Journal of Uncertain Systems
Vol.5, No.4, pp.257-262, 2011
Online at: www.jus.org.uk

Variable Precision Rough Set Model Based on Uncertain Measure

Liping Li[1], Minghu Ha[2,*], Chao Wang[3]
[1]College of Mathematics and Computer Sciences, Hebei University, Baoding 071002, China
[2] College of Science, Hebei University of Engineering, Handan 056038, China
[3]College of Physics Science & Technology, Hebei University, Baoding 071002, China

Received 31 May 2011; Revised 10 July 2011

Abstract

By confining the parameter α in confident threshold $1/2 < \alpha \le 1$ in rough set model based on uncertain measure, variable precision rough set model based on uncertain measure is given. The lower approximation and upper approximation are defined in approximation uncertainty space, and their properties are discussed.

Keywords: variable precision, rough sets, lower and upper approximation, uncertain measure

1 Introduction

In order to deal with imprecise, uncertain and incomplete data, the rough set was introduced by Pawlak [3] in 1982, which was a branch of uncertain mathematics. Since the rough set has been proposed, the rough set attracts more and more people's attention and is widely used in practice. Now, it becomes a popular branch of information and systems science [9]. The extension of rough set model is a main research direction of the theory of rough sets [8].

Considering the shortcoming of Pawlak rough set model in [3], the probabilistic rough set model based on probability measure is established [6]. Because the condition of additivity of probability measure is more restrictive, probabilistic rough set model is confined in some applications [7]. Therefore, many scholars studied rough set model based on non-additivity measures [1], which expanded the rough set model. For example, Tian et al. [5] proposed rough set model based on uncertain measure.

The rough set model in [5] contains two parameters α,β , and their restrictive condition is same; the constraint between α and β is not strict simultaneously. With different values of α,β , we have a different understanding a notion of the universe [4]. Therefore, decision risk may increase in decision-making practical application. In order to deal with the above problem, variable precision rough set model based on uncertain measure is proposed in this paper.

2 Preliminaries

In this section, we will introduce the notations of uncertain measure, conditional uncertain measure, and the approximations of rough set model based on uncertain measure.

Definition 1([2]) Let Γ be a nonempty set and L be a σ -algebra on Γ. Each element $\Lambda \in \mathrm{L}$ is called an event. An extended real valued set function M defined on L is called an uncertain measure, if it satisfies:

(1) (Normality) $\mathrm{M}(\Gamma)=1$;

(2) (Monotonicity) $\mathrm{M}(\Lambda_1) \le \mathrm{M}(\Lambda_2)$ whenever $\Lambda_1, \Lambda_2 \in \mathrm{L}$, and $\Lambda_1 \subseteq \Lambda_2$;

(3) (Self-Duality) $\mathrm{M}(\Lambda)+\mathrm{M}(\Lambda^c)=1$ for every $\Lambda \in \mathrm{L}$;

(4) (Countable Subadditivity)

[*] Corresponding author. Email: minghuha@yahoo.com.cn (M. Ha).

$$M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \le \sum_{i=1}^{\infty} M(\Lambda_i).$$

For every countable sequence $\{\Lambda_i\}$ of sets in L. The triplet (Γ, L, M) is an uncertainty space.

Theorem 1([2]) Let M be an uncertainty measure and be defined on (Γ, L). If $A_1, A_2 \in L$, then

$$M(A_1 \cup A_2) \le M(A_1) + M(A_2).$$

Definition 2([2]) Let (Γ, L, M) be an uncertainty space, $A, B \in L$. Then the conditional uncertain measure of A given B is defined by

$$M(A|B) = \begin{cases} \dfrac{M(A\cap B)}{M(B)}, & \text{if } \dfrac{M(A\cap B)}{M(B)} < 0.5 \\ 1 - \dfrac{M(A^C\cap B)}{M(B)}, & \text{if } \dfrac{M(A^C\cap B)}{M(B)} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Definition 3([5]) Let U be a nonempty set, R be an equivalence relation on U, and the equivalence classes of R be $U/R = \{X_1, X_2, \cdots, X_n\}$. The equivalence class containing x is denoted by $[x]$. Let M be an uncertain measure on σ-algebra assembled by subsets of U. Then the triplet $A_M = (U, R, M)$ is called an approximation uncertainty space. Any subset of U is called a notion, which represents a random event. $M(A|B)$ is the conditional uncertain measure of A given B.

