0, 2\theta/c) \) of system (16) is asymptotically stability when \( a = 0.9, b = 0.2, c = 1.2 \).

**Proof:** The Jacobi matrix of system (17) is
\[
\begin{bmatrix}
1/c - k_1 - k_2 & -\theta & 1 \\
2\theta & -b & 0 \\
-1 & 0 & -c
\end{bmatrix}. \tag{18}
\]

The characteristic equation of matrix (18) is
\[
\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0,
\]
where
\[
B_1 = k_1 + k_2 + k_3 + b + c - 1/c,
B_2 = bc - b/c + bk_1 + bk_2 + ck_2 + 2\theta^2,
B_3 = bck_2 + 2c\theta^2.
\]

According to Routh-Hurwitz criterion, the real parts \( \lambda \) are negative if and only if \( B_1 > 0, B_2 > 0, B_3 > 0, \) and \( k_i(i = 1, 2) > 0. \)

When \( a = 0.9, b = 0.2, c = 1.2 \) and \( k_1, k_2 > 0, \) the controlled chaotic system (16) is asymptotically stable at equilibrium point \( P_1(-2\theta, 0, 2\theta/c) \).

5 Numerical Results

To verify the effectiveness and feasibility of the control approach, by using Matlab program, the numerical simulations have been completed. In the simulations, we choose the parameters \( a=0.9, b=0.2, c=1.2. \)

5.1 Linear Feedback Control

The initial states of the controlled system (5) are selected as \( (4.5, 5.5, 3.5) \), and the corresponding feedback coefficients are given by \( k_1 = 1.5, k_2 = 2.5, \) the behaviors of the states \( (x, y, z) \) of the controlled chaotic system (5) with time are displayed in Fig.2 (a).

5.2 Speed Feedback Control

The initial states of the controlled system (8) are selected as \( (900, 500, 900) \), and the corresponding coefficient is given by \( k_1 = 1.5, \) the behaviors of the states \( (x, y, z) \) of the controlled chaotic system (5) with time are displayed in Fig.2 (b).

![Figure 2: The behaviors of the states \( (x, y, z) \) of the controlled chaotic system (5)](a) (b)
5.3 The Selection of Gain Matrix Control

The initial states of the controlled system (13) are selected as \((0.5, 0.2, 0.6)\), and the corresponding feedback coefficient is given by \(k_1 = k_3 = 2, k_2 = 6.75\), the behaviors of the states \((x, y, z)\) of the controlled chaotic system (13) with time are displayed in Fig.3(a).

5.4 The Revision of Gain Matrix Control

The initial states of the controlled system (17) are selected as \((2, 5.5, 2)\), and the corresponding feedback coefficient are given by \(k_1 = 0.5, k_2 = 1.5\), the behaviors of the states \((x, y, z)\) of the controlled chaotic system (17) with time are displayed in Fig.3(b).

![Figure 3: The behaviors of the states \((x, y, z)\) of the controlled chaotic system](image)

6 Conclusion

The control problem of a chaotic finance system is investigated. The selection of gain matrix control and its revision of gain matrix control are used to suppress chaos to unstable equilibrium point \(P_1(-2\theta, 0, 2\theta / c)\), while speed feedback control and linear feedback control are used to suppress chaos to unstable equilibrium point \(P_2(0, 0, 0)\). The Routh-Hurwitz criterion is used to study the conditions of the asymptotic stability of the controlled chaotic system. Furthermore, Numerical simulations are presented to verify the effectiveness of the proposed controllers.

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