Definition 4([5]) Let $0 \le \beta < \alpha \le 1$ and $X \subseteq U$. Then the lower and upper approximations with respect to the approximation uncertainty space $A_M = (U, R, M)$ according to parameters α, β are defined as follows:

$$\underline{M}_\alpha(X) = \{x \in U \mid M(X|[x]) \ge \alpha\};$$
$$\overline{M}_\beta(X) = \{x \in U \mid M(X|[x]) > \beta\}.$$

Definition 5([5]) If $0 \le \beta < \alpha \le 1$, then the approximation accuracy $\eta(X, \alpha, \beta)$ of X with respect to approximation uncertainty space $A_M = (U, R, M)$ according to α, β is defined by

$$\eta(X, \alpha, \beta) = \frac{|\underline{M}_\alpha(X)|}{|\overline{M}_\beta(X)|}.$$

The numbers of the elements contained in X are expressed by $|X|$.

3 Variable Precision Rough Set Model Based on Uncertain Measure

Definition 6 Let U be a nonempty set, R be an equivalence relation on U, and the equivalence classes of R be $U/R = \{X_1, X_2, \cdots, X_n\}$. The equivalence class containing x is denoted by $[x]$. Let M be an uncertain measure over σ-algebra assembled by subsets of U. Then the triplet $A_M = (U, R, M)$ is called an approximation uncertainty space. Let $1/2 < \alpha \le 1$ and $\forall X \subseteq U$. Then its lower and upper approximations with respect to the approximation uncertainty space $A_M = (U, R, M)$ according to parameters α are defined as follows:

$$\underline{M}_\alpha(X) = \{x \in U \mid M(X|[x]) \ge \alpha\},$$
$$\overline{M}_\alpha(X) = \{x \in U \mid M(X|[x]) > 1 - \alpha\}.$$

The positive region, negative region and boundary region of X with respect to $A_M = (U, R, M)$ according to parameters α are defined by:

$$pos(X, \alpha) = \underline{M}_\alpha(X) = \{x \in U \mid M(X\|[x]) \ge \alpha\},$$
$$bn(X, \alpha) = \{x \in U \mid 1 - \alpha < M(X|[x]) \le \alpha\},$$

$$\text{neg}(X,\alpha) = U/\overline{M}_\alpha(X) = \{x \in U \mid M(X \mid [x]) \le 1-\alpha\}.$$

Obviously, the above definitions are equivalent to the following equations:

$$\underline{M}_\alpha(X) = \cup\{[x] \mid M(X \mid [x]) \ge \alpha,\ x \in U\},\ \overline{M}_\alpha X = \cup\{[x] \mid M(X \mid [x]) > 1-\alpha,\ x \in U\},$$

$$\text{pos}(X,\alpha) = \underline{M}_\alpha(X) = \cup\{[x] \mid M(X \mid [x]) \ge \alpha,\ x \in U\},$$

$$\text{bn}(X,\alpha) = \cup\{[x] \mid 1-\alpha < M(X \mid [x]) \le \alpha,\ x \in U\},$$

$$\text{neg}(X,\alpha) = U/\overline{M}_\alpha(X) = \cup\{[x] \mid M(X \mid [x]) \le 1-\alpha,\ x \in U\}.$$

The positive region, negative region and boundary region of X with respect to $A_M = (U,R,M)$ according to parameter α form a division of U, which is

$$\overline{M}_\alpha(X) = \text{pos}(X,\alpha) \cup \text{bn}(X,\alpha),\ \text{bn}(X,\alpha) = \overline{M}_\alpha(X)/\text{pos}(X,\alpha).$$

When $\overline{M}_\alpha(X) = \underline{M}_\alpha(X)$, or $\text{bn}(X,\alpha) = \phi$, X is called a defined set with respect to $A_M = (U,R,M)$ according to parameter α, otherwise X is called a rough set with respect to $A_M = (U,R,M)$ according to parameter α.

Theorem 2 Let $A_M = (U,R,M)$ be an approximation uncertainty space. If $1/2 < \alpha \le 1$, $X \subseteq U$ and $\sim X = U - X$, then

$$\text{pos}_\alpha(\sim X) = \text{neg}_\alpha(X).$$

Proof: Because

$$\text{pos}_\alpha(X) = \underline{M}_\alpha(X) = \{x \in U \mid M(X \| [x]) \ge \alpha\},$$

$$\overline{M}_\alpha(X) = \{x \in U \mid M(X \| [x]) > 1-\alpha\},$$

$$\text{neg}_\alpha(X) = U/\overline{M}_\alpha(X) = \{x \in U \mid M(X \| [x]) \le 1-\alpha\},$$

and $\sim X = U - X$,
so

$$\text{pos}_\alpha(\sim X) = \text{pos}_\alpha(U - X) = U/\overline{M}_\alpha(X).$$

We have

$$\text{pos}_\alpha(\sim X) = \text{neg}(X).$$

Example 1([5]) Let $U = \{x_1, x_2, x_3, x_4\}$ be the universe and R be an equivalence relation on U. Let $M\{x\}$ be the uncertain measure defined on a σ-algebra over U. Its equivalence class be $U/R = \{E_1, E_2, E_3\}$, $E_1 = \{x_1, x_2\}$, $E_2 = \{x_3\}$, $E_3 = \{x_4\}$, $X = \{x_1, x_4\}$ is one of the subsets of U. We define

$$M(E_1) = 0.4,\ M(X \cap E_1) = 0.2,\ M(E_3) = 0.3,\ M(X \cap E_2) = 0.$$

Then

$$M(X|E_1) = \frac{M(X \cap E_1)}{M(E_1)} = \frac{1}{2},\ M(X|E_2) = \frac{M(X \cap E_2)}{M(E_2)} = 0,$$

$$M(X|E_3) = \frac{M(X \cap E_3)}{M(E_3)} = \frac{3}{4}.$$

According to the above Definition 4, we can obtain the lower and supper approximate sets of X with respect to R according to the parameters α, β in rough set based on uncertain measure.

Let $\alpha = 0.3, \beta = 0$.

$$\underline{M}_{0.3}(X) = E_1 \cup E_3,\ \overline{M}_0 = E_1 \cup E_3.$$

Let $\alpha = 0.6, \beta = 0.4$.

$$\underline{M}_{0.6}(X) = E_3,\ \overline{M}_{0.4} = E_1 \cup E_3.$$

According to the above Definition 6, we can obtain the lower and supper approximate sets of X with respect to R according to the parameters α, β in variable precision rough set based on uncertain measure.

Let $\alpha = 0.7, 1-\alpha = 0.3$.

$$\underline{M}_{0.7}(X) = E_3,\ \overline{M}_{0.7} = E_1 \cup E_3.$$

Let $\alpha = 0.9, 1-\alpha = 0.1$.

$$\underline{M}_{0.9}(X) = \phi, \overline{M}_{0.9} = E_1 \cup E_3.$$

According to the above Example 1, we obtain: With different values of α, β, both $\overline{M}_\alpha(X)$ and $\underline{M}_\alpha(X)$ have different values in rough set based on uncertain measure; if for any α $(1/2 < \alpha \le 1)$, then $\overline{M}_\alpha(X) = E_1 \cup E_3$ and $\underline{M}_\alpha(X) \subseteq \overline{M}_\alpha(X)$ are permanent establishment in variable precision rough set based on uncertain measure. By comparing with the model in Definition 4, the model in Definition 6 decreases the decision risk in dealing with practical problems.

Definition 7 Let U be a nonempty set and $A_M = (U, R, M)$ be an approximation uncertainty space. If for any $X \subseteq U$ and $\alpha = 1/2$, the boundary of X with respect to $A_M = (U, R, M)$ according to parameters α is defined by:

$$bn_\alpha(X) = \left\{ x \in U \,\middle|\, M(X \| [x]) = \frac{1}{2} \right\},$$

then $bn_{1/2}(X)$ is the boundary region according to the parameters α in $A_M = (U, R, M)$.

Theorem 3 Let $1/2 < \alpha \le 1$ and $X, Y \subseteq U$. Then the approximation operators satisfy the flowing dual properties:

(1) $\underline{M}_\alpha(X) \subseteq X, \underline{M}_\alpha(X) \subseteq \overline{M}_\alpha(X)$;

(2) $\underline{M}_\alpha(\phi) = \overline{M}_\alpha(\phi) = \phi, \underline{M}_\alpha(U) = \overline{M}_\alpha(U) = U$;

(3) $\overline{M}_\alpha(X) \cup \overline{M}_\alpha(Y) \subseteq \overline{M}_\alpha(X \cup Y)$;

(4) $\underline{M}_\alpha(X) \cap \underline{M}_\alpha(Y) \supseteq \underline{M}_\alpha(X \cap Y)$;

(5) $\underline{M}_\alpha(X) \cup \underline{M}_\alpha(Y) \subseteq \underline{M}_\alpha(X \cup Y)$;

(6) $\overline{M}_\alpha(X) \cap \overline{M}_\alpha(Y) \supseteq \overline{M}_\alpha(X \cap Y)$;

(7) $\underline{M}_\alpha(\sim X) = \sim \overline{M}_\alpha(X), \overline{M}_\alpha(\sim X) = \sim \underline{M}_\alpha(X)$;

(8) If $X \subseteq Y$, then

$$\underline{M}_\alpha(X) \subseteq \underline{M}_\alpha(Y), \overline{M}_\alpha(X) \subseteq \overline{M}_\alpha(Y);$$

(9) If $\alpha_1 \le \alpha_2$, then

$$\overline{M}_{\alpha_1}(X) \subseteq \overline{M}_{\alpha_2}(X), \underline{M}_{\alpha_1}(X) \supseteq \underline{M}_{\alpha_2}(X), bn(X, \alpha_1) \subseteq bn(X, \alpha_2),$$
$$neg(X, \alpha_2) \subseteq neg(X, \alpha_1).$$

Proof: (1) and (2) can be obtained directly from Definition 6.

(3) Because

$$\overline{M}_\alpha(X) = \{ x \in U | M(X \| [x]) > 1-\alpha \},$$
$$\overline{M}_\alpha(Y) = \{ x \in U | M(Y \| [x]) > 1-\alpha \},$$

if $x \in \overline{M}_\alpha(X) \cup \overline{M}_\alpha(Y)$, then $x \in \overline{M}_\alpha(X)$ or $\overline{M}_\alpha(Y)$.

According to Theorem 1 we can obtain that

$$M((X \cup Y) \| [x]) > \mathbf{max}\{ M(X \| [x]), M(Y \| [x]) \}.$$

Then $M((X \cup Y) \| [x]) > 1-\alpha$, $x \in \overline{M}_\alpha(X \cup Y)$. So $\overline{M}_\alpha(X) \cup \overline{M}_\alpha(Y) \subseteq \overline{M}_\alpha(X \cup Y)$.

(4) Let $x \in \underline{M}_\alpha(X \cap Y)$. Then $M((X \cap Y) \| [x]) > \alpha$.

According to the nature of uncertainty measure we can obtain that

$$M((X \cap Y) \| [x]) < \mathbf{min}\{ M(X \| [x]), M(Y \| [x]) \}.$$

Then

$$\underline{M}_\alpha(X) = \{ x \in U | M(X \| [x]) \ge \alpha \},$$
$$\underline{M}_\alpha(Y) = \{ x \in U | M(Y \| [x]) \ge \alpha \},$$
$$x \in \underline{M}_\alpha(X) \cap \underline{M}_\alpha(Y).$$

So $\underline{M}_\alpha(X)\cap\overline{M}_\alpha(Y)\supseteq\underline{M}_\alpha(X\cap Y)$.

Similarly, we can conclude that (5) holds true.

(6) is proved according to the proof of (4).

(7) is proved according to Theorem 2.

(8) and (9) are obtained according to the definitions of the lower and upper approximations.

Theorem 4 Let U be a nonempty set and $A_M=(U,R,M)$ be an approximation uncertainty space. If $X\subseteq U$ and $\alpha\to 1/2$, then

(1) $\lim\limits_{\alpha\to 1/2}\underline{M}_\alpha(X)=\bigcup\limits_{\alpha>1/2}\underline{M}_\alpha(X)=\underline{M}_{1/2}(X)$;

(2) $\lim\limits_{\alpha\to 1/2}\overline{M}_\alpha(X)=\bigcap\limits_{\alpha>1/2}\overline{M}_\alpha(X)=\overline{M}_{1/2}(X)$;

(3) $\lim\limits_{\alpha\to 1/2}\mathrm{bn}(X,\alpha)=\bigcap\limits_{\alpha\to 1/2}\mathrm{bn}(X,\alpha)=\mathrm{bn}\left(X,\frac{1}{2}\right)$;

(4) $\lim\limits_{\alpha\to 1/2}\mathrm{neg}_\alpha(X)=\bigcup\limits_{\alpha\to 1/2}\mathrm{neg}_\alpha(X)=\mathrm{neg}_{1/2}(X)$.

Proof: (1) As $1/2<\alpha\le 1$, we have

$$\underline{M}_\alpha(X)=\left\{x\in U\,\middle|\,M(X\|[x])\ge\alpha\right\},$$

$$\underline{M}_{1/2}(X)=\left\{x\in U\,\middle|\,M(X\|[x])\ge\frac{1}{2}\right\}.$$

And according to the properties of the uncertain measure and the feature of the set we can obtain that

$$\underline{M}_\alpha(X)\subseteq\underline{M}_{1/2}(X).$$

Then

$$\lim_{\alpha\to 1/2}\underline{M}_\alpha(X)=\bigcup_{\alpha>1/2}\underline{M}_\alpha(X)\subseteq\underline{M}_{1/2}(X).$$

If $\exists x_0\in\underline{M}_{1/2}(X)\setminus\bigcup\limits_{\alpha>1/2}\underline{M}_\alpha(X)$, then $x_0\in\underline{M}_{1/2}(X)$, so $M(X\|[x_0])\ge 1/2$.

Because $x_0\notin\underline{M}_\alpha(X)$, $M(X\|[x_0])<\alpha$. And because $\alpha\to 1/2$, this is a contradiction.

Because $M(X\|[x_0])<\alpha$, $M(X\|[x_0])>1/2$. So

$$\bigcup_{\alpha>1/2}\underline{M}_\alpha(X)=\underline{M}_{1/2}(X).$$

Then

$$\lim_{\alpha\to 1/2}\underline{M}_\alpha(X)=\bigcup_{\alpha>1/2}\underline{M}_\alpha(X)=\underline{M}_{1/2}(X).$$

Similarly, (2), (3) and (4) are proved.

In order to describe the approximation of X with respect to $A_M=(U,R,M)$ according to parameters α, then approximation accuracy $\eta(X,\alpha)$ and approximation roughness $\rho(X,\alpha)$ are defined by

$$\rho(X,\alpha)=1-\frac{\left|\underline{M}_\alpha(X)\right|}{\left|\overline{M}_\alpha(X)\right|},$$

$$\eta(X,\alpha)=1-\rho(X,\alpha)=\frac{\left|\underline{M}_\alpha(X)\right|}{\left|\overline{M}_\alpha(X)\right|}.$$

$\left|\underline{M}_\alpha(X)\right|$ represents the number of elements contained in the lower approximation.

Theorem 5 Let $\eta(X,\alpha)$ be approximation accuracy in an approximation uncertainty space $A_\alpha=(U,R,M)$. Then

$$0\le\eta(X,\alpha)\le 1.$$

Proof: Because X is definite set, $\underline{M}_\alpha(X)=\overline{M}_\alpha(X)$, $\mathrm{bn}(X,\alpha)=\phi$, we obtain $\eta(X,\alpha)=1$.

If X is rough set,

$$\underline{M}_\alpha(X)\ne\overline{M}_\alpha(X),\ \mathrm{bn}(X,\alpha)\ne\phi,\ \underline{M}_\alpha(X)\subseteq\overline{M}_\alpha(X).$$

And because

$$\eta(X,\alpha)=\frac{|\underline{M}_{\alpha}(X)|}{|\overline{M}_{\alpha}(X)|}.$$

Then $0 \le \eta(X,\alpha) < 1$. If and only if $\underline{M}_{\alpha}(X)=\phi$ and $\overline{M}_{\alpha}(X)=\mathrm{bn}(X,\alpha)$,

$$\eta(X,\alpha)=0.$$

Therefore, the conclusion is proved.

4 Conclusions

Rough set model based on uncertain measure considers the uncertainty of available information in the approximation uncertainty space, but the range of its parameter values is imprecise. Therefore, decision risk maybe increased in decision-making practical application. Comparing with rough set model based on uncertain measure, the parameter α in confident threshold $1/2 < \alpha \le 1$ is contained in variable precision rough set model based on uncertain measure. So the model decreases the decision risk in dealing with practical problems.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 60773062; 61073121) and Natural Science Foundation of Hebei University (No. 2008-125).

References

[1] Ha, M.H., L.Z. Yang, and C.X. Wu, *The Introduction of Generalized Fuzzy Set Valued Measure*, Beijing, Science Press, 2009.

[2] Liu, B., *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.

[3] Pawlak, Z., Rough sets, *International Journal of Computer and Information Science*, vol.11, pp.341–356, 1982.

[4] Sun, B.Z., and Z.T. Kong, Variable precision probabilistic rough set model, *Journal of Northwest Normal University*, vol.41, no.4, pp.23–27, 2005.

[5] Tian, D.Z., L. Wang, J. Wu, and M.H. Ha, Rough set model based on uncertain measure, *Journal of Uncertain Systems*, vol.3, no.4, pp.252–255, 2009.

[6] Wang, J.Y., and L.M. Xu, Probabilistic rough set models, *Computer Science*, vol.29, no.8, pp.76–78, 2002.

[7] Yao, Y.Y., and S.K.M. Wong, Generalized probabilistic rough set models, *Expert Systems*, vol.20, no.5, pp.287–297, 2003.

[8] Zhang, W.X., and W.Z. Wu, Rough set model based on random set, *Journal of Xi'an Jiaotong University*, vol.34, no.12, pp.75–79, 2000.

[9] Zhang, W.X., W.Z. Wu, J.Y. Liang, and D.Y. Li, *Rough Set Theory and Methods*, Science Press, Beijing, 2008